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# Girth and planarity of the generalized Sierpiński gasket $S[G, t]$

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**Abstract.** Sierpiński gasket graphs have many applications and are studied in diverse areas including fractal theory, topology, dynamic systems and chemistry. In this paper we study and determine the girth of generalized Sierpiński gasket $S[G, t]$, in terms of the girth of arbitrary simple graph $G$. Moreover, we determine the planarity of $S[G, t]$ for some fames families of graphs.

*Keywords*: Girth, Planarity, Generalized Sierpiński gasket.
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## 1 Introduction

Decomposition into special substructures that inherit remarkable features is an important method used for the investigation of some mathematical structures, specifically when the regarded structures have self-similarity features. In these cases, we usually only need to study the substructures and the way that they are related to each other. Sierpiński type graphs appear naturally in diverse areas of mathematics and other scientific fields, see for example [4] and [9]. Sierpiński gasket graphs introduced in 1944 by Scorer, Grundy and Smith [22], are one of the most important families of such graphs which are obtained after a finite number of iterations and have a significant role in dynamic systems, random walks, Lipscomb's space, probability and psychology, see [8, 10–13, 16, 18, 26, 27] and references there in for more details. Let $G = (V, E)$

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be a simple graph with vertex set $V(G)$ and edge set $E(G)$, then an embedding of $G$ in the plane is a function $\varphi$ which assigns each vertex of $G$ a distinct point in the plane and assigns to each edge $e$ with ends $u$, $v$ a simple rectifiable curve with ends $\varphi(u)$ and $\varphi(v)$ so that this curve minus its ends is disjoint from the image of $V(G) \cup (E(G) \setminus \{e\})$. Also, a plane graph is a graph $G$ together with an embedding of $G$ in the plane. A graph is planar if there exists an embedding of it in the plane. It is well known that if $G$ is a planar graph with girth $gr(G)$, then, $|E(G)| \leqslant \frac{gr(G)}{gr(G)-2}(|V(G)|-2)$. Motivated by topological studies of the Lipscomb's space, Klavžar et al. introduced the Sierpiński graph $S(K_n, t)$ in which the base graph is the complete graph $K_n$, see [16] and [17]. More generally, see [6], the $t$-th generalized Sierpiński of an arbitrary graph $G = (V, E)$, denoted by $S(G, t)$, is the graph with vertex set $V^t$ (the set of all words of length $t$ on the alphabet $V$) and two vertices $\mathbf{u} = u_1u_2 \ldots u_t$ and $\mathbf{v} = v_1v_2 \ldots v_t$ are adjacent in it if and only if there exist $i \in \{1, \ldots, t\}$ such that
$(i)$ $u_j = v_j$ $if$ $j < i$,
$(ii)$ $u_i \neq v_i$ $and$ $u_iv_i \in E(G)$,
$(iii)$ $u_j = v_i$ $and$ $v_j = u_i$ $if$ $j > i$.
In this case, $\mathbf{uv}$ is considered as a linking edge appeared at step $i$. For convenient, we usually let $V = \{1, 2, ..., n\}$, see Figure 1. In general, $S(G, t)$ can be constructed recursively from the base

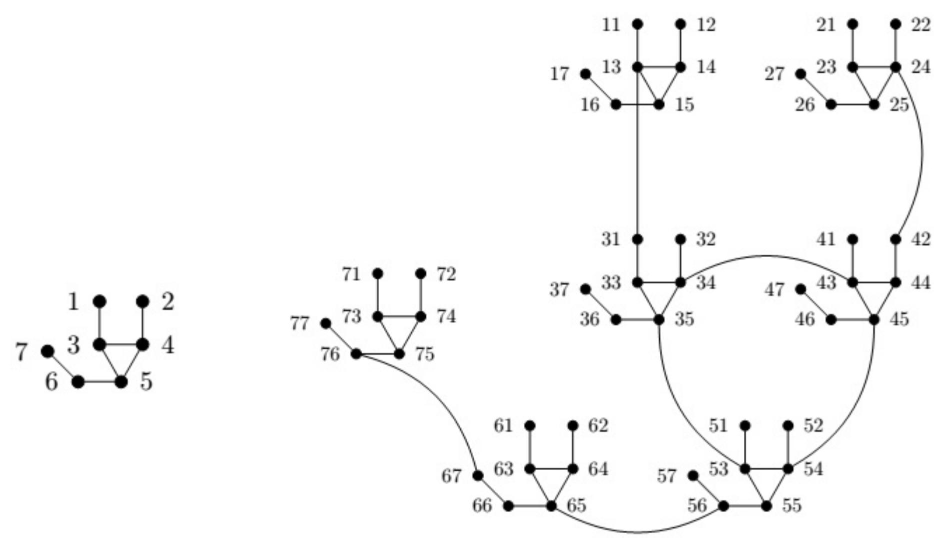


**Figure 1:** A graph $G$ and its generalized Sierpiński $S(G, 2)$.

graph $G$ with the following process: $S(G, 1) = G$ and, for $t \geq 2$, we copy $n$ times $S(G, t-1)$ and add the letter $i$ at the beginning of each label of the vertices belonging to the copy of $S(G, t-1)$ corresponding to vertex $i$. Then for each edge $ij \in E(G)$, we add an edge between two vertices $ijj...j$ and $jii...i$ (a linking edge). Vertices of the form $ii...i$ (where $1 \leq i \leq n$) are called extreme vertices. For each $i \in \{1, 2, \ldots, n\}$ let $S_i(G, t)$ be the subgraph of $S(G, t)$ induced by the vertices of the form $i...$ . Note that $S_i(G, t)$ is isomorphic to $S(G, t-1)$ and consequently, $S(G, t)$ contains $n^{t-1}$ copies of the graph $S(G, 1) = G$. Also, if $ij \in E(G)$, then the vertex $ijj...j$ in the copy $S_i(G, t)$ is adjacent to the vertex $jii...i$ in copy $S_j(G, t)$ and this is the unique edge between these two copies. It is shown in [20] that the order of generalized Sierpiński graph $S(G,t)$ is $n^t$ and its size is $|E(G)|\,\frac{n^t-1}{n-1}$. Sierpiński graphs $S(K_n,t)$ are almost regular and possess many appealing properties, as for instance several coding and several metric properties [7]. The graph $S(K_3,t)$ is isomorphic to the Tower of Hanoi game graph with $t$ disks, [1, 16, 21]. Polymer networks and WK-recursive networks can be modeled by generalized Sierpiński graphs, see [19]. The degree sequence of generalized Sierpiński graphs is completely determined in terms of the degree sequence of $G$ in [14]. As mentioned before, one of the most important families of fractal like graphs is the the family of Sierpiński gasket graphs (Sierpiński fractals) which appears frequently and has many applications in different areas. The Sierpiński gasket graph $S_t$ is just a step from the Sierpiński graph $S(K_3,t)$ and is constructed from the Sierpiński graph $S(K_3,t)$ by contracting every edge of $S(K_3,t)$ that lies in no triangle $K_3$ (i.e. by contracting all newly added edges, linking edges, during the iterations), see [15]. This connection (type) was already observed back in 1970 and in a psychological literature by Sydow [23]. The Sierpiński and the Sierpiński gasket of $G = K_3$ at step $t = 3$ are shown in Figure 2. Sierpiński gasket is

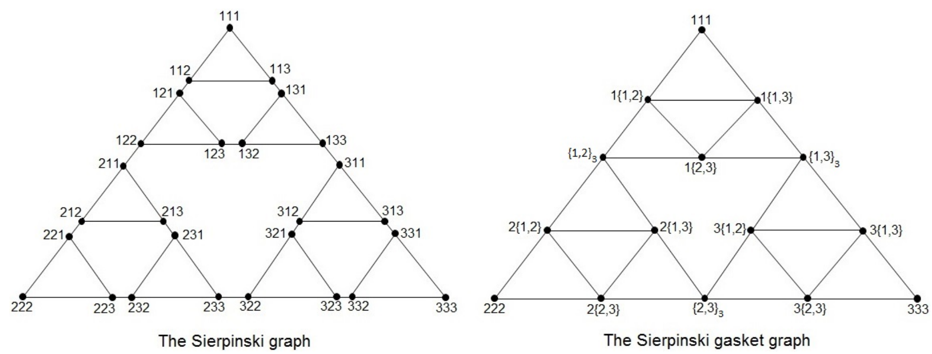


**Figure 2:** The Sierpiński graph and the Sierpiński gasket graph at step 3.

a finite structure obtained by iteration in a finite number of times which is a self-similar object and its (capacity) fractal dimension is $\frac{\log 3}{\log 2}$, [5]. Note that $S_{t+1}$ consists of three attached copies of $S_t$. In 2006 Teguia and Godbole investigated some properties of the Sierpiński gasket graphs and proved that they are pancyclic (i.e have cycles of all possible sizes) and Hamiltonian with diameter $2^{n-1}$ and domination number $3^{n-2}$ for $n \geq 3$, see [24]. Infinite Sierpiński gaskets and their spectra is considered in [25]. In [11] the same construction method is applied for any Sierpiński graph $S(K_n,t)$ by contracting edges that lie in no induced complete subgraphs $K_n$ (i.e. by contracting all linking edges) and the resulting graph is called a generalized Sierpiński gasket graph which is denoted by $S[K_n,t]$. Also, Jakovac in [11] studied several properties of graphs $S[K_n,t]$ including hamiltonicity and chromatic number. He generalized the results of Tequia and Goshow and show that $\chi(S[K_n,t]) = n$ and that $S[K_n,t]$ is Hamiltonian.

## 2 Generalized Sierpiński Gasket $\mathbf{S[G,t]}$

Since generalized Sierpiński gasket graphs $S[K_n,t]$ are important and are naturally derived from Sierpiński graphs $S(K_n,t)$ by contracting the linking edges, we apply the same for each (base) graph $G$ to construct the (more) generalized Sierpiński gasket graph $S[G,t]$.

**Definition 1.** *Let $t$ be a positive integer and $G$ be a simple graph of order $n \geq 2$ with the vertex set $V(G) = \{1, 2, \ldots n\}$. The (more) generalized Sierpiński gasket graph $S[G,t]$ is obtained by contracting all of linking edges in the generalized Sierpiński $S(G,t)$ during the iteration processes.*

Similar to the structures of Sierpiński gaskets and generalized Sierpiński graph $S(G,t)$, the generalized Sierpiński gasket $S[G,t]$ can be constructed by using all the vertices of $G$ replaced by the copies of $S[G,t-1]$ and then contracting all linking edges between these copies. Let $S_i[G,t]$ be the copy of $S[G,t-1]$ in $S[G,t]$ corresponding to the vertex $i \in V = \{1, 2, \ldots n\}$. Note that if $ij \in E(G)$, then the linking edge between two vertices $ijj \ldots j$ and $jii \ldots i$ in $S(G,t)$ is contracted in $S[G,t]$. This (newly contracted) vertex will be denoted by $\{i,j\}_t$ in $S[G,t]$ $\big($or, for convenient by $\{i,j\}$ when $t = 2\big)$ and we say that the expanded forms of $\{i,j\}_t$ are $ijj \ldots j$ and $jii \ldots i$. Some times and for convenient, we identify a contracted vertex with its expanded form representations. In fact $\{i,j\}_t$ is the unique common vertex of two copies $S_i[G,t]$ and $S_j[G,t]$, see Figure 3. Since $|V(S[G,t])| = n|V(S[G,t-1])| - |E(G)|$ and $|E(S[G,t])| = n|E(S[G,t-1])|$, the order of $S[G,t]$ is

$$n^t - (n^{t-2} + \cdots + n + 1)|E(G)| = n^t - |E(G)|\,\frac{n^{t-1} - 1}{n - 1}$$

and its size is $|E(G)|\, n^{t-1}$, see [2]. Also, when $ij \in E(G)$, two vertices $\mathbf{u} = u_1u_2 \ldots u_r ijj \ldots j$

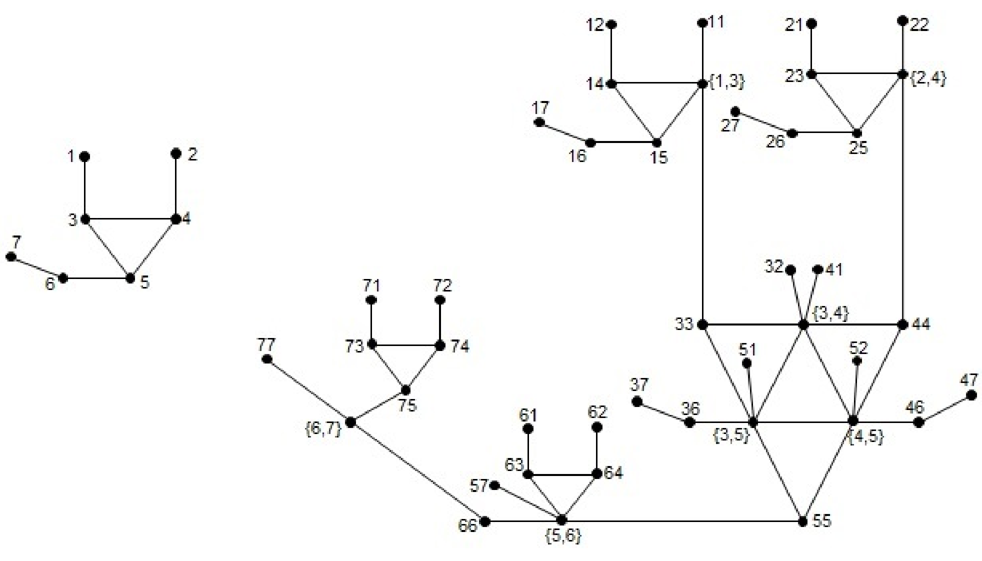


**Figure 3:** A graph $G$ and its generalized Sierpiński gasket $S[G,2]$.

and $\mathbf{v} = u_1u_2 \ldots u_r jii \ldots i$ (in which $0 \leq r \leq t-2$ and $u_l \in V$ for each $1 \leq l \leq r$) are adjacent in $S(G,t)$ and the linking edge $\mathbf{uv}$ $\big($ which is actually produced at step $t - r$ in the iteration process of $S(G,t)\big)$ is contracted in $S[G,t]$ and this (contracted) vertex may be denoted by $u_1u_2 \ldots u_r\{i,j\}_{t-r}$ $\big($or, for convenient by $u_1u_2 \ldots u_r\{i,j\}$ when $t - r = 2\big)$ whose expanded forms are $u_1u_2 \ldots u_r ijj \ldots j$ and $u_1u_2 \ldots u_r jii \ldots i$ . In [2], the degree sequence of $S[G,t]$ is determined in terms of the degree sequence of $G$ and $t$. Moreover, the general first Zagreb index of $S[G,t]$ is calculated and it is determined whether $S[G,t]$ is Eulerian or Hamiltonian. Also, in [3] chromatic and clique numbers of this graph are determined.

# 3 Main results

**Theorem 1.** *For every graph $G$, we have*

$$gr(G) = gr(S[G,t]).$$

*Proof.* Since $G$ is a subgraph of $S[G,t]$, we have $gr(S[G,t]) \leqslant gr(G)$. Now we use induction on $t$ to show that, if there exists a cycle $C$ of length $|E(C)|$ in $S[G,t]$, then, there exists a cycle of length at most $|E(C)|$ in $G$ (note that $E(C)$ is the edge set of the cycle C). Then, this implies that $gr(G) \leqslant gr(S[G,t])$, which completes the proof. Let $C$ be a cycle in $S[G,t]$. If $t = 1$, then $C$ is a cycle in $G$. Thus, we assume that $t \geqslant 2$. If $C$ is in $S_i[G,t]$, $(1 \leqslant i \leqslant n)$, then $C$ is in $S[G,t-1]$, hence, completes the proof. Now, let $C$ has passed through the copies of $i_1, i_2, \dots, i_s$, respectively. Since both copies share at most one point and $C$ is a closed cycle, hence $S \geqslant 3$ $(i_1 < i_2 < \cdots < i_s)$. Since $\{i_1, i_2\}$ is the unique common vertex between two copies $i_1$ and $i_2$, hence $i_1 i_2 \in E(G)$. Similarly, we have $i_2 i_3 \in E(G), \dots, i_s i_1 \in E(G)$. Thus, we have cycle $C' = i_1 i_2 \dots i_s$ in $G$. Therefore, $gr(G) \leqslant |C'| = S \leqslant |C|$. □

**Theorem 2.** *Let $T$ be a tree. For any positive integer $t$, $S[T,t]$ is also a tree.*

**Proposition 1.** *Let $T$ be a tree, then $S[T,t]$ is planar.*

*Proof.* By Theorem 2, since every tree is planar, hence $S[T,t]$ is planar. □

**Proposition 2.** *For each $n \geqslant 3$, there exists an embedding of $S[C_n,t]$ in the plane such that all of extreme vertices appear on the boundary of external face. Specially, $S[C_n,t]$ is planar.*

*Proof.* We use induction on $t$. For $t = 1$, we have $S[C_n,1] = C_n$ and the usual drawing of $C_n$ is planar embedding in which all of its (extreme) vertices appear on the boundary of external face. Assume that there exists an embedding of $S[C_n,t-1]$ in the plane such that all of extreme vertices appear on the boundary of external face. According to the structure of this graph, we put $n$ copies of $S[C_n,t-1]$ clockwise in the plane. We know that for each $i \in \{1,2,\dots n\}$ two vertices $i$ and $i+1$ in $C_n$ are adjacent ($i+1$ is considered in modul $n$). Since extreme vertices appear on the boundary of external face in $S[C_n,t-1]$, hence the vertex $i(i+1)(i+1)\dots(i+1)$ in $i$-th copy amalgamates with the vertex $(i+1)ii\dots i$ in $(i+1)$-th copy. Therefore, two copies corresponding to vertices $i$ and $i+1$ share the vertex $\{i,i+1\}_t$ in $S[C_n,t]$. Hence, their edges do not intersect in $S[C_n,t]$, See Figure 4. □

Recall that a subdivision of a graph is a graph obtained by inserting some new vertices on some edges. Kuratowski's theorem states that a (finite) graph is planar if and only if it contains no subgraph isomorphic to a subdivision of $K_5$ or $K_{3,3}$.

**Theorem 3.** *$S[K_n,t]$ is planar if and only if $n \leqslant 3$ or ($n = 4$ and $t \leqslant 2$).*

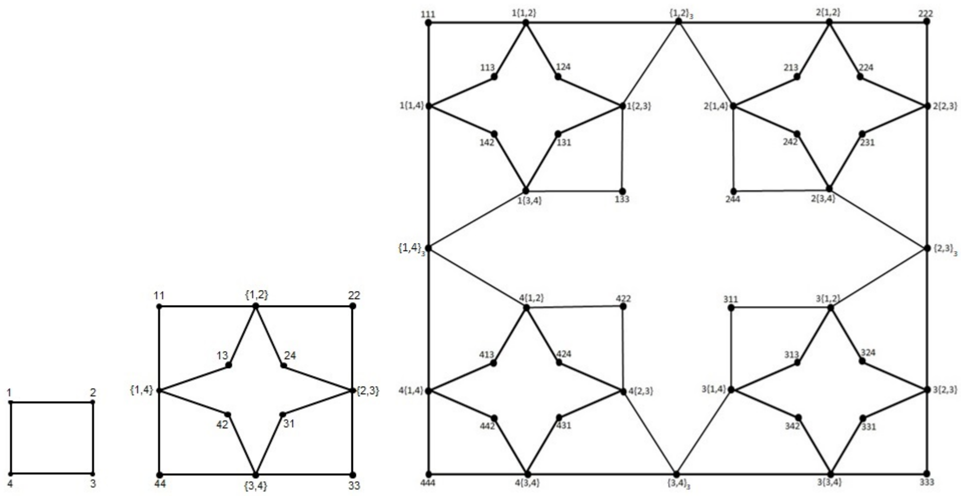


**Figure 4:** $C_4$, $S[C_4, 2]$ and $S[C_4, 3]$.

*Proof.* If $n \leqslant 2$, then $K_n$ is a path and by Theorem 1, $S[K_n, t]$ is a tree, hence it is planar. For $n = 3$, $K_3$ is a cycle and by Proposition 2, $S[K_3, t]$ is planar for each $t \geqslant 1$. For $n = 4$, we have $S[K_4, 1] = K_4$ which is planar, and $S[K_4, 2]$ is also planar as shown in Figure 5. Now we prove that $S[K_4, 3]$ contains a subdivision of $K_{3,3}$ and hence is not planar. Let $x_1 = 1\{3, 4\}$, $x_2 = \{1, 4\}_3$, $x_3 = \{1, 2\}_3$, $y_1 = 1\{2, 4\}$, $y_2 = 1\{1, 4\}$ and $y_3 = 3\{1, 2\}_2$. Then we have the following nine paths in $S[K_4, 3]$, see Figure 6.
$P_1 : x_1 = 1\{3, 4\},\ y_1 = 1\{2, 4\}$
$P_2 : x_1 = 1\{3, 4\},\ y_2 = 1\{1, 4\}$
$P_3 : x_1 = 1\{3, 4\},\ \{1, 3\}_3,\ y_3 = 3\{1, 2\}$
$P_4 : x_2 = \{1, 4\}_3,\ y_1 = 1\{2, 4\}$
$P_5 : x_2 = \{1, 4\}_3,\ y_2 = 1\{1, 4\}$
$P_6 : x_2 = \{1, 4\}_3,\ 4\{1, 4\},\ 4\{3, 4\},\ \{3, 4\}_3,\ 3\{2, 4\},\ y_3 = 3\{1, 2\}$
$P_7 : x_3 = \{1, 2\}_3,\ y_1 = 1\{2, 4\}$
$P_8 : x_3 = \{1, 2\}_3,\ 1\{1, 2\},\ y_2 = 1\{1, 4\}$
$P_9 : x_3 = \{1, 2\}_3,\ 2\{1, 3\},\ \{2, 3\}_3,\ y_3 = 3\{1, 2\}$
Since these paths are internally disjoint, we have a subdivision of $K_{3,3}$ in $S[K_4, 3]$ with partite sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$. Therefore, $S[K_4, 3]$ is not planar. Now since $S[K_4, 3]$ is a subgraph of $S[K_4, t]$, thus $S[K_4, t]$ is not planar for each $t \geqslant 3$. Also, if $n \geqslant 5$, obviously $K_5 \leqslant S[K_n, t]$ and since $K_5$ is not planar, thus $S[K_n, t]$ is not planar. □

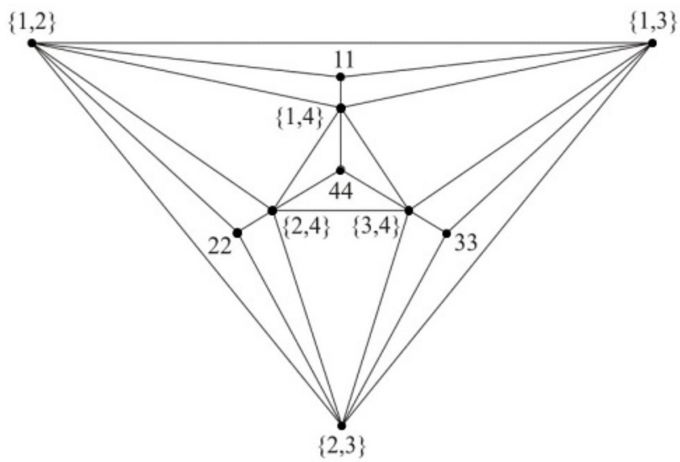


**Figure 5:** A planar embeding of $S[K_4, 2]$.

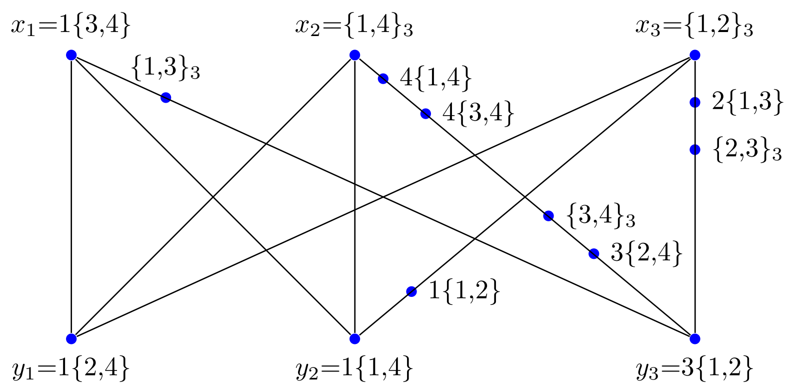


**Figure 6:** A subdivision of $K_{3,3}$ in the generalized Sierpiński gasket $S[K_4, 3]$.

**Theorem 4.** *Consider the complete bipartite graph $K_{m,n}$. Then, $S[K_{m,n}, t]$ is planar if and only if $\min\{m, n\} = 1$ or $n = m = 2$.*

*Proof.* If $m, n \geqslant 3$, then $K_{m,n}$ contains $K_{3,3}$ and hence $S[K_{m,n}, t]$ is not planar. Thus, assume that $\min\{m, n\} = m$ and $m \in \{1, 2\}$.
If $m = 1$, then $K_{m,n}$ is a star and hence, it is a tree. Now Theorem 1 implies that $S[K_{m,n}, t]$ is planar for each $t \geqslant 1$.
If $m = 2$ and $n = 2$, then $K_{2,2} = C_4$ and Proposition 2 implies that $S[K_{2,2}, t]$ is planar for each $t \geqslant 1$.
For $n = 3$ we prove that $S[K_{2,3}, 2]$ contains a subdivision of $K_{3,3}$ and hence, is not planar. Let $x_1 = \{1, 3\}$, $x_2 = \{1, 4\}$, $x_3 = \{1, 5\}$, $y_1 = 11$, $y_2 = 12$ and $y_3 = 22$. Then we have the following nine paths in $S[K_{2,3}, 2]$, see Figure 7.
$P_1 : x_1 = \{1, 3\},\ y_1 = 11$
$P_2 : x_1 = \{1, 3\},\ y_2 = 12$
$P_3 : x_1 = \{1, 3\},\ 33,\ \{2, 3\},\ y_3 = 22$

$P_4 : x_2 = \{1, 4\},\ y_1 = 11$
$P_5 : x_2 = \{1, 4\},\ y_2 = 12$
$P_6 : x_2 = \{1, 4\},\ 44,\ \{2, 4\},\ y_3 = 22$
$P_7 : x_3 = \{1, 5\},\ y_1 = 11$
$P_8 : x_3 = \{1, 5\},\ y_2 = 12$
$P_9 : x_3 = \{1, 5\},\ 55,\ \{2, 5\},\ y_3 = 22$
Since $S[K_{2,3}, 2]$ is a subgraph of $S[K_{2,3}, t]$ for each $t \geqslant 2$, thus $S[K_{2,3}, t]$ is not planar. □

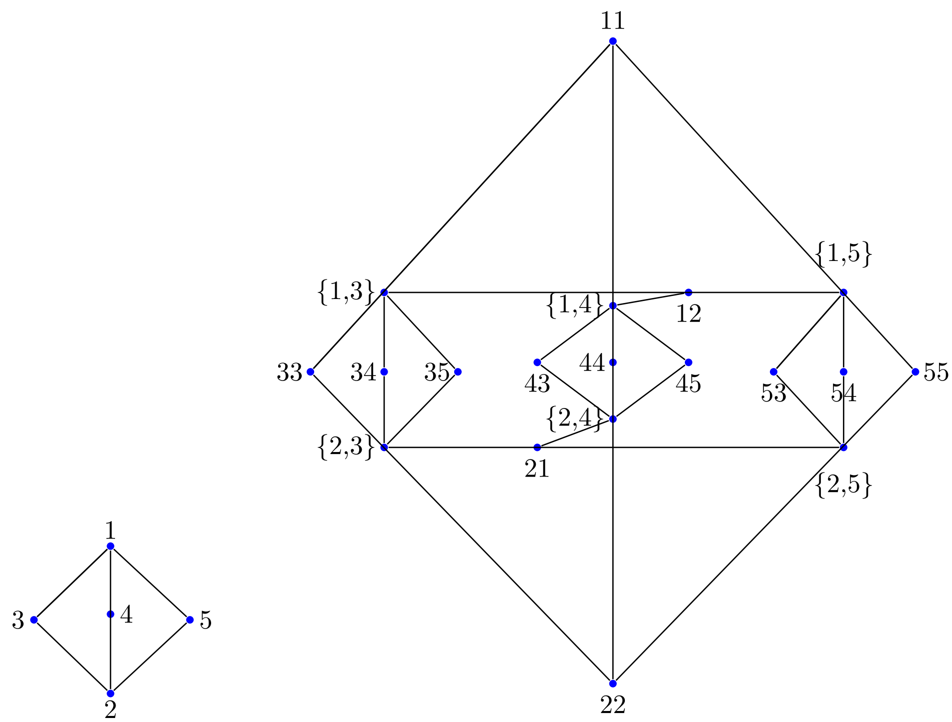


**Figure 7:** $K_{2,3}$ and $S[K_{2,3}, 2]$

Recall that the hypercube $Q_n$ is a geometric figure that develops the concept of a cube into higher dimensions. Since cube is a three-dimensional object composed of squares, a hypercube is an n-dimensional object composed of cubes. It is often depicted as a set of cubes connected at their corresponding vertices. The number of vertices, edges and faces of a hypercube growths exponentially with each additional dimension. A 1-dimensional hypercube, also known as a line segment, has 2 vertices and 1 edge. A 2-dimensional hypercube, a square, has 4 vertices, 4 edges, and 1 face. Similarly, a 3-dimensional hypercube has 8 vertices, 12 edges, 6 faces. Also, a 4-dimensional hypercube has $2^4 = 16$ vertices, $4.2^{4-1} = 32$ edges, 24 faces. Figure 8 shows four hypercubes $Q_1$, $Q_2$, $Q_3$, and $Q_4$. Also, $Q_3$ cube is sometimes drawn as shown in Figure 9.

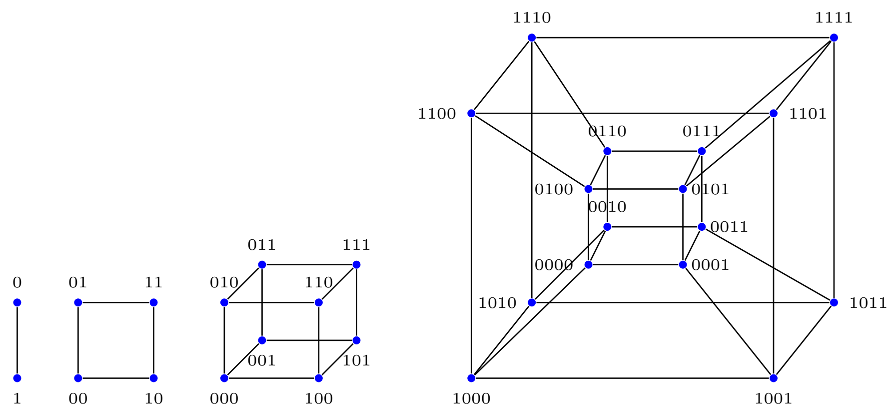


**Figure 8:** Hypercube structures for $n = 1, 2, 3, 4$.

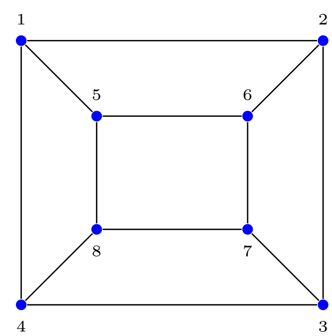


**Figure 9:** Hypercube $Q_3$.

**Theorem 5.** *For the hypercube $Q_n$, $S[Q_n, t]$ is planar if and only if $n \leqslant 2$ or ($n = 3$ and $t = 1$).*

*Proof.* If $n = 1$, then $Q_1$ is a tree and by Theorem 1, $S[Q_1, t]$ is planar for each $t \geqslant 1$. If $n = 2$, then $Q_2$ is the cycle $C_4$, and by Proposition 2, $S[Q_2, t]$ is planar for each $t \geqslant 1$. Now assume that $n = 3$. As shown in Figure 9, $Q_3 = S[Q_3, 1]$ is planar. In the following we show that $S[Q_3, 2]$ contains a subdivision of $K_{3,3}$ and hence, $S[Q_3, 2]$ is not planar. Let $x_1 = \{1, 2\}$, $x_2 = \{1, 4\}$, $x_3 = \{1, 5\}$, $y_1 = 11$, $y_2 = 13$ and $y_3 = 44$. Then we have the following nine paths in $S[Q_3, 2]$.
$P_1 : x_1 = \{1, 2\},\ y_1 = 11$
$P_2 : x_1 = \{1, 2\},\ y_2 = 13$
$P_3 : x_1 = \{1, 2\},\ 22,\ \{2, 3\},\ 33,\ \{3, 4\},\ y_3 = 44$
$P_4 : x_2 = \{1, 4\},\ y_1 = 11$
$P_5 : x_2 = \{1, 4\},\ y_2 = 13$
$P_6 : x_2 = \{1, 4\},\ y_3 = 44$
$P_7 : x_3 = \{1, 5\},\ y_1 = 11$
$P_8 : x_3 = \{1, 5\},\ 16,\ 17,\ y_2 = 13$
$P_9 : x_3 = \{1, 5\},\ 55,\ \{5, 8\},\ 88,\ \{4, 8\},\ y_3 = 44$
Since these paths are internally disjoint, we have a subdivision of $K_{3,3}$ in $S[Q_3, 2]$ with partite sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ as shown in Figure 10. Therefore, $S[Q_3, 2]$ is not planar. Since $S[Q_3, 2]$ is a subgraph of $S[Q_3, t]$ for each $t \geqslant 3$, $S[Q_3, t]$ is not planar. Also, we know that the

girth of $Q_4$ is 4, $|V(Q_4)| = 16$ and $|E(Q_4)| = 32$. Since $|E(Q_4)| = 32 \nleqslant \frac{4}{4-2}(16-2)$, $Q_4$ is not planar. Hence, $Q_n$ is not planar for each $n \geqslant 5$, because $Q_4$ is a subgraph of $Q_n$. Therefore, for each $n \geqslant 4$ and for each $t \geqslant 1$, the graph $S[Q_n, t]$ is not planar. □

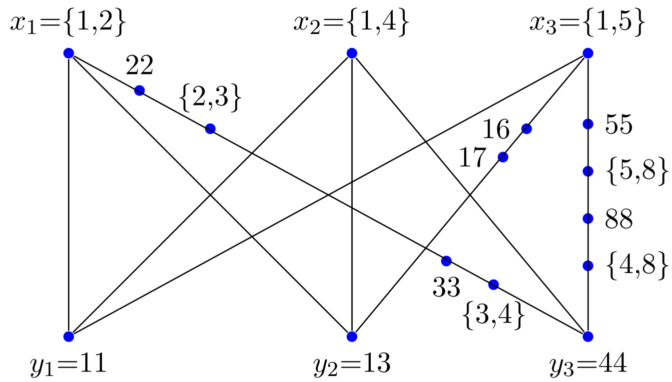


**Figure 10:** A subdivision of $K_{3,3}$ in $S[Q_3, 2]$.

Recall that a wheel $W_n$ consists of a cycle $C_n$ whose vertices are joined to a new vertex. Hence, $W_n$ is a graph of order $n+1$, see Figure 11.

**Theorem 6.** *The generalized Sierpiński Gasket wheel of $(n+1)$-vertex graph $W_n$ i.e., $S[W_n, t]$, is planar if and only if $t = 1$ or $(n, t) = (3, 2)$.*

*Proof.* For $t = 1$, we have $S[W_n, 1] = W_n$ which is obviously planar. Hence, assume that $t \geqslant 2$. If $n = 3$ and $t = 2$, then $W_3$ is isomorphic to $K_4$ and $S[W_3, 2]$ is planar by Theorem 3. Also, when $n = 3$ and $t \geqslant 3$, Theorem 3 implies that $S[W_3, t]$ is not planar. Now assume that $n \geqslant 4$ and consider the structure of $W_n$ as illustrated in Figure 11 for $W_5$. Let $x_1 = \{1, 2\}$, $x_2 = \{1, n\}$, $x_3 = \{1, 0\}$, $y_1 = 11$, $y_2 = 13$ and $y_3 = 00$. Then we have the following nine paths in $S[W_n, 2]$, see Figure 12.
$P_1 : x_1 = \{1, 2\},\ y_1 = 11$
$P_2 : x_1 = \{1, 2\},\ y_2 = 13$
$P_3 : x_1 = \{1, 2\},\ \{2, 0\},\ y_3 = 00$
$P_4 : x_2 = \{1, n\},\ y_1 = 11$
$P_5 : x_2 = \{1, n\},\ 1n-1,\ 1n-2,\ \ldots,\ 14,\ y_2 = 13$
$P_6 : x_2 = \{1, n\},\ \{n, 0\},\ y_3 = 00$
$P_7 : x_3 = \{1, 0\},\ y_1 = 11$
$P_8 : x_3 = \{1, 0\},\ y_2 = 13$
$P_9 : x_3 = \{1, 0\},\ y_3 = 00$
Since these paths are internally disjoint, we have a subdivision of $K_{3,3}$ in $S[W_n, 2]$ with partite sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$. Therefore, $S[W_n, 2]$ is not planar. Since $S[W_n, 2]$ is a subgraph of $S[W_n, t]$ for each $t \geqslant 3$, $S[W_n, t]$ is not planar. □

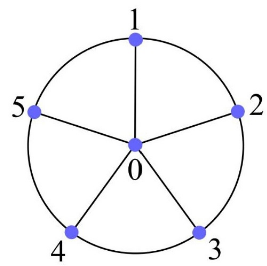


**Figure 11:** The wheel graph $W_5$.

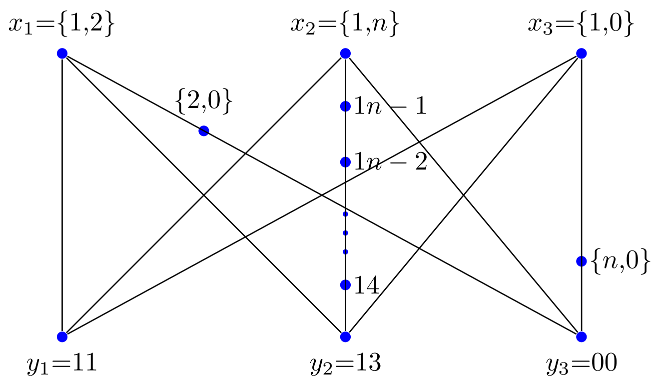


**Figure 12:** A subdivision of $K_{3,3}$ in the $S[W_n, 2]$.

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