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Dedekind N-group and uniform N-group in near-rings

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Abstract. In this paper the authors have defined Dedekind N-groups, prime N-groups, uniform N-groups in near rings and have found the relationship between them. For example they have showed the notions uniform N-group and Dedekind N-group in near rings are equivalent.

Keywords: Dedekind N-group, Prime N-group, Uniform N-group.
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1 Introduction

In this paper the word near ring N, means a commutative near ring with 1, and M is an unitary N-group. It should be noted that all near rings are right near rings. A right (left) near-ring is a non-empty set N together with two binary operations "+" and "." such that

(a) $(N,+)$ is a group (not necessarily abelian),

(b) $(N,\cdot)$ is a semigroup,

(c) $(n_1+n_2)n_3 = n_1n_3 + n_2n_3 (n_1(n_2+n_3) = n_1n_2 + n_1n_3)$, $\forall n_1, n_2, n_3 \in N$ [6].

Let $(M,+)$ be a group with zero and N be a near ring. Consider the map $\mu : N \times M \longrightarrow M$ such that $\mu(n,m) = nm$, $\forall n \in N$, $\forall m \in M$. Then M is called right (left) N-group if M satisfies the following conditions:

(i) $(n+n')m = nm + n'm\,(m(n+n') = mn + mn')$,

(ii) $(nn')m = n(n'm)\,(m(nn') = (mn')n)$, $\forall n, n' \in N$, $\forall m \in M$ [3].

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If M is the left and right N-group, then M is N-group. It is written N^M for the N-group above.

Let M be a group. Then $N^g = \{N^M | N \text{ is a near ring}\}$.

$N^M \in N^g$ is called unitary, if $\forall m \in M : 1 \cdot m = m$.

A subgroup Δ of N-group M with, $N\Delta \subseteq \Delta$ is said to be an N-subgroup of M (notation: $\Delta \leq_N M$).

A normal subgroup I of $(N, +)$ is called ideal of N $(I \trianglelefteq N)$ if

(i) $IN \subseteq I$,

(ii) $n(n' + i) - nn' \in I, \ \forall n, n' \in N, \ \forall i \in I$.

The normal subgroup R of $(N, +)$ with (i) is called the right ideal of $N(R \trianglelefteq_r N)$, while normal subgroup L of $(N, +)$ with (ii) is said to be the left ideal $(L \trianglelefteq_l lN)$.

A normal subgroup Δ of N-group M is called ideal of M $(\Delta \trianglelefteq_N M)$, if Δ satisfies in $n(m + \delta) - nm \in \Delta; \ \forall m \in M, \ \forall n \in N$. Let N be a near ring and I be an ideal of N. Then in [4], authors defined addition and multiplication of these N/I as follows: $(n + I) + (n' + I) = (n + n') + I \ , \ (n + I) \cdot (n' + I) = n \cdot n' + I, \text{ for each } n, n' \in N.$

An N-homomorphism is a mapping h of N-group M to N-group M' such that:

(i) $h(m_1 + m_2) = h(m_1) + h(m_2)$,

(ii) $h(nm_1) = nh(m_1), \ \forall m_1, m_2 \in M, \ \forall n \in N$ [3].

Let N, N' be near rings. We say that, N is embedded in N', if there exists an monomorphism $N \to N'$, and write $N \hookrightarrow N'$.

Let M be an N-group, N be a near ring and S be a closed subset of $(N, \cdot)$. An N-group M_S is called an N-group left (right) fraction of M if

(a) $M_S \in N^g$ is unitary,

(b) $M \hookrightarrow M_S$,

(c) s is invertible in $(N, \cdot), \ \forall s \in S$,

(d) $\forall q \in M_S \quad \exists s \in S, \ \ \exists m \in M$ such that $q = h(m)s^{-1}(q = s^{-1}h(m))$ [5].

Let M be an N-group, N be a near ring and S be a closed subset of $(N, \cdot)$. An N-group M_S is called an N-group left (right) fraction of M if

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Let M be an N-group, N be a near ring and S be a closed subset of $(N,\cdot)$. In [3], authors defined an equivalence relation $\sim$ on $M\times S$ by $(m,s)\sim(m',s')\Leftrightarrow\ \exists s_1,s_2\in S$ such that $ms_1=m's_2$ and $ss_1=s's_2$. $[m,s]$ denoted the equivalence relation class of (m,s) and M_S denoted the set of equivalence relation class. Also, in [3] authors defined addition and multiplication of these fractions as follows:

$$(m,s)+(m',s')=(ms_1+m's_2,ss_1),\ n\cdot(m,s)=(nm,s);$$

$$m,m'\in M,\ n\in N,s,s'\in S,\ (\exists s_1,s_2\in S).$$

Let N be a near-ring and S be a subsemigroup of $(N,\cdot)$. A near-ring N_s is called a quotient left (right) of N if

(a) $N_s\in\eta_1$,

(b) $N\hookrightarrow N_S$,

(c) $\forall s\in S$: s is invertible in $(N_S,\cdot)$,

(d) $\forall q\in N_S\quad\exists s\in S,\quad\exists n\in N$ such that $q=h(n)s^{-1}(q=s^{-1}h(n))$ [4].

Let N be an near ring, S be a closed subset of $(N,\cdot)$. In [5], authors defined an equivalence relation $\sim$ on $N\times S$ by $(n,s)\sim(n',s')\Leftrightarrow\ \exists(n_1,s_1)\in(N\times S)$ such that $ss_1=s'n_1$ and $ns_1=n'n_1$. The equivalence relation class of (n,s) is denoted by $[n,s]$ and the set of equivalence relation classes is denoted by N_S. Also, in [2], authors defined addition and multiplication of these quotient as follows:

$$(n,s)+(n',s')=(ns_1+n'n_1,ss_1),\ (n,s)\cdot(n',s')=(n'n_1,ss_1);$$

$$n,n'\in N,\ s,s'\in S,\ (\exists s_1,s_2\in S,\ n_1\in N).$$

Let N be a near-ring and S be sub-semigroup of $(N,\cdot)$. Then N is called the left ore condition, if $\forall(s,s')\in S\times S$ then there exists $(t_1,t_2)\in S\times S$ such that $st_2=s't_1$ [4].
Let P be an N-subgroup of M. Then $(P:M)=\{x\in N|xM\subseteq P\}$. $(0:M)$ is called the annihilator of M if $(0:M)=\{x\in N|xm=0,\ \forall m\in M\}$.

Let P be an N-subgroup of M. Then $(P:M)=\{x\in N|xM\subseteq P\}$. $(0:M)$ is called the annihilator of M if $(0:M)=\{x\in N|xm=0,\ \forall m\in M\}$.
A near ring N is called integral if N has no non-zero divisors of zero [1].

Let N be a near-ring. N is called prime near-ring if $xNy=0$, for each $x,y\in N$, then $x=0$ or $y=0$ [1].

Theorem 1. *[3] Let M be an N-group. If P and P' are N-subgroups of M. Then*

(i) $(P+P')_S=P_S+P'_s$,

(ii) $(P\cap P')_S=P_S\cap P'_s$.

2 Dedekind N-group in near-ring

In this section, the notions of Dedekind N-group, prime N-group and uniform N-group will be introduced. Our purpose is to find the relation between Dedekind N-group, prime N-group, uniform N-group in near ring. Also, we prove some theorems.

Definition 1. *A non-zero N-subgroup K of M is called invertible if $K'K = M$, where $K' = \{x \in M_S | x \cdot K = K \cdot x \subseteq M\}$.*

Definition 2. *Let M be a non-zero N-group. Then M is called Dedekind N-group if each non-zero N-subgroup of M is invertible.*

Lemma 1. *Let N be a near-ring and S be a subsemigroup of $(N, \cdot)$., we define $T = \{t \in S | tm \neq 0,\ \forall 0 \neq m \in M\}$. Then T is a closed subset of $(N, \cdot)$.*

Proof. Let $t_1, t_2 \in T$ such that $t_1 m \neq 0$, $t_2 m \neq 0$, for each $m \in M$. $(t_1 \cdot t_2)m = t_1(t_2 m) \neq 0$, $\forall m \in M$. Then T is a closed subset of $(N, \cdot)$. □

Example 1. Let $N = \left\{ \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{b} & \bar{c} \end{pmatrix} \middle| \bar{a}, \bar{b}, \bar{c} \in Z_2 \right\}$ with addition and multiplication of Matrices be a near-ring and $M = N^N$ be an N-group. We show that M is a Dedekind N-group. By simple calculation we see that

$$M_S = \left\{ \frac{\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}}{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}}, \frac{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}}{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}}, \frac{\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix}}{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}}, \frac{\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}}{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}}, \right.$$
$$\left. \frac{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix}}{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}}, \frac{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}}{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}}, \frac{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}}{\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}} \right\},$$

$$K_1 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix} \right\},$$

$$K_2 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix} \right\},$$

$$K_3 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix} \right\},$$

$$K_4 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix} \right\},$$

and $M = N^N$ are the only non-zero N-subgroups of M. By Definition 1, we have $K'_1 = \{x \in M_S | xK_1 \subseteq M\}$ and for each $a \in K_1$, $x = \frac{r}{t} \in M_S$ implies that there exists $m_a \in M$ such that $ra = m_a t$. In the following we show that $K'_1 = M_S$

$$\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$$

$$\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}$$

$$\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}$$

$$\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$$

$$\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}$$

$$\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$$

$$\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix} \cdot \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}.$$

Similarly, we get that $K'_2 = K'_3 = K'_4 = M' = M_S$.

Therefore, K_1, K_2, K_3, K_4 and M are invertible N-subgroups. Thus M is a Dedekind N-group.

Proposition 1. *Let M be a Dedekind N-group. Then every N-subgroup of M is a Dedekind N-group.*

Proof. Let P be an N-subgroup of M and K be an N-subgroup of P. It is enough to show that K is an invertible N-subgroup. Consider $K' = \{x \in P_S | \ x \cdot K = K \cdot x \subseteq P\}$. It is clear that, $K'K \subseteq P$. We show that $P \subseteq K'K$. Since K is N-subgroup of M and M is a Dedekind N-group, we have $K'K = M$, $P'P = M$ and therefore, $K'K = P'P$. Again have $1_{P'} \in P'$. Therefore, $P \subseteq P'P = K'K$. Hence P is a Dedekind N-group. $\square$

Theorem 2. *Every image homomorphism of a Dedekind N-group is a Dedekind N-group.*

Proof. Consider $\varphi : M \to M'$. We show that if M is a Dedekind N-group, then $\varphi(M)$ is a Dedekind N-group. Let M be a Dedekind N-group. If K is a non-zero N-subgroup of M, by Definition 1, there exists N-subgroup K' of M, such that $K' \cdot K = M$. Thus, $\varphi(K' \cdot K) = \varphi(M)$, i.e., $\varphi(K')\varphi(K) = \varphi(M) = M'$. Therefore, $\varphi(M) = M'$ is a Dedekind N-group. $\square$

Lemma 2. *Let P be an N-subgroup of M_S. Then there exists an N-subgroup K of M, such that $P = K_S$.*

Proof. Let $K = \{x \in M | \exists t \in S,\ \text{such that } \frac{x}{t} \in P\}$. Then the proof is clear. $\square$

Theorem 3. *Let K, K' be N-subgroups of M. Then $(K' \cdot K)_S = K'_S \cdot K_S$.*

Proof. By [3], $\sim$ is an equivalent relation. We define the multiplication of this fraction as follows:

$$\cdot\ : M_S \times M_S \longrightarrow M_S,$$

$$(\frac{m}{s}, \frac{m'}{s'}) \longrightarrow \frac{m}{s} \cdot \frac{m'}{s'} = \frac{m' s_2}{s s_1}\ (\exists s_1, s_2 \in S).$$

We show that this multiplication is well-define. If

$$(\frac{m}{s}, \frac{m'}{s'}) = (\frac{a}{t}, \frac{b}{t'}),\ \forall \frac{m}{s}, \frac{m'}{s'}, \frac{a}{t}, \frac{b}{t'} \in M_S,$$

then

$$\frac{m}{s} = \frac{a}{t} \Rightarrow \exists s_1, s_2 \in S \text{ such that } m s_1 = a s_2,\ s s_1 = t s_2,$$

$$\frac{m'}{s'} = \frac{b}{t'} \Rightarrow \exists s_3, s_4 \in S \text{ such that } m' s_3 = b s_4,\ s' s_3 = t' s_4,$$

therefore, we have the followings:

$$\begin{cases} m s_1 = a s_2 \Rightarrow m = a s_2 s_1^{-1}, \\ s s_1 = t s_2 \Rightarrow s = t s_2 s_1^{-1}, \\ m' s_3 = b s_4 \Rightarrow m' = b s_4 s_3^{-1}, \\ s' s_3 = t' s_4 \Rightarrow s' = t' s_4 s_3^{-1}. \end{cases}$$

Consider $(s', s) \in S \times S$ be the left ore condition, so there exists $(t_1, t_2) \in S \times S$ such that $s' t_2 = s t_1$.
Then, by considering $\big((x = s_2 s_1^{-1} t_1), (z = s_4 s_3^{-1} t_2)\big)$,

$$\frac{m}{s} \cdot \frac{m'}{s'} = \frac{m' t_2}{s t_1} = \frac{(b s_4 s_3^{-1}) t_2}{(t s_2 s_1^{-1}) t_1} = \frac{b (s_4 s_3^{-1} t_2)}{t (s_2 s_1^{-1} t_1)} = \frac{bz}{tx}.$$

Now, we show that

$$\frac{bz}{tx} = \frac{a}{t} \cdot \frac{b}{t'}.$$

Consider $(tx, tz) \in S \times S$. Since S is the left ore condition, there exists $(r_1, r_2) \in S \times S$ such that $t x r_2 = t z r_1 \Rightarrow x r_2 = z r_1$, therefore,

$$\frac{a}{t} \cdot \frac{b}{t'} = \frac{b r_2}{t r_1}.$$

Now, we show that

$$\frac{bz}{tx} = \frac{b r_2}{t r_1}.$$

By considering ($r' = r_1,\ r'' = x$), we get that

$$\begin{cases} (bz)r' = (br_2)r'', \\ txr' = tr_1r'' \Rightarrow xr' = r_1r''. \end{cases}$$

Therefore,

$$\frac{bz}{tx} = \frac{br_2}{tr_1}.$$

Then,

$$\frac{m}{s} \cdot \frac{m'}{s'} = \frac{a}{t} \cdot \frac{b}{t'}.$$

So the above multiplication is well-defined. Now, we show that $(K' \cdot K)_S = K'_S \cdot K_S$.
Let $x \in (K'K)_S$ then there exists $y_i \in K',\ t_i \in S,\ i \in N$ such that $x = \sum_{i=1}^{n}(\frac{y_i \cdot k_i}{t_i})$.
Without loss of generality, consider $n = 2$. let $x = \frac{y_1 \cdot k_1}{t_1} + \frac{y_2 \cdot k_2}{t_2} \in (K' \cdot K)_S$. Since $y_1, y_2 \in K'$, by Definition 1, there exists $m, m' \in M$, $t, t' \in S$ such that $y_1 = \frac{m}{t}$, $y_2 = \frac{m'}{t'}$. Thus,

$$x = \frac{y_1 \cdot k_1}{t_1} + \frac{y_2 \cdot k_2}{t_2} = \frac{\frac{m}{t} \cdot \frac{k_1}{1}}{t_1} + \frac{\frac{m'}{t'} \cdot \frac{k_2}{1}}{t_2}.$$

Also, consider $(1, t) \in S \times S$. Since S is the left ore condition, there exists $(s_1, s_2) \in S \times S$ such that $s_2 = ts_1$. Put $(1 \cdot t') \in S \times S$. Since S is the left ore condition, there exists $(s_3, s_4) \in S \times S$ such that $s_4 = ts_3$. Thus,

$$x = \frac{y_1 \cdot k_1}{t_1} + \frac{y_2 \cdot k_2}{t_2} = \frac{\frac{m}{t} \cdot \frac{k_1}{1}}{t_1} + \frac{\frac{m'}{t'} \cdot \frac{k_2}{1}}{t_2} = \frac{\frac{k_1 s_2}{ts_1}}{t_1} + \frac{\frac{k_2 s_4}{ts_3}}{t_2} = \frac{\frac{k_1 ts_1}{ts_1}}{t_1} + \frac{\frac{k_2 ts_3}{ts_3}}{t_2} = \frac{k_1}{t_1} + \frac{k_2}{t_2}.$$

Consider $(t_2, t_1) \in S \times S$. Since S is the left ore condition, there exists $(r_1, r_2) \in S \times S$ such that $t_2 r_2 = t_1 r_1$. Thus, $\frac{k_1}{t_1} + \frac{k_2}{t_2} = \frac{k_1 r_1 + k_2 r_2}{t_1 r_1}$. It is enough to show that

$$\frac{k_1 r_1 + k_2 r_2}{t_1 r_1} = \frac{1_{K'}}{1} \cdot \frac{k_1 r_1 + k_2 r_2}{t_1 r_1} = \frac{k_1 r_1 + k_2 r_2}{t_1 r_1}.$$

Consider $(t_1 r_1, 1) \in S \times S$. Since S is the left ore condition, there exists $(r_3, r_4) \in S \times S$ such that $t_1 r_1 r_4 = r_3$. We have $\frac{1_{K'}}{1} \cdot \frac{k_1 r_1 + k_2 r_2}{t_1 r_1} = \frac{(k_1 r_1 + k_2 r_2) r_4}{r_3}$. We show that

$$\frac{k_1 r_1 + k_2 r_2}{t_1 r_1} = \frac{(k_1 r_1 + k_2 r_2) r_4}{r_3}.$$

By choosing ($r'' = r_4^{-1},\ r' = 1$), we get that

$$\begin{cases} (k_1 r_1 + k_2 r_2) r' = \big((k_1 r_1 + k_2 r_2) r_4\big) r'' \\ t_1 r_1 r' = r_3 r'' = t_1 r_1 r_4 r'' \Rightarrow r' =_1 r_4 r''. \end{cases}$$

Therefore,

$$\frac{k_1 r_1 + k_2 r_2}{t_1 r_1} = \frac{1_{K'}}{1} \cdot \frac{k_1 r_1 + k_2 r_2}{t_1 r_1} \in K'_S \cdot K_S.$$

Hence, $(K'K)_S \subseteq K'_S \cdot K_S$. Conversely, let $x \in K'_S \cdot K_S$ then there exists $K'_i \in K'$, $k_i \in K$, $t_i, t'_i \in S$, $i \in N$ such that $x = \sum_{i=1}^{n}(\frac{k'_i}{t'_i} \cdot \frac{k_i}{t_i}) \in K'_S \cdot K_S$, without loss of generality, consider $n = 2$. $(\frac{k'_1}{t'_1} \cdot \frac{k_1}{t_1}) + (\frac{k'_2}{t'_2} \cdot \frac{k_2}{t_2}) \in K'_S \cdot K_S$ thus, there exists $s_1, s_2, s'_1, s'_2 \in S$ such that $(\frac{k'_1}{t'_1} \cdot \frac{k_1}{t_1}) + (\frac{k'_2}{t'_2} \cdot \frac{k_2}{t_2}) = \frac{k_1 s_2}{t'_1 s_1} + \frac{k_2 s'_2}{t'_2 s'_1} = \frac{(k_1 s_2)s_3 + (k_2 s'_2)s_4}{(t'_1 s_1)} = \frac{1_{K'} \cdot ((k_1 s_2)s_3 + (k_2 s'_2)s_4)}{(t'_1 s_1)s_3} \in (K'K)_S$. Then, $(K' \cdot K)_S = K'_s \cdot K_S$. □

Theorem 4. *Let M be a Dedekind N-subgroup. Then M_S is a Dedekind N-group.*

Proof. Let P be a non-zero N-subgroup of M_S. Therefore, by Lemma 2, there exists N-subgroup K of M, such that $P = K_S$. Since M is a Dedekind N-group, K is an invertible N-subgroup of M. Therefore, there exists K' such that $KK' = M$. By Theorem 3, $(KK')_S = K'_S \cdot K_S = M_S$. Thus, M_S is a Dedekind N-group. □

Definition 3. *The N-group M is called prime, if for every non-zero N-subgroup K of M, $ann_N(K) = ann_N(M)$.*

Example 2. [2] Consider the near ring $(N, +, \cdot)$ where $(N, +)$ is kleins, four group $\{0, a, b, c\}$ and " $\cdot$ " defined as the following:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	c
b	0	0	0	0
c	0	a	0	c

By simple calculation we see that $I = \{0, a\}$, $J = \{0, 2\}$, $K = \{0, c\}$, and $M = N^N$ are the only non-zero N-subgroups of M. We see that

$$ann_N(I) = \{x \in N | xi = 0, \ \forall i \in I\} = \{0, b\},$$

$$ann_N(J) = \{x \in N | xj = 0, \ \forall j \in J\} = \{0, b\}.$$

$$ann_N(K) = \{x \in N | xk = 0, \ \forall k \in K\} = \{0, b\},$$

$$ann_N(M) = \{x \in N | xm = 0, \ \forall m \in M\} = \{0, b\}.$$

Since $ann_N(I) = ann_N(J) = ann_N(K) = ann_N(M)$, M is a prime N-group.

Theorem 5. *Let M be a Dedekind N-group. Then M is a prime N-group.*

Proof. Let $a \in (0 : M)) \Rightarrow aM = \{0\}$. Let K is N-sub-group of M, so $aK = 0 \Rightarrow a \in (0 : K)$. Therefore, $ann_N(M) \subseteq ann_N(K)$. Now we show that $ann_N(K) \subseteq ann_N(M)$. Since M is a Dedekind N-group, then $K \cdot K' = M$, for every non-zero N-subgroup K of M. Let $a \in ann_N(K)$. It means $aK = 0$, this implies that $aM = a(KK') = (aK)K' = 0K' = 0$. Therefore, $a \in ann_N(M)$. So $ann_N(K) \subseteq ann_N(M)$. Consequently, $ann_N(K) = ann_N(M)$. □

Remark 1. *If N is not a commutative near ring, then the above theorem may not be true. Considering Example 1, we see that M is a Dedekind N-group and N is not commutative, but M is not a prime N-group, because,*

$$ann_N(K_1) = ann_N(K_2) = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix} \right\}.$$

and,

$$ann_N(K_3) = ann_N(M) = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}.$$

So, $ann_N(K1) = ann_N(K2) \neq ann_N(K3) = ann_N(M)$.

Theorem 6. *Let M be a Dedekind N-group. Then M_S is an N-simple N_S-group.*

Proof. Let P be an N-subgroup of M_S such that $P \neq N0$, by Lemma 2, there exists N-subgroup K of M such that $P = K_S$. Since M is a Dedekind N-group, by Theorem 8, M_S is a Dedekind N-group. We know $K_S \subseteq M_S$, it is enough to show that $M_S \subseteq K_S$. Let $x \in M_S = K'_S \cdot K_S$ then there exists $k'_i \in K',\ k_i \in K,\ t_i, t'_i \in S$, $i \in N$ such that $x = \sum_{i=1}^{n}(\frac{k'_i}{t'_i} \cdot \frac{k_i}{t_i}) \in K'_S \cdot K_S$. Without loss of generality, consider $n = 2$. Let $x = (\frac{k'_1}{t'_1} \cdot \frac{k_1}{t_1}) + (\frac{k'_2}{t'_2} \cdot \frac{k_2}{t_2}) \in K'_S \cdot K_S$. Thus, there exists $s_1, s_2, s'_1, s'_2 \in S$ such that $x = (\frac{k'_1}{t'_1} \cdot \frac{k_1}{t_1}) + (\frac{k'_2}{t'_2} \cdot \frac{k_2}{t_2}) = \frac{k_1 s_2}{t'_1 s_1} + \frac{k_2 s'_2}{t'_2 s'_1} = \frac{(k_1 s_2)s_3 + (k_2 s'_2)s_4}{(t'_1 s_1)} \in K_S$. Hence, $M_S \subseteq K_S$. Consequently, $P = K_S = M_S$. So, M_S is an N-simple N-group. Now, we show that M_S is a Dedekind N_S-group.

Let $\mu : N_S \times M_S \to M_S$ such that $\mu(\frac{n}{s}, \frac{m}{t}) = \frac{nm}{st}$ for each $\frac{n}{s} \in N_S$, $\frac{m}{t} \in M_S$. We show that this μ is well-defined.

Let $(\frac{n}{s}, \frac{m}{t}) = (\frac{n'}{s'}, \frac{m'}{t'})$.

$$\begin{cases} \frac{n}{s} = \frac{n\prime}{s\prime} \Rightarrow \exists (s_1, n_1) \in S \times N \text{ s.t } ss_1 = s'n_1,\ ns_1 = n'n_1 \\ \frac{m}{t} = \frac{m\prime}{t\prime} \Rightarrow \exists t_1, t_2 \in T \text{ s.t } mt_1 = m't_2,\ tt_1 = t't_2. \end{cases}$$

Since S is invertible in N, we have

$$\begin{cases} ss_1 = s'n_1 \Rightarrow s = s'n_1 s_1^{-1} \\ ns_1 = n'n_1 \Rightarrow n = n'n_1 s_1^{-1} \\ mt_1 = m't_2 \Rightarrow m = m't_2 t_1^{-1} \\ tt_1 = t't_2 \Rightarrow t = t't_2 t_1^{-1}. \end{cases}$$

Since $ss_1 = s'n_1$ we have $n_1 s_1^{-1} = s'^{-1}s$. Now we show that

$$\left(\frac{n}{s} \cdot \frac{m}{t}\right) = \left(\frac{n'}{s'} \cdot \frac{m'}{t'}\right).$$

Consider $(\frac{n}{s}\cdot\frac{m}{t}) = \frac{(n'n_1s_1^{-1})(m't_2t_1^{-1})}{(s'n_1s_1^{-1})(t't_2t_1^{-1})} = \frac{n'm'(n_1s_1^{-1}t_2t_1^{-1})}{s't'(n_1s_1^{-1}t_2t_1^{-1})} = \frac{n'm'(s'^{-1}st_2t_1^{-1})}{s't'(s'^{-1}st_2t_1^{-1})} = \frac{n'm'}{s't'} = (\frac{n'}{s'}\cdot\frac{m'}{t'})$.

Consequently, $(\frac{n}{s}\cdot\frac{m}{t}) = (\frac{n'}{s'}\cdot\frac{m'}{t'})$ thus μ is well-defined. Now, we show that M_S is an N_S-subgroup.

(i) $(M_S,+)$ is a group.

(ii) We show that $N_S\cdot M_S \subseteq M_S$.
$\sum_{i=1}^{n}(\frac{n_i}{s_i}\cdot\frac{m_i}{t_i}) = \sum_{i=1}^{n}(\frac{n_i\cdot m_i}{s_i\cdot t_i}) \subseteq M_S$, for every $\frac{n_i}{s_i}\in N_T$, $\frac{m_i}{t_i}\in M_S$. Then M_S is N_S-group.

□

Definition 4. *An N-group M is called a uniform N-group if for every non-zero N-subgroups of M, $P_1\cap P_2 \neq 0$.*

Example 3. Let $N = \left\{ \begin{pmatrix} \bar{0} & \bar{\bar{a}} \\ \bar{0} & \bar{\bar{b}} \end{pmatrix} \middle| \bar{a},\bar{b}\in Z_2 \right\}$ with addition and multiplication of Matrices be a near-ring and $M = N^N$ be an N-group. Then M is a uniform N-group. Because,

$$M = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{pmatrix} \right\},$$

and by simple calculation, we see that $I = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$, and $M = N^N$ are the only non-zero N-subgroups of M. Then $I\cap M = I \neq 0$.

Proposition 2. *Let M be a Dedekind N-group and T satisfies in Lemma 2.3 . Then M is a uniform N-group.*

Proof. Let P, P' be two non-zero N-subgroups of M. We will show that $P\cap P' \neq 0$. Let $P\cap P' = 0$ and $0 \neq m \in P$ be such that $m \notin P'$. Now we show that, there exists $s\in T$ such that $sm \in P'$. Since M is Dedekind N-group hence, P' is invertible N-group of M, i.e. $P''P' = M$, where $P'' = \{x \in M_T | xP' \subseteq M\}$.

Since $m\in M$ there exists $p_i''\in P''$, $p_i'\in P'$ such that $x = \sum_{i=1}^{n} p_i''\cdot p_i'$. Without loss of generality, consider $n = 2$. Suppose that $m = p_1''\cdot p_1' + p_2''\cdot p_2'$. Since $p_1'', p_2''\in P''$ there exists $\frac{m_1}{t_1}, \frac{m_2}{t_2} \in M_T$ such that $p_1'' = \frac{m_1}{t_1}$, $p_2'' = \frac{m_2}{t_2}$ therefore,

$$m = p_1''\cdot p_1' + p_2''\cdot p_2' = \frac{m_1}{t_1}\cdot\frac{p_1'}{1} + \frac{m_2}{t_2}\cdot\frac{p_2'}{1}.$$

So, (there exists$s_1, s_2, s_3, s_4, r, r' \in T$) such that

$$\frac{m_1}{t_1}\cdot\frac{p_1'}{1} + \frac{m_2}{t_2}\cdot\frac{p_2'}{1} = \frac{p_1's_2}{t_1s_1} + \frac{p_2's_4}{t_2s_3} = \frac{p_1's_2r + p_2's_4r'}{t_1s_1r}$$

By choosing $(s = t_1s_1r)$ we have

$$sm = p_1's_2r + p_2's_4r' \in P'.$$

Since P is an N-subgroup, $sm\in P$. Thus, $sm = 0$ and $m = 0$ is a contradiction. Hence for each non-zero N-subgroup P, P' of M, $P\cap P' \neq 0$. □

Proposition 3. *If $ann_{N_S}(m) = P_S$, for each non-zero $m \in M$, then $ann_N(m) = P$.*

Proof. Let $x \in ann_N(m)$, then $\frac{x}{1} \in ann_{N_S}(m) = P_S$. Therefore, $\frac{x}{1} \in P_S$. Then there exists $\frac{p_1}{t_1} \in P_T$ such that $\frac{x}{1} = \frac{p_1}{t_1}$. So, there exists $s_1, s_2 \in S$ such that $xs_1 = p_1 s_2$, $s_2 = t_1 s_2$. Thus $x = p_1 s_2 s_1^{-1} \in P$. then $x \in P$. So $ann_N(m) \subseteq P$.

Let $x \in P$, then $\frac{x}{1} \in P_S$. Therefore, $\frac{x}{1} \in ann_{N_S}(m)$, then for each $m \in M$, $(\frac{x}{1})m = 0$ So, $\frac{xm}{1} = xm = 0$. Hence,$x \in ann_N(m)$. Consequently, $P \subseteq ann_N(m)$. □

Theorem 7. *Let every N-subgroup of M be an ideal of M and M be a Dedekind N-group. Then for each non-zero $m \in M$, $ann_N(m) = ann_N(M)$ and $ann_N(M)$ is a prime ideal of N.*

Proof. It is clear that for each non-zero $m \in M$, $ann_N(m) = ann_N(M)$. We show that $ann_N(M)$ is a prime ideal of N. Let $0 \neq m \in M$, by Theorem 6, since M is Dedekind N-group. Then M_S is a N-simple N_S-group. Therefore, there exists a maximal ideal K of N_S such that $ann_{N_S}(m) = K$. We know that $K = P_S$ for some prime ideal P. Thus $ann_{N_S}(m) = P_S$. Consequently, by Proposition 3, $ann_N(m) = P$. Since for each $m \in M$, $ann_N(m) = P$, Then $ann_N(m) = ann_N(M) = P$. □

Theorem 8. *Let every N-subgroup of M be an ideal of M and M be a Dedekind N-group. Then $N/ann_N(M)$ is a prime near-ring.*

Proof. Since M is a Dedekind N-group, therefore by Theorem 5, M is a prime N-group. We show that $N/ann_N(M)$ is a prime near-ring. Let $(x + ann_N(M)) \cdot (N/ann_N(M)) \cdot (y + ann_N(M)) = ann_N(M)$ for each $x + ann_N(M)$, $y + ann_N(M) \in N/ann_N(M)$.

For all $n + ann_N(M) \in N/ann_N(M)$ we have $(x + ann_N(M)) \cdot (n + ann_N(M)) \cdot (y + ann_N(M)) = ann_N(M))$. So, $(xny + ann_N(M)) = ann_N(M)$. Therefore, $xny \in ann_N(M)$. By Theorem 7, $ann_N(M)$ is a prime ideal of N.

Thus, $x \in ann_N(M)$ or $y \in ann_N(M)$. Therefore, $(x + ann_N(M)) = ann_N(M)$ or $(y + ann_N(M)) = ann_N(M)$. Hence, $N/ann_N(M)$ is a prime near-ring. □

Lemma 3. *Every prime near-ring is an integral domain.*

Proof. It is clear. □

Corollary 1. *Let every N-subgroup of M be ideal of M and M be a Dedekind N- group. Then $N/annN(M)$ is an integral domain.*

Proof. The proof is clear by Lemma 3 and Theorem 8. □

Theorem 9. *Let M and M' be Dedekind N-groups. Then $M \cap M'$ is a Dedekind N-group.*

Proof. Suppose P is a non-zero N-subgroup of $M \cap M'$. Now we show that P is invertible. Consider $P' = \{x \in (M \cap M')_S | xP \subseteq (M \cap M')\}$. It is clear that $PP' \subseteq M \cap M'$. It is enough to show that $M \cap M' \subseteq P'P$. Suppose $x \in M \cap M'$. So, $x \in M$ and $x \in M'$. Since M, M' are Dedekind N-groups, there exist P', P'' such that $P'P = M$, $P''P = M'$. Therefore, $x \in P'P$, $x \in P''P$. So, $x \in P'P \cap P''P \subseteq P'P$. Consequently, $M \cap M' \subseteq P'P$ is a Dedekind N-group.

□

Proposition 4. *Every uniform N-group is a Dedekind N-group.*

Proof. Let M be a uniform N-group and P be a non-zero N-subgroup of M. Then we show that P is an invertible N-subgroup. Consider $P' = \{x \in M_S | xP \subseteq M\}$, it is clear that $P'P \subseteq M$. Let $0 \neq x \in M$, since M is a uniform N-group, there exists an N-subgroup P of M such that $x \in P \cap M$, since $1_{P'} \in P'$ therefore, $x = 1 \cdot x \in P'(P \cap M)$. Therefore, $P'P \subseteq M$. Consequently, M is a Dedekind N-group. □

Corollary 2. *Every uniform N-group is a prime N-group.*

Proof. The proof is clear by Proposition 4 and Theorem 5. □

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