Journal of Algebra and Related Topics
Vol. 13, No. 01, 2025, pp. 99-107.

Some additive results for the g-Drazin inverses of operators

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Abstract. The motivation of this article, is to establish new additive results for the g-Drazin inverse of linear operators over Banach spaces. Following the applicability of the g-Drazin inverse of operator matrices in solving the systems of linear differential equations, we then apply our results to operator matrices and obtain some results on generalized Drazin inverse of block operator matrices.

Keywords: Drazin inverse, Additive property, Operator matrix, Banach algebra.
AMS Subject Classification 2010: 15A09, 32A65, 16E50.

1 Introduction

Let X and Y be complex Banach spaces. Denote by $\mathcal{B}(X,Y)$ the Banach algebra of all bounded linear operators from X into Y with an identity I and abbreviate $\mathcal{B}(X,X)\ to\ \mathcal{B}(X)$. An element a in a Banach algebra is quasi-nilpotent if and only if $\| a^n \|^{\frac{1}{n}} \to 0$. By applying quasi-nilpotent elements, Koliha [12] introduced the notion of generalized Drazin inverse (g-Drazin inverse for short) for an operator $T \in \mathcal{B}(X)$ which is the unique element $T^d \in \mathcal{B}(X)$ such that

$$TT^d = T^dT, T^dTT^d = T^d, \quad T - T^2T^d\ is\ quasi-nilpotent.$$

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Received: 31 October 2023/ Revised: 11 May 2025/ Accepted: 13 May 2025
DOI: 10.22124/JART.2025.25909.1595

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We always use $\mathcal{B}(X)^d$ to denote the set of all elements which have g-Drazin inverse in $\mathcal{B}(X)$. The g-Drazin inverse for bounded linear operators on Banach spaces is interesting for several authors duo to their applicable properties and wide applications. Especially several authors have been investigeted the g-Drazin invertibility of operator matrices [3, 4, 6–8]. It is interesting for authors to investigate the conditions under which the sum of two g-Drazin invertible elements in a Banach algebra is g-Drazin invertible. In 1996, Koliha [12] gave the representations of g-Drazin inverse of the sum of two g-Drazin invertible elements a and b under the conditions $ab = ba = 0$. Hartwig, Wang and Wei gave the formula for $(P + Q)^d$ under the condition $PQ = 0$ The g-Drazin inverse of the sum has been studied extensively in [6, 9–11]. The g-Drazin inverse of operator matrices have various applications in singular differential and difference equations, Markov chains, and iterative methods (see [1, 4, 15]). So several authors study the g-Drazin inverse of operator matrices under different conditions (see [2, 3, 5, 8]). In Section 2, we establish new additive results which are different from those in the literature. Let A, B be two linear operators such that $I - A$ and $I - B$ are g-Drazin invertible and A is regular. If $AB^2 = ABA = 0$, we prove that $A + B \in I + \mathcal{B}(X)^d$.

We next consider the g-Drazin inverse of a 2×2 operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad (*)$$

where $A \in \mathcal{B}(X), B \in \mathcal{B}(Y, X), C \in \mathcal{B}(X, Y)$ *and* $D \in \mathcal{B}(Y)$ are g-Drazin invertible and X, Y are complex Banach spaces. Here, M is a bounded operator on $X \oplus Y$. In Section 3, we present some g-Drazin inverses for a 2×2 operator matrix M under a number of different conditions, which generalize [9, Lemma 2.2] and [10, Theorem 2.3].

If $T \in \mathcal{B}(X)$ has g-Drazin inverse T^d, the element $T^\pi = I - TT^d$ is called the spectral idempotent of T. In Section 3, we further consider the g-Drazin inverse of a 2×2 operator matrix M under the conditions related to spectral idempotents.

2 Additive results

The purpose of this section is to investigate the additive results for two linear operators p, *and* q such that $I - p$ *and* $I - q$ are g-Drazin invertible. We start with the following simple lemmas.

Lemma 1. *If $A \in \mathcal{B}(X)$ and $D \in \mathcal{B}(Y)$ are g-Drazin invertible, $B \in \mathcal{B}(Y, X)$ and $C \in \mathcal{B}(X, Y)$, then $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ are also g-Drazin invertible.*

Proof. See [9, Lemma 2.2]. □

Lemma 2. *Let $A \in M_{m\times n}(\mathcal{B}(X)), B \in M_{n\times m}(\mathcal{B}(X))$. If $I_m + AB \in M_m(\mathcal{B}(X))$ has g-Drazin inverse, then $I_n + BA \in M_n(\mathcal{B}(X))$ has g-Drazin inverse.*

Proof. See [14, Corollary 2.7]. □

Lemma 3. *Let $A \in \mathcal{B}(X)$. If $A \in \mathcal{B}(X)$ has g-Drazin inverse, then $-A \in \mathcal{B}(X)$ has g-Drazin inverse.*

Proof. By hypothesis, there exists $B \in comm(A)$ such that $A - A^2B \in \mathcal{B}(X)^{qnil}$ and $B = BAB$. Hence, $(-A) - (-A)^2(-B) \in \mathcal{B}(X)^{qnil}$ and $-B = (-B)(-A)(-B)$. Therefore $-A \in \mathcal{B}(X)^d$, as desired. $\square$

We are now ready to prove one of our main results in this paper.

Theorem 1. *Let $P, Q \in I + \mathcal{B}(X)^d$. If $PQ = 0$, then $P + Q \in I + \mathcal{B}(X)^d$.*

Proof. Set
$$M = -I_2 + \begin{pmatrix} P \\ I \end{pmatrix} (I, Q).$$
Then
$$M = \begin{pmatrix} P - I & 0 \\ I & Q - I \end{pmatrix}.$$
In light of Lemma 1, M has g-Drazin inverse. It follows by Lemma 3 that $-M$ has g-Drazin inverse. Thus,
$$I_2 - \begin{pmatrix} P \\ I \end{pmatrix} (I, Q) \in M_2(\mathcal{B}(X))^d.$$
In view of Lemma 2,
$$I - (I, Q) \begin{pmatrix} P \\ I \end{pmatrix} \in \mathcal{B}(X)^d.$$
By using Lemma 3 again, $P + Q - I = -I + (I, Q) \begin{pmatrix} P \\ I \end{pmatrix}$ has g-Drazin inverse. This completes the proof. $\square$

Corollary 1. *Let $E \in \mathcal{B}(X)$ be an idempotent, and let $Q \in I + \mathcal{B}(X)^d$. If $EB = 0$, then $E + B \in I + \mathcal{B}(X)^d$.*

Proof. Since $(I - E)^d = I - E$, we see that $E - I \in \mathcal{B}(X)^d$ by Lemma 3. We complete the proof by replacing $P = E$ in Theorem 1. $\square$

In the following lemma, we investigate the g-Drazin invertibility of block operator matrices under several conditions.

Lemma 4. *Let $A, D \in I + \mathcal{B}(X)^d$. If $BC = 0$ and $BD = 0$, then $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in I_2 + M_2(\mathcal{B}(X))^d$.*

Proof. It is evident that
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q,$$
where
$$P = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}.$$
Clearly, $P, Q \in I_2 + M_2(\mathcal{B}(X))^d$. It is easy to verify that
$$PQ = \begin{pmatrix} BC & BD \\ 0 & 0 \end{pmatrix} = 0.$$
In light of Theorem 1, $P + Q \in I_2 + M_2(\mathcal{B}(X))^d$, as desired. $\square$

In the following theorem, the g-Drazin invertibility of the sum of two operators are investigated.

Theorem 2. *Let $A, B \in I + \mathcal{B}(X)^d$, and let A be regular. If $AB^2 = 0$ and $ABA = 0$, then $A + B \in I + \mathcal{B}(X)^d$.*

Proof. Set
$$M = \begin{pmatrix} A \\ I \end{pmatrix} (I, B).$$
Since A is regular, we have $X \in \mathcal{B}(X)$ such that $A = AXA$. Then
$$\begin{aligned} M &= \begin{pmatrix} 0 & 0 \\ I - AX & 0 \end{pmatrix} + \begin{pmatrix} A & AB \\ AX & B \end{pmatrix} \\ &:= P + Q. \end{aligned}$$
Since $AB(AX) = 0$ and $(AB)B = 0$, it follows by Lemma 4 that $Q \in I_2 + M_2(\mathcal{B}(X))^d$. Clearly, $P \in I_2 + M_2(\mathcal{B}(X))^d$. Since
$$PQ = \begin{pmatrix} 0 & 0 \\ (I - AX)A & (I - AX)AB \end{pmatrix} = 0,$$
by using Theorem 1, $M \in I_2 + M_2(\mathcal{B}(X))^d$. In light of Lemma 2, $A + B \in I + \mathcal{B}(X)^d$, as required. $\square$

Corollary 2. *Let $E \in \mathcal{B}(X)$ be an idempotent, and let $B \in I + \mathcal{B}(X)^d$. If $EB^2 = 0$ and $EBE = 0$, then $E + B \in I + \mathcal{B}(X)^d$.*

Proof. It is clear by replacing $A = E$ in Theorem 2. $\square$

3 Operator matrices

Let $A \in \mathcal{B}(X), D \in \mathcal{B}(Y)$ be g-Drazin invertible and M be given by $(*)$. The aim of this section is to investigate the g-Drazin invertiblity of 2×2 operator matrix M under several conditions. Using different splitting of the operator matrix M as $M = P + Q$, we will apply Theorem 1 to obtain various conditions for an operator matrix M, which extend [9, Lemma 2.2] and [10, Theorem 2.3].

Theorem 3. *If $CB = 0$ and $(A + I)B = 0$, then M has g-Drazin inverse.*

Proof. It is evident that
$$M' = \begin{pmatrix} A + I & B \\ C & D + I \end{pmatrix} = P + Q,$$
where
$$P = \begin{pmatrix} A + I & 0 \\ C & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ 0 & D + I \end{pmatrix}.$$

Clearly, $P, Q \in I_2 + \mathcal{B}(X \oplus Y)^d$. We easily check that

$$PQ = \begin{pmatrix} 0 & (A+I)B \\ 0 & CB \end{pmatrix} = 0.$$

In light of Theorem 1, $P + Q - I_2$ has g-Drazin inverse. Therefore $M = M' - I_2$ has g-Drazin inverse, as asserted. □

Now we study the g-Drazin invertibility of the matrix M under different conditions.

Theorem 4. *If $BC = 0$ and $B(D+I) = 0$, then M has g-Drazin inverse.*

Proof. We easily see that

$$M' = \begin{pmatrix} A+I & B \\ C & D+I \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A+I & B \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & D+I \end{pmatrix}.$$

Clearly, $P, Q \in I_2 + \mathcal{B}(X \oplus Y)^d$. One can easily check that

$$PQ = \begin{pmatrix} BC & B(D+I) \\ 0 & 0 \end{pmatrix} = 0.$$

In light of Theorem 1, $P + Q - I_2$ has g-Drazin inverse. Therefore $M = M' - I_2$ has g-Drazin inverse, as asserted. □

Also we can apply the privious result to prove one of the main result of this paper.

Theorem 5. *If $CB = 0$ and $C(A+I) = 0$, then M has g-Drazin inverse.*

Proof. It is obvious that

$$M' = \begin{pmatrix} A+I & B \\ C & D+I \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} 0 & 0 \\ C & D+I \end{pmatrix}, Q = \begin{pmatrix} A+I & B \\ 0 & 0 \end{pmatrix}.$$

Clearly, $P, Q \in I_2 + \mathcal{B}(X \oplus Y)^d$. We easily check that

$$PQ = \begin{pmatrix} 0 & 0 \\ C(A+I) & CB \end{pmatrix} = 0.$$

In light of Theorem 1, $P + Q - I_2$ has g-Drazin inverse. Therefore $M = M' - I_2$ has g-Drazin inverse, as asserted. □

By the other splitting approach, we derive the following theorem.

Theorem 6. *If $BC = 0$ and $(D + I)C = 0$, then M has g-Drazin inverse.*

Proof. We check that

$$M' = \begin{pmatrix} A+I & B \\ C & D+I \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A+I & B \\ 0 & D+I \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

Clearly, $P, Q \in I_2 + \mathcal{B}(X \oplus Y)^d$. It is obvious that

$$PQ = \begin{pmatrix} BC & 0 \\ (D+I)C & 0 \end{pmatrix} = 0.$$

In light of Theorem 1, $P + Q - I_2$ has g-Drazin inverse. Therefore $M = M' - I_2$ has g-Drazin inverse, as asserted. □

If $B = 0$ (or $C = 0$), then the known results for g-Drazin invertibility of block matrix M which is given in [9, Lemma 2.2] for a bounded linear operator and in [10, Theorem 2.3] for arbitrary elements in a Banach algebra is obtained. In fact, these are the special cases of our theorems in this Section. We have,

We note that Theorem 3 is a nontrivial generalization of [9, Lemma 2.2] and [10, Theorem 2.3], as the following shows.

Example 1. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \text{ and } D = 0$$

be complex matrices. Then $A^d = A, D^d = 0$. Moreover, we have $CB = 0$ and $(A + I)B = 0$. In light of Theorem 3, M is g-Drazin invertible. In this case, $B \neq 0$.

Let M be an operator matrix M given by $(*)$. It is of interest to consider the g-Drazin inverse of M with zero generalized Schur complement, that is, $D = CA^dB$.

Theorem 7. *Let $A \in \mathcal{B}(X)^d, B \in \mathcal{B}(Y, X), C \in \mathcal{B}(X, Y)$ and $D \in \mathcal{B}(Y)^d$ and M be given by $(*)$. If $A^dBCA^d = 0, (A + I)A^\pi B = 0$ and $CA^\pi B = 0$, then $M \in \mathcal{B}(X \oplus Y)^d$.*

Proof. We easily see that

$$M' = \begin{pmatrix} A+I & B \\ C & CA^dB+I \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A+I & AA^dB \\ C & CA^dB+I \end{pmatrix}, Q = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}.$$

Clearly, $Q \in I_2 + \mathcal{B}(X \oplus Y)^d$. We easily check that

$$PQ = \begin{pmatrix} 0 & (A+I)A^\pi B \\ 0 & CA^\pi B \end{pmatrix} = 0.$$

Moreover, we see that

$$\begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix} = P_1 + P_2,$$
$$P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}, \ P_2 = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}$$

and $P_2P_1 = 0$. Clearly, P_2 is quasinilpotent, and so it has g-Drazin inverse. Furthermore, we have

$$P_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A & AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A & AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = A^2A^d + AA^dBCA^d = A^2A^d.$$

By using Cline's formula (see [13, Theorem 2.1]) and [12, Theorem 5.4 and Theorem 5.5], P_1 has g-Drazin inverse. By virtue of [9, Theorem 2.3] $P \in I_2 + \mathcal{B}(X \oplus Y)^d$. According to Theorem 1, M has g-Drazin inverse, as asserted. □

As a direct consequence of Theorem 7, we have the following result.

Corollary 3. *Let $A \in \mathcal{B}(X)^d, B \in \mathcal{B}(Y,X), C \in \mathcal{B}(X,Y)$ and $D \in \mathcal{B}(Y)^d$ and M be given by $(*)$. If $ABCA = 0, (A+I)A^\pi B = 0$ and $CA^\pi B = 0$, then $M \in \mathcal{B}(X \oplus Y)^d$.*

Proof. Since $A^d = A(A^d)^2$, we have $A^dBCA^d = (A^d)^2(ABCA)(A^d)^2 = 0$. Therefore we complete the proof by Theorem 7. □

In the following theorem we establish other conditions for the g-Drazin invertibility of M.

Theorem 8. *Let $A \in \mathcal{B}(X)^d, B \in \mathcal{B}(Y,X), C \in \mathcal{B}(X,Y), D \in \mathcal{B}(Y)^d$ and M be given by $(*)$. If $A^dBCA^d = 0, CA^\pi(A+I) = 0$ and $CA^\pi B = 0$, then $M \in \mathcal{B}(X \oplus Y)^d$.*

Proof. Clearly, we have

$$M' = \begin{pmatrix} A+I & B \\ C & CA^dB + I \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A+I & B \\ CAA^d & CA^dB + I \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}$$

with $Q \in I_2 + \mathcal{B}(X \oplus Y)^d$. We easily check that

$$QP = \begin{pmatrix} 0 & 0 \\ CA^\pi(A+I) & CA^\pi B \end{pmatrix} = 0.$$

Further, we have

$$\begin{pmatrix} A & B \\ CAA^d & CA^dB \end{pmatrix} = P_1 + P_2,$$

$$P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}, \; P_2 = \begin{pmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{pmatrix}$$

and $P_1P_2 = 0$. Clearly, P_2 is quasinilpotent, and so it has g-Drazin inverse. As in the proof of Theorem 7, we see that

$$\begin{pmatrix} A & AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = A^2A^d.$$

In view of Cline's formula, P_1 has g-Drazin inverse. By virtue of [9, Theorem 2.3] $P \in I_2 + \mathcal{B}(X \oplus Y)^d$. According to Theorem 1, M has g-Drazin inverse, as desired. □

As in the proof of Corollary 3, by using Theorem 8, we now derive the following corollary.

Corollary 4. *Let $A \in \mathcal{B}(X)^d, B \in \mathcal{B}(Y, X), C \in \mathcal{B}(X, Y), D \in \mathcal{B}(Y)^d$ and M be given by $(*)$. If $ABCA = 0, CA^\pi(A + I) = 0$ and $CA^\pi B = 0$, then $M \in \mathcal{B}(X \oplus Y)^d$.*

Acknowledgments

The authors would like to thank the referee for careful reading.

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