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Rings in which every regular ideal is projective

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Abstract. In this paper, we introduce a new class of ring called regular hereditary ring, which is a weak version of hereditary ring property. Any hereditary ring is naturally a regular hereditary ring, and in the domain context, these two forms coincide to become a Dedekind domain. We study the transfer of this notion to various context of commutative ring extensions such as localization, direct product, trivial ring extensions and pullbacks. Our results generate new families of examples of non-hereditary regular hereditary rings.

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1 Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. Let R denote such a ring, we denote by $Reg(R)$ and $Z(R)$ the set of all regular elements of R and the set of all zero-divisors of R respectively. By a "local" ring we mean a (not necessarily Noetherian) ring with a unique maximal ideal.

Recall that a ring R is called a hereditary ring if every ideal of R is projective and called a Dedekind domain when it is an integral domain. In this paper, we introduce a weak version of hereditary that we call regular hereditary property. A ring is called regular hereditary, if every regular ideal is projective. A hereditary ring is naturally a regular hereditary ring, and in the domain context, these two forms coincide and which is a Dedekind domain.

Let A be a ring and E an A-module. Then $A \propto E$, the trivial ring extension of A by E, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction

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is also known by other terminology and other notation, such as the idealization $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [8, 9]. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R; ideals of R need not be of this form [13, Example 2.5]. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A [9, Theorem 25.1(3)]. If (A, M) is a local ring with maximal ideal M and E an A-module with $ME = 0$, then $R := A \propto E$ is local total ring of fractions from [13, Proof of Theorem 2.6]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties and for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [1, 4–9, 11–14, 16, 17].

Let T be a ring and let M be an ideal of T. Denote by π the natural surjection $\pi : T \longrightarrow T/M$. Let D be a subring of T/M. Then, $R := \pi^{-1}(D)$ is a subring of T and M is a common ideal of R and T, such that $D = R/M$. The ring R is known as the pullback associated to the following pullback diagram:

$$\begin{array}{ccc} R := \pi^{-1}(D) & \xrightarrow{\pi|_R} & D = R/M \\ \downarrow i & & \downarrow j \\ T & \xrightarrow{\pi} & T/M \end{array}$$

where i and j are the natural injections.
A particular case of this pullback is the $D + M$-construction, when the ring T is of the form $K + M$, where K is a field and M is a maximal ideal of T, and R takes the form $D + M$. See for instance [2, 3, 8, 15].

In this paper, we investigate the possible transfer of regular hereditary property to the direct product of rings and various trivial extension constructions. Also, we examine the transfer of regular hereditary property to a particular pullbacks. Using these results, we construct several classes of examples of non-hereditary regular hereditary rings.

2 Main results

A ring is called regular hereditary and noted reg-hereditary, if every regular ideal is projective.

Now we give the following natural results.

Proposition 1. *Let R be a ring. Then :*

1. *R is a reg-hereditary ring provided that R is a hereditary ring.*
2. *Assume that R is an integral domain. Then R is reg-hereditary if and only if R is hereditary if and only if R is a Dedekind domain.*

3. *A total ring is reg-hereditary.*

Proof. Straightforward. □

First, we construct a non-local non-hereditary reg-hereditary rings.

Example 1. Let $R = \mathbb{Z} \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$, where $(\mathbb{Z}/3\mathbb{Z})^{\infty}$ is a $(\mathbb{Z}/3\mathbb{Z})$-vector space with infinite rank. Then:

1. R is a non-local reg-hereditary ring.
2. R is non-hereditary.

Proof. 1. Let $R = \mathbb{Z} \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$. It is clear that R is non-local since so is $\mathbb{Z}$. Now, we claim that $R - Z(R) = \{(n,e) \in R \mid n \notin 3\mathbb{Z}$ and $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty}\}$.
Indeed, we have $(0,e)(0,e) = 0$ and $(3m,e)(0,e) = 0_R$ for every $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty}$ and $m \in \mathbb{Z}$. Hence, $R - Z(R) \subseteq \{(n,e) \in R \mid n \in \mathbb{Z} - 3\mathbb{Z}$ and $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty}\}$. Conversely, let $(n,e) \in R$ such that $n \in \mathbb{Z} - 3\mathbb{Z}$ and $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty}$ and let $(m,f) \in R$ such that $(n,e)(m,f) = (0,0)$. Hence, $(0,0) = (n,e)(m,f) = (nm, nf + me)$ and so $nm = 0$ and $nf + me = 0$. Since $n \in \mathbb{Z} - 3\mathbb{Z}$ and $nm = 0$, then $m = 0$ and so $nf = 0$. On the other hand, since $n \in \mathbb{Z} - 3\mathbb{Z}$, two cases are then possible:

Case 1: $n = 3p + 1$ for some $p \in \mathbb{Z}$.
Hence, $0 = nf = (3p + 1)f = 3pf + f = f$ and so $(m,f) = (0,0)$, as desired.

Case 2: $n = 3p + 2$ for some $p \in \mathbb{Z}$.
Hence, $0 = nf = (3p + 2)f = 3pf + 2f = 2f$ and so $f = 4f = 2.2f = 2.0 = 0$. Hence, $(m,f) = (0,0)$, as desired.

In all cases, we have $(m,f) = (0,0)$, as desired.

Now, our aim is to show that R is a reg-hereditary ring. Let I be a regular ideal of R. Then, there exists $(p,e) \in I$ such that $p \in \mathbb{Z} - 3\mathbb{Z}$ and $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty}$. Hence, for every $f \in (\mathbb{Z}/3\mathbb{Z})^{\infty}$, we have $(p,e)(0,f) = (0,pf)$. Therefore, $0 \propto (\mathbb{Z}/3\mathbb{Z})^{\infty} \subseteq I$ (since $p \in \mathbb{Z} - 3\mathbb{Z}$) and so $I = J \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$, where J is an ideal of $\mathbb{Z}$, that is $J = n\mathbb{Z}$, where $n \in \mathbb{Z} - 3\mathbb{Z}$ since I is a regular ideal of R. Hence, $I = n\mathbb{Z} \propto (\mathbb{Z}/3\mathbb{Z})^{\infty} = R(n,0) \cong R$, as desired.

2. We claim that the ideal $0 \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$ is not projective. Deny. Let $S =: \{(2,0)^n \mid n \in \mathbb{N}\}$ be a multiplicatively closed subset of R. Hence, $S^{-1}(0 \propto (\mathbb{Z}/3\mathbb{Z})^{\infty})$ is a projective ideal of a local ring $S^{-1}R$ (since $S^{-1}R = (\mathbb{Z}_{2\mathbb{Z}}) \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$) and so it is a principal ideal generated by a regular element of R, a desired contradiction since $S^{-1}(0 \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}) = 0 \propto (\mathbb{Z}/3\mathbb{Z})^{\infty})$ and $(0,e)(0 \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}) = (0,0)$ for every $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty} - \{0\}$.
Therefore, the ideal $0 \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$ is not projective and so R is non-hereditary, as desired.

□

Now, we construct a non-hereditary local reg-hereditary rings.

Example 2. Let $A = K[[X_1, \ldots, X_n, \ldots]] = K + M$ be a power series local ring with infinite indeterminates $(X_i)_{i=1,\ldots,n,\ldots}$ over a field K with maximal ideal M and set $R := A/M^2$. Then:

1. R is a reg-hereditary ring.

2. R is a non-hereditary ring.

Proof. 1. R is a local total ring with maximal ideal M/M^2. In particular, R is a reg-hereditary ring.

2. We claim that R is a non-hereditary ring. Deny. Then, R is a valuation domain since R is a local, a desired contradiction since R is a total ring with maximal ideal M/M^2. Therefore, R is a non-hereditary ring, as desired.

□

Now, we study the transfer of reg-hereditary notion to a direct product.

Proposition 2. *Let $R := \prod_{i=1}^{n} R_i$ the direct product of a rings R_i. Then R is a reg-hereditary ring if and only if so is R_i, for every $i = 1, \ldots, n$.*

Proof. By induction, it suffices to show the proof for $n = 2$. Assume that R_1 and R_2 are reg-hereditary rings and let J be a regular ideal of R. Then, it is easy to see that $J = I_1 \times I_2$, where I_i is a regular ideal of R_i for $i = 1, 2$. Hence, I_i is a projective ideal of R_i and so $J := I_1 \times I_2$ is a projective ideal of R, as desired.
Conversely, assume that R is reg-hereditary and let I_1 be a regular ideal of R_1. Then, $I_1 \times R_2$ is a regular ideal of a reg-hereditary ring R, hence $I_1 \times R_2$ is a projective ideal of R. Therefore, I_1 is a projective ideal of R_1, as desired.
By the same argument, we show that R_2 is also a reg-hereditary ring which completes the proof. □

We know that a localization of a hereditary ring is hereditary. Now, we give an example showing that the localization of a reg-hereditary ring is not always a reg-hereditary.

Example 3. Let $A = K[[X_1, \ldots, X_n, \ldots]] = K + M$ be a local power series ring with infinite indeterminates $(X_i)_{i=1,\ldots,n,\ldots}$ over a field K, where M is its maximal ideal generated by $(X_i)_{i=1,\ldots,n,\ldots}$ over a field K. Set $E := (A/M)^\infty (= K^\infty)$ be a K-vector space with infinite rank and set $R = A \propto E$ be the trivial ring extension of A by E. Let $S_0 := \{X_1^n / n \in \mathbb{N}\}$ be a multiplicative set of A and set $S := S_0 \propto 0$ a multiplicative set of R. Then:

1. R is reg-hereditary since R is a total ring.

2. $S^{-1}R \cong S_0^{-1}A$ is a non-Dedekind domain. In particular, $S^{-1}R$ is a non-reg-hereditary ring.

Proof. 1. Straightforward.

2. If we take $S_0 = \{X_1^n / n \in \mathbb{N}\}$ and $S = S_0 \propto 0$, we have $S^{-1}R \cong S_0^{-1}A = [S_0^{-1}(K[X_1])]$ $[X_2, \ldots, X_n, \ldots]$ which is a non-Dedekind domain. Hence, $S^{-1}R$ is a non-reg-hereditary ring, as desired.

□

But a localization by a multiplicative set $S \subseteq Reg(R)$ of a reg-hereditary ring is reg-hereditary.

Proposition 3. *Let S be a multiplicative set of a ring R such that $S \subseteq Reg(R)$. If R is reg-hereditary, then so is $S^{-1}R$.*

Proof. Remark that if $S \subseteq Reg(R)$, then x/s is a regular element of $S^{-1}R$ if and only if x is a regular element of R, for every $x \in R$ and $s \in S$. The rest of the proof is straightforward. □

Now, we study the transfer of reg-hereditary property in trivial ring extension.

Theorem 1. *Let A be a ring, E be an A-module and set $R := A \propto E$ be the trivial ring extension of A by E. Then:*

1. *Assume that $A \subseteq B$ be an extension of domains, $K := qf(A)$ and $E := B$. Then:*
   1) *$R := A \propto B$ is a reg-hereditary ring if and only if A is a Dedekind domain and $K \subseteq B$.*
   2) *$R := A \propto B$ is a non-reg-hereditary ring.*

2. *Assume that A be an integral domain, $K := qf(A)$ and E be a K-vector space. Then:*
   1) *$R := A \propto E$ is a reg-hereditary ring if and only if A is a Dedekind domain.*
   2) *$R := A \propto B$ is a non-reg-hereditary ring.*

3. *Assume that (A, M) is a local ring and E is an (A/M)-vector space. Then R is reg-hereditary.*

Proof. 1. **1)** Assume that A is a Dedekind domain, $K \subseteq B$ and let J be a proper regular ideal of R. Then there exists $(a, e) \in J$ such that $a \neq 0$ (since $(0 \propto E)(0, e) = 0$). Since $(a, e)R = aA \propto E$ and $aE = aB = B = E$ (since $a \in A \subseteq K$ and $K \subseteq B$), $J := I \bigotimes_A R = IR = I \propto E$ for some proper ideal I of A. Hence, I is a projective ideal of A (since A is a Dedekind domain) and so $J := I \bigotimes_A R = IR = I \propto E$ is a projective ideal of R since R is a flat A-module, as desired.
Conversely, assume that R is a reg-hereditary ring. Our aim is to show that $K \subseteq B$.
First, we wish to show that $K \subseteq B$ in the case when A is local. Let $x \neq 0 \in A$ and let $I := ((x, 0), (x, 1))R$, a regular ideal of R. Then I is projective and hence principal (since R is local too). Write $I = (a, b)R$ for some $a \in A$ and $b \in B$. Clearly, $a = ux$ for some invertible element u in A, hence $I = (ux, b)R = (x, u^{-1}b)R$. Further $(x, 0) \in I$ yields $u^{-1}b = b'x$ for some $b' \in B$. It follows that $I = (x, b'x)R = (x, 0)(1, b')R = (x, 0)R$, since $(1, b')$ is invertible. But $(x, 1) \in I$ yields $1 = xb''$ for some $b'' \in B$. Therefore $K \subseteq B$.
Now, suppose that A is not necessarily local and let $q \in Spec(B)$ and $p := q \cap A$. Clearly, $S := (A \setminus p) \times 0$ is a multiplicatively closed subset of R with the feature that $\frac{r}{1}$ is regular in $S^{-1}R$ if and only if r is regular in R. So regular ideals of $S^{-1}R$ originate from regular ideals of R. Hence $A_p \propto B_p = S^{-1}R$ is a reg-hereditary ring. Whence $K = qf(A_p) \subseteq B_p \subseteq B_q$.

It follows that $K \subseteq B = \bigcap B_q$, where q ranges over $Spec(B)$, as desired.
It remains to show that A is a Dedekind domain and let I be a proper ideal of A. Hence, $J := I \bigotimes_A R = IR = I \propto E$ is a regular ideal of a reg-hereditary ring R, so J is a projective ideal of R. Therefore, I be a projective ideal of A since R is a faithfully flat A-module and $J := I \bigotimes_A R = IR = I \propto E$, as desired.
2) We claim that $R := A \propto B$ is a non-hereditary ring. Indeed, set $I := R(0, e)$, where $e \in E - \{0\}$ and set $T := S \propto 0$ be a multiplicatively closed subset of R, where $S := A - \{0\}$. Remark that $T^{-1}R = K \propto S^{-1}E$ is a local ring and $T^{-1}I = 0 \propto Ke$ is a non-projective ideal of a local ring $T^{-1}R$ since $T^{-1}I(= 0 \propto Ke)$ is not principal generated by a regular element (since $(0, e)T^{-1}I = 0$). Therefore, I is a non-projective ideal of R since $pd_{T^{-1}R}(T^{-1}I) \leqslant fd_R(I)$, as desired.

2. Argue as 1) above.

3. Straightforward since R is a (local) total ring and this completes the proof of Theorem 1. □

The following corollary is an immediate consequence of Theorem 1.

Corollary 1. *Let A be a domain, $K := qf(A)$, and $R := A \propto K$. Then the following statements are equivalent:*

1. *R is a reg-hereditary ring.*

2. *A is a Dedekind domain.*

Now, we construct a new examples of non-hereditary reg-hereditary rings by using Theorem 1.

Example 4. Let $R := \mathbb{Z} \propto \mathbb{Q}$ be the trivial ring extension of $\mathbb{Z}$ by $\mathbb{Q}$. Then:

1. R is a reg-hereditary ring.

2. R is non-hereditary ring.

Now, we study the transfer of reg-hereditary property in a particular case of pullbacks.

Theorem 2. *Let $T = K + M$ be a local ring, where K is a field and M is a maximal ideal of T such that for each $m \in M$, there exists $n \in M$ such that $mn = 0$ (take for instance $M^n = 0$ for some a positive integer n). Let $D \subseteq K$ be a subring of K and set $R = D + M$. Then R is a reg-hereditary ring if and only if D is a Dedekind domain.*

Proof. Assume that R is a reg-hereditary ring and let I be a proper ideal of D. Set $J = IR = I + M$ (since $aM = M$ for every $a \in K$) an ideal of R and we claim that J is a regular ideal of R. Indeed, let $d \in I - \{0\} \subseteq J$ and let $a + m \in R$ such that $d(a + m) = 0$, where $a \in D$ and $m \in M$. Then $0 = da + dm$ and so $da = 0$ in D and $dm = 0$ in M. Therefore, $a = 0$ since D is an integral domain and $d \in D - \{0\}$ and $m = 0$ since $0 = dm \in M$ and d is invertible in K, hence d is a regular element in J. Therefore, J is a projective ideal of R since R is reg-hereditary and so I a projective ideal of D. Hence, D is a Dedekind domain.

Conversely, assume that D is a Dedekind domain and let J be a proper regular ideal of R. Then $J \subsetneq M$ since J is a regular ideal of R and so there exists $d+m \in J$, where $d \in D-\{0\}$ and $m \in M$. Hence, $J \supseteq (d+m)M = dM + mM = M$ (since $mM \subseteq M = dM$) and so $J = I + M$, where I is a proper ideal of D. Hence, I is a projective ideal of a Dedekind domain D. Therefore, J is a projective ideal of R and so R is a reg-hereditary ring which completes the proof of Theorem 2. □

Now, we construct a non-total non-hereditary reg-hereditary ring by using the above Theorem 2.

Example 5. Let $T = \frac{\mathbb{Q}[[\mathbb{X}]]}{<X^n>} = \mathbb{Q} + XT$, where X is an indeterminates over $\mathbb{Q}$, $\mathbb{Q}[[\mathbb{X}]]$ is the power series ring over $\mathbb{Q}$, and $< X^n >= X^n\mathbb{Q}[[X]]$ where n is a positive integers. Set $R = \mathbb{Z} + XT$. Then:

1. R is a reg-hereditary ring.
2. R is non-total and non-hereditary.

Proof. 1. R is a reg-hereditary ring by Theorem 2.

2. The ring R is non-total since every $n \in \mathbb{Z} - \{0\}$ is regular in R. Also, we claim that the ideal XT is not projective. Deny. Let $S := \{2^n / n \in \mathbb{N}\}$ be a multiplicative subset of R. Hence, $S^{-1}(XT)$ is a projective ideal of a local ring $S^{-1}R$ (since $S^{-1}R = (\mathbb{Z}_{2\mathbb{Z}}) \propto XT$), a desired contradiction since $X^{n-1}(XT) = 0$.
   Therefore, the ideal XT is not projective and so R is non-hereditary, as desired.

□

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