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Characterization of rings by some filters

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Abstract. Let $R = \prod_{i \in I} R_i$ be the product of an infinite family of rings $\{R_i\}_{i \in I}$. In this study, we investigate the direct sum $\bigoplus_{i \in I} R_i$. Special attention is paid to the relationship between the ideal $\bigoplus_{i \in I} R_i$ and the $\mathcal{F}_r$ Frechet filter in I, also we show a new characterization of $\bigoplus_{i \in I} R_i$ by the $\mathcal{F}_r - \lim$.

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1 Introduction

Consider a commutative ring R with an identity element. Let $\text{Max}(R)$ and $\beta(I)$ represent the sets of all maximal ideals of R and the collection of ultrafilters on I, respectively. It is noteworthy that R is called a zero dimensional ring when all its prime ideals are maximal.

Now, let $\{R_i\}_{i \in I}$ be a nonempty family of rings, and their product denoted as $R = \prod_{i \in I} R_i$. Previous research has focused on characterizing the prime ideals of R using algebraic or logical methods (refer to [5], [7]). In the article [6], the authors have explored the correlation between ultrafilters on I and maximal ideals in the product of $\{R_i\}_{i \in I}$.

In the article [7], the authors provide a characterization of the ideals of $\text{Spec}\left(\prod_{i \in I} R_i\right)$ through $\mathcal{F}$-limit and establish a connection between the elements of $\text{Spec}\,(R_i)$ and $\text{Spec}\left(\prod_{i \in I} R_i\right)$ using $\mathcal{F}$-limit. They conclude by introducing a novel condition for a product of zero-dimensional rings to be itself zero-dimensional.

The paper is structured as follows: the first section presents properties of the $\mathcal{F}$-limit for a collection of subsets of a set A. Building upon the maximal ideals of rings R_i, a maximal ideal

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of $\prod_{i\in I} R_i$ is introduced. In the second section, a characterization of the direct sum $\bigoplus_{i\in I} R_i$ through the ideals of $\prod_{i\in I} R_i$ is provided, employing the Frechet filter.

2 Definitions and notations

We will be working in at least ZFC, that is, Zermelo-Frankel set theory with the axiom of choice. In certain cases we will use additional axioms.

We recall that $\mathcal{F}$ is a filter on a set I if it is a subset of the power set of I that satisfies the following conditions:

1. $\emptyset \notin \mathcal{F}$ and $I \in \mathcal{F}$;
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
3. If $A \in \mathcal{F}$ and $A \subset A' \subset I$, then $A' \in \mathcal{F}$.

A filter $\mathcal{F}$ on I is called an ultrafilter if $\mathcal{F}$ is maximal with respect to being a filter, or equivalently, whenever $A \subset I$, then either $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$. If $\mathcal{F}$ is an ultrafilter on an infinite set I and $J \in \mathcal{F}$, then $\mathcal{F}|J = \{J \cap F : F \in \mathcal{F}\}$ is an ultrafilter on J. An ultrafilter $\mathcal{F}$ is called principal if there exists an element $i_0 \in I$ such that $\mathcal{F}$ consists of all subsets of I that contain i_0. Other ultrafilters are called non-principal. We denote the collection of all ultrafilters on a set I by $\beta(I)$.

Definition 1 ([2])**.** *Let A be a set, $S(A)$ be the set of all subsets of A and let I be an infinite set. Let $\{S_i\}_{i\in I}$ be a family of $S(A)$, and $\mathcal{F}$ be a filter on I, then we define the $\mathcal{F}-\lim$ of $\{S_i\}_{i\in I}$ by:*

$$\mathcal{F}-\lim_{i\in I} S_i := \{a \in A : \{i \in I \ : \ a \in S_i\} \in \mathcal{F}\} = \bigcup_{X\in\mathcal{F}} (\bigcap_{i\in X} S_i).$$

We note that the set $\mathcal{F}-\lim_{i\in I} S_i$ is a subset of A.

Let $\{R_i\}_{i\in I}$ be a nonempty family of rings, $\prod_{i\in I} R_i$ denotes their product. The elements of the product are frequently considered in two different ways. A more rigorous way is to consider $\prod_{i\in I} R_i$ as the set of all functions $a : I \to \bigcup_{i\in I} R_i$ such that $a_i \in R_i$ for each $i \in I$, with addition and multiplication defined pointwise: $(a+b)(i) = a(i)+b(i)$ and $(ab)(i) = a(i)b(i)$.

In the following definition, we give a construction of the ideals of $\prod_{i\in I} R_i$ starting from the ideals of R_i using the ultrafilters.

Definition 2. *Let $R = \prod_{i\in I} R_i$, where I is an infinite set and each R_i is nonzero rings, recall that $\hat{i}$ denotes the principal ultrafilter on I generated by $\{i\}$, we define for any ideal N_i of R_i the set $(\hat{i}, N_i) := \{r \in R \ : \ r_i \in N_i\}$.*

It is easy to prove that $(\hat{i}, N_i)$ is also an ideal of $\prod_{i\in I} R_i$.

Notice that

$$(\hat{i}, N_i) = \{r \in R \ : \ \{j \in I \ : \ r_j \in N_j\} \in \hat{i}\}.$$

Let $\{R_i\}_{i\in I}$ be a family of rings and $\prod_{i\in I} R_i$ their product, the direct sum ideal of $\prod_{i\in I} R_i$ denoted by $\bigoplus_{i\in I} R_i$, is the set of $a \in \prod_{i\in I} R_i$ that are finitely nonzero.

Definition 3. *Let I an infinite set. The Frechet filter $\mathcal{F}_r$ on I consists of the co-finite sets of I, i.e. $\mathcal{F}_r := \{A \in \mathcal{P}(I) \ : \ I \setminus A \ is \ finite\}$.*

Remark 1. *The Frechet filter is an example of a free filter that is not an ultrafilter. It is used in the literature to describe topological concepts (see [2–4]).*

Example 1. Let $\mathbb{R}$ be the set of real numbers, and let $\mathbb{R}^+$ and $\mathbb{R}^-_*$ respectively be the subsets of the positive and negative stress numbers of $\mathbb{R}$. It is clear that $\mathbb{R}^+ \bigcap \mathbb{R}^-_* = \varnothing$ and $\mathbb{R}^+ \bigcup \mathbb{R}^-_* = \mathbb{R} \in \mathcal{F}_r$, but neither $\mathbb{R}^+$ nor $\mathbb{R}^-_*$ belongs to $\mathcal{F}_r$.

Definition 4. *Let $\{R_i\}_{i\in I}$ be a nonempty family of commutative unitary rings indexed by a set I and $R = \prod_{i\in I} R_i$. For $f \in R$, let $\mathcal{Y}(f) := \{i \in I \ : \ f(i) \in R_i \setminus U(R_i)\}$, where $U(R_i)$ denotes the set of all unit elements of R_i. For an ideal J of R, let $\mathcal{Y}(J) = \{\mathcal{Y}(f) \ : \ f \in J\}$.*

3 General Properties

In this part, we give some basic properties of the $\mathcal{F} - \lim$ of a collection of subsets.

Theorem 1. *Let A be a set, $\{S_i \ : \ i \in I\} \subseteq S(A)$, where I is an infinite set, and $\mathcal{F}$ an ultrafilter on I.*

1. *$\hat{k} - \lim_{i\in I} S_i = S_k$ for each $k \in I$, with $\hat{k} = \{X \subseteq I \ : \ k \in X\}$.*
2. *If $J \in \mathcal{F}$, then $\mathcal{F} - \lim_{i\in I} S_i = \mathcal{F} \mid_J - \lim_{i\in J} S_i$.*
3. *Let Γ be an infinite set, and $\sigma : \Delta \to \Gamma$ be a surjective function. For each $j \in \Gamma$ put $T_j = S_i$ if $\sigma(i) = j$, then $\mathcal{F} - \lim_{i\in\Delta} S_i = \mathcal{C} - \lim_{j\in\Gamma} T_j$, where $\sigma(\mathcal{F}) = \{\sigma[F] \ : \ F \in \mathcal{F}\} = \mathcal{C}$.*
4. *$(\mathcal{F} - \lim_{i\in I} S_i)^c = \mathcal{F} - \lim_{i\in I} S_i^c$.*

Proof. Using the definition 1, it is easy to show that (1) holds. Using a proof similar to that of Theorem 2.1 in [1], we can conclude that (2) and (3) hold.
(4) By definition 1, we have

$$\mathcal{F} - \lim_{i\in I} S_i^c = \{a \in A : \{i \in I : a \in S_i^c\} \in \mathcal{F}\}$$

If $a \in A$, then we can conclude that

$$\{i \in I : a \in S_i^c\} \in \mathcal{F} \Leftrightarrow \{i \in I : a \in S_i\} \notin \mathcal{F}$$

since $\mathcal{F}$ is ultrafilter on I. Thus,

$$\mathcal{F} - \lim_{i\in I} S_i^c = A \setminus \{a \in A : \{i \in I : a \in S_i\} \in \mathcal{F}\}$$

which implies that

$$\left(\mathcal{F} - \lim_{i\in I} S_i\right)^c = \mathcal{F} - \lim_{i\in I} S_i^c$$

□

Corollary 1. *Let $\{M_i\}_{i\in I}$ be a family of multiplicatively closed subsets of a commutative ring R and $\mathcal{F}$ an ultrafilter in I, then $\mathcal{F}-\lim_{i\in I} M_i$ is a multiplicatively closed subset of R.*

Proof. According to [7, Example 2.5], the $\mathcal{F}-\lim$ of a collection of prime ideals is a prime ideal, and by applying Theorem 1 (4) we get the desired result. □

The following proposition gives a simple characterization of the $\mathcal{F}-\lim$.

Proposition 1. *Let R be a set, I be an infinite set, and $\mathcal{F}$ be an ultrafilter on I. Suppose $\{R_i : i \in I\} \subseteq S(R)$ such that $R_i \subseteq R_j$ or $R_j \subseteq R_i$ for all $i, j \in I$. Then, $\mathcal{F}-\lim_{i\in I} R_i$ is equal to either $\bigcap_{i\in A} R_i$ or $\bigcup_{i\in B} R_i$ for some $A, B \in F$.*

Proof. Clearly, one of the sets $A_1 = \{i \in I \ : \{j \in I \ : \ R_i \subset R_j\} \in \mathcal{F}\}$, $A_2 = \{i \in I \ : \{j \in I \ : \ R_i \subset R_j\} \in \mathcal{F}\}$, or $A_3 = \{i \in I \ : \{j \in J \ : \ R_i = R_j\} \in \mathcal{F}\}$ is in $\mathcal{F}$. Suppose $A = A_1 \in \mathcal{F}$, it is easy to see $\mathcal{F}-\lim_{i\in I} R_i = \cup_{i\in A} R_i$. Suppose that $A = A_3 \in \mathcal{F}$, it is simple to see that $\mathcal{F}-\lim_{i\in I} R_i = \cup_{i\in X} R_i = \cap_{i\in A} R_i$. Finally, we can assume that $A = A_2 \in \mathcal{F}$. We claim $\mathcal{F}-\lim_{i\in I} R_i = \cap_{i\in A} P_i$. This is obvious if we suppose that $a \in \mathcal{F}-\lim_{i\in I} R_i$. We have $Z = \{i \in I \ : \ a \in R_i\} \in \mathcal{F}$. Let $i \in A$ be fixed but arbitrary. So we know that $Y = \{j \in I \ : \ R_j \subset R_i\} \in \mathcal{F}$. Thus $Z \cap Y \in \mathcal{F}$ and hence, for any $j \in Z \cap Y$, we have $a \in R_j \subset R_i$ and so $a \in R_i$. Thus, $a \in \cap_{i\in A} R_i$. □

The following result shows that collections of subrings closed under $\mathcal{F}-\lim$ always have a maximum and a minimum.

Proposition 2. *Let C be a collection of subrings of a ring R closed under $\mathcal{F}-\lim$. Then, C has both maximal and minimal elements with respect to containment.*

Proof. The application of Zorn's lemma shows that every non-empty subring T of C, which is linearly ordered by containment, has an upper bound within C. □

Corollary 2. *Let R be a subring of a ring T and let $\mathcal{Z}(R, T)$ be the set of all zero-dimensional subrings of T containing R such that $\mathcal{Z}(R, T) \neq \emptyset$, then $\mathcal{Z}(R, T)$ has maximal and minimal elements.*

Proposition 3. *Let $R = \prod_{i\in I} R_i$, where I is an infinite set and each R_i is a nonzero ring and M_i is a maximal ideal of R_i for each $i \in J$ for some $J \in \mathcal{F}$, then $\mathcal{F}-\lim(\hat{i}, M_i)$ is a maximal ideal of R.*

Proof. Let $f \in \prod_{i\in I} R_i$ such that $f \notin \mathcal{F}-\lim(\hat{i}, M_i)$, then $\{i \in I \ : \ f(i) \in M_i\} \notin \mathcal{F}$, we set $X = \{i \in I \ : \ f(i) \notin M_i\}$, then $X \in \mathcal{F}$ and, consequently, we can find that for each $i \in X$ an $g(i) \in R_i$ such that $f(i)g(i) - 1 \in M_i$. Thus, $fg - 1 \in \mathcal{F}-\lim(\hat{i}, M_i)$, then $\mathcal{F}-\lim(\hat{i}, M_i)$ is a maximal ideal of $\prod_{i\in I} R_i$. □

We know that the $\mathcal{F}-\lim$ of a family of prime ideals is a prime ideal, but this property is generally false for maximal ideals (see, [7, Remark 2.11]). Therefore, [7, Theorem 3.1] cannot hold for a family of maximal ideals, since, based on the above statement and the proof of [7, Theorem 3.1], we give the following theorem:

Theorem 2. *Let $R = \prod_{i\in I} R_i$, where I is infinite and each R_i is nonzero commutative ring and M is a prime ideal of R, then there is an infinite set K and an ultrafilter $\mathcal{E}$ on K such that for each $k \in K$, there are maximal ideals of R of the form $M_i = \mathcal{F}_k - \lim_{i\in I}(\hat{i}, M_{i,k})$, where $\mathcal{F}_k$ is an ultrafilter on I and $M_{i,k}$ are maximal ideals in R_i, such that:*

$$M = \mathcal{E} - \lim_{k\in K} M_k.$$

Proof. Just using the proof of [7, Theorem 3.1], but since M is maximal, we only need half of the above machinery, since by maximality one we have that $M \subseteq \mathcal{E} - \lim_{k\in K} M_k$, then they will be equal. Let $T = M$ and let K be the set of all non-empty finites of T. For $k \in K$, consider the ideal (k) generated by $k \in R$. Using the above proposition and the proof of [7, Lemma 3. 2] we can find a maximal ideal $M_i = \mathcal{F}_k - \lim_{i\in I}(\hat{i}, M_{i,k})$, where $\mathcal{F}_k$ is an ultrafilter on I and $M_{i,k}$ are maximal ideals in R_i such that $(k) \subseteq M_i$.
However, for $t \in T$, we put $\hat{t} = \{k \in K \ : \ t \in k\}$ and $G = \{\hat{t} \ : \ t \in T\}$. It is easy to see that G has f.i.p., so it can be extended to an ultrafilter $\mathcal{E}$ on K.
Now let $x \in M$. So $x \in T$ and therefore $\hat{x} = \{k \in K \ : \ x \in k\} \in G$. Then for $k \in \hat{x}$ we get $x \in (k) \subseteq M_k$. So $\{k \in K \ : \ x \in M_k\} \in \mathcal{E}$ and $x \in \mathcal{E} - \lim_{k\in K} M_k$. Consequently, $M = \mathcal{E} - \lim_{k\in K} M_k$.. □

4 Characterization of the product of rings by the Frechet filter

Theorem 3. *Let I an infinite set, an ultrafilter $\mathcal{F}$ on I is free if and only if $\mathcal{F}_r \subset \mathcal{F}$, where $\mathcal{F}_r$ is the Frechet filter on I.*

The following lemma explains how the frechet filter characterizes this sum.

Lemma 1. *Let $\{R_i\}_{i\in I}$ be a family of rings, and $\mathcal{F}_r$ the Frechet filter on I, then :*

$$\bigoplus_{i\in I} R_i = \mathcal{F}_r - \lim_{i\in I}(\hat{i}, 0).$$

Proof. By definition of $\bigoplus_{i\in I} R_i$ we have :

$$\bigoplus_{i\in I} R_i = \{a \in \prod_{i\in I} R_i \ : \ \{i \in I \ : \ a_i \neq 0\} \ is \ finite\}.$$

Which implies that

$$\bigoplus_{i\in I} R_i = \{a \in \prod_{i\in I} R_i \ : \ \{i \in I \ : \ a_i = 0\} \ is \ co-finite\}.$$

By Definition 3 and Definition 2, we can easily conclude that

$$\bigoplus_{i\in I} R_i = \mathcal{F}_r - \lim_{i\in I}(\hat{i}, 0).$$

□

Corollary 3. *Let $R = \prod_{i\in I} R_i$, where I is infinite and each R_i is nonzero quasilocal rings, then a prime ideal P in R is contained in a unique maximal ideal if and only if $\bigoplus_{i\in I} R_i \subset P$.*

Proposition 4. *Let R be a ring and $\mathcal{F}_r$ frechet filter in $\mathbb{N}$ and let $R^{(n)} = \{a \in R^{\mathbb{N}} \ : \ a_{n+1} = a_{n+2} = a_{n+3} = ...\}$ then $\mathcal{F}_r - \lim_{n\in\mathbb{N}} R^{(n)} = R^{\mathbb{N}}$.*

To prove this result, we need the following lemma.

Lemma 2. *Let $A \subseteq \mathbb{N}$. Then $A \in \mathcal{F}_r$ if and only if there exists $p \in \mathbb{N}$ such that $\{p, p+1, p+2, ...\} \subseteq A$.*

Proof of Proposition 4

Let $R^{\mathbb{N}}$ be a countable direct product of copies of a ring R. We consider $R^{(n)} = \{a \in R^{\mathbb{N}} \ : a_{n+1} = a_{n+2} = a_{n+3} = ...\}$ the subring of $R^{\mathbb{N}}$.
By the definition of $\mathcal{F} - \lim$, we have

$$\mathcal{F}_r - \lim_{n\in\mathbb{N}} R^{(n)} = \{a \in R^{\mathbb{N}} : \ \{n \in \mathbb{N} \ : \ a \in R^{(n)}\} \in \mathcal{F}_r\},$$

which implies that

$$\mathcal{F}_r - \lim_{n\in\mathbb{N}} R^{(n)} = \{a \in R^{\mathbb{N}} : \ \{n \in \mathbb{N} \ : \ a_{n+1} = a_{n+2} = a_{n+3} = ...\} \in \mathcal{F}_r\}.$$

By Lemma 2, $\{n \in \mathbb{N} \ : \ a_{n+1} = a_{n+2} = a_{n+3} = ...\} \in \mathcal{F}_r$ if and only if there exists $p \in \mathbb{N}$ such that $\{p, p+1, p+2, ...\} \subseteq \{n \in \mathbb{N} \ : \ a_{n+1} = a_{n+2} = a_{n+3} = ...\}$.
On the other hand, $\forall n \in \mathbb{N}\ \exists p = n+1$ such that $\{p, p+1, p+2, ...\} \subseteq \{n \in \mathbb{N} \ : \ a_{n+1} = a_{n+2} = a_{n+3} = ...\}$. Then, $\forall a \in R^{\mathbb{N}}\ a \in \mathcal{F}_r - \lim_{n\in\mathbb{N}} R^{(n)}$. □

Remark 2. *$R^{\mathbb{N}} \subset \mathcal{F} - \lim_{n\in\mathbb{N}} R^{(n)}$, for each ultrafilter $\mathcal{F}$ on $\mathbb{N}$.*

We now present another connection between the Frechet filter and the direct sum of commutative rings.

Proposition 5. *Let $\{R_i\}_{i\in I}$ be a family of rings and $\mathcal{F}_r$ the Frechet filter on I, then $\mathcal{Y}(\bigoplus_{i\in I} R_i) = \mathcal{F}_r$.*

Proof. By the definition 4, we have $\mathcal{Y}(\bigoplus_{i\in I} R_i) = \{\mathcal{Y}(f) \ : \ f \in \bigoplus_{i\in I} R_i\}$, then

$$\begin{aligned} F \in \mathcal{Y}(\bigoplus_{i\in I} R_i) &\Leftrightarrow \exists f \in \bigoplus_{i\in I} R_i; F = \mathcal{Y}(f) \\ &\Leftrightarrow \exists f \in \bigoplus_{i\in I} R_i; F = \{i \in I \ : \ f(i) \in R_i \setminus U(R_i)\} \\ &\Leftrightarrow \exists f \in \bigoplus_{i\in I} R_i; I \setminus F = \{i \in I \ : \ f(i) \in U(R_i)\} \end{aligned}$$

From the definition of $\bigoplus_{i\in I} R_i$, we obtain $I \setminus F$ is a finite set, hence $\mathcal{Y}(\bigoplus_{i\in I} R_i)$ is a frechet filter.

□

Lemma 3. *Let $\{R_i\}_{i\in I}$ be a family of quasilocal rings and $\mathcal{F}_r$ the frechet filter on I, then $\bigoplus_{i\in I} R_i$ is the unique largest ideal of $\prod_{i\in I} R_i$ such that $\mathcal{Y}(\bigoplus_{i\in I} R_i) = \mathcal{F}_r$.*

Proof. Direct consequence of Proposition 5 and [7, Remark 2.1 (2)] □

Acknowledgments

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