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Monitoring a Two-Stage Process Using Adaptive Control Charts with a Markov - Monte Carlo Chain Approach

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ABSTRACT

One of the effective approaches to quality improvement is the application of statistical science within the framework of Total Quality Management (TQM). Statistical Process Control (SPC), as a key component of TQM, utilizes tools such as control sheets, histograms, Pareto charts, cause-and-effect diagrams, defect concentration charts, correlation diagrams, and control charts to detect and prevent defective products. This study focuses on control charts as instruments for identifying variations and out-of-control conditions in process means. In traditional methods, there is usually a delay between the occurrence of a process change and its detection on Shewhart control charts. This research aims to minimize such delay by proposing the use of adaptive control charts based on Markov chain models, which enhance the capability of rapid detection of assignable causes. To evaluate the proposed approach, one of the machines in a tea bag production company-characterized by a two-stage production process—was selected for case analysis. Sampling was conducted in two modes: once with fixed sample sizes and intervals, and again using adaptive sampling with variable sizes and intervals, to compare the efficiency of the proposed method.

1.Introduction

One of the major issues that developing countries face is the absence of a healthy and competitive market. In these countries, the products manufactured do not encounter significant obstacles due to the lack of market saturation, and they are often sold regardless of the quality. As the quality expert Dr. Juran states, "In times of scarcity, the first thing to be sacrificed is quality." This principle is particularly evident in countries that lack a healthy competitive market [8,9]. Quality

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is a key competitive characteristic in today's world, and control charts, as one of the most important and widely used tools in statistical process control (SPC), play a crucial role in enhancing process quality by monitoring and identifying changes that occur within them. Control charts, in a way, are central to the quality process. Fortunately, in recent years, with the shift in the mindset of organizations and individuals involved in production, it has become increasingly evident that improving quality does not only avoid increased costs but can also contribute to cost reduction. In every production process, despite proper design or maintenance, a certain degree of inherent variability exists. This inherent variability or disturbance is caused by the accumulation of numerous small, unavoidable deviations. Variability in key quality characteristics typically stems from three sources: incorrect machine settings, operator errors, and defective raw materials. One of the main objectives of statistical process control is to quickly detect the presence of special-cause variations or changes in the process mean, so that corrective actions can be taken before a large number of defective products are produced. The goal of SPC is to eliminate variation in the process mean. While it is impossible to eliminate all process variability, control charts can serve as an effective tool to reduce process variation [13].

A control chart is a method of presenting a quality characteristic, measured or calculated based on the chart's data, in terms of sample number or time. There is a strong connection between control charts and hypothesis testing. Essentially, a control chart is a hypothesis test used to assess whether a process is in statistical control.To monitor the process, mean, the Shewhart $\bar{X}$ control chart was developed for easier implementation and has been widely used in industrial projects. However, until today, Shewhart $\bar{X}$ control charts have mostly monitored processes in which equal-sized samples are taken at regular time intervals. They are often slow to detect small to moderate changes in the process mean, or in some cases, they may fail to detect these changes. As a result, several alternative solutions have been proposed and developed in recent years to improve the performance of Shewhart control charts. One such method to enhance detection capability is to use variable sample size (VSS) or variable sample intervals (either in combination or separately) rather than fixed sample size or fixed sample intervals as used in the traditional approach. Whenever there are signs that a process parameter may have changed, the subsequent sample should be larger or the sample interval should be shorter. Conversely, if no indication of change in a parameter is observed, the subsequent sample should be smaller or the sample interval should be larger [17].

In many previous studies, it was assumed that only single-stage processes were involved, while many of the current production processes are multi-stage dependent processes. The goal of this research is to provide statistical solutions to minimize the drawbacks of traditional methods and to more efficiently utilize new two-stage control charts in practice. Furthermore, the research aims to demonstrate the differences in results obtained from these new control charts compared to traditional ones and to present the implications of these differences for production managers [7-9].

2. Theoretical Framework of the Study

Statistics comprises a set of powerful and effective techniques that can significantly assist in decision-making related to a process or population, based on the analysis of data derived from a representative sample. Statistical methodologies play a pivotal and foundational role in quality improvement initiatives. These methods provide the essential principles for designing sampling schemes, conducting statistical tests, evaluating results, and utilizing the acquired data to monitor and enhance production processes. Within the field of quality management, it is widely acknowledged by experts that quality cannot be incorporated into the final product merely through inspection and testing. The product must be manufactured correctly from the very beginning. This concept underscores the necessity for a stable production process, wherein all stakeholders involved—including machine operators, process engineers, quality assurance personnel, and managerial staff—must consistently aim to improve process performance and minimize variability in critical process parameters.Statistical Process Control (SPC) during manufacturing represents the principal tool to achieve such objectives. Among SPC tools, control charts are regarded as one of the simplest yet most effective instruments for real-time process monitoring. Adaptive control charts have emerged as viable alternatives to conventional Shewhart control charts, especially in dynamic production environments. Nevertheless, several practical challenges hinder the implementation of adaptive charts-particularly those developed for monitoring process mean values. Despite their proven advantages, such as significant reductions in detection time and monitoring costs compared to fixed-parameter control charts, resistance to their deployment often persists among operational staff and, by extension, quality control management.A comprehensive review of the literature reveals that, with few exceptions, the majority of existing studies have overlooked the practical implementation of adaptive control charts. Most real-world applications in quality control revolve around the detection of assignable causes, which induce shifts in the statistical parameters of processes that are otherwise considered to be in control.The present study endeavors to address these practical limitations by statistically modeling the interrelationships among subprocesses and leveraging estimation techniques to mitigate, and if possible, eliminate current inefficiencies. Furthermore, the research aims to empirically demonstrate the superior operational efficiency of novel two-stage control charts over their traditional counterparts, thereby providing valuable insights for process improvement and strategic decision-making in quality management [17,18].

3. Research Methodology

In this article, using samples taken from a tea packaging machine with two dependent stages, the time difference in detecting the process being out of control is monitored and examined in two traditional and adaptive control charts. In the first phase, historical data are analyzed to determine whether the process is under statistical control. Simultaneously, process control parameters are estimated. In the second phase, the control charts established in the previous phase are monitored to identify and detect deviations in statistical parameters. Additionally, the proposed control charts with variable sample size (VSS) and variable sampling interval (VSI) are also monitored [2].

3.1. Two-Stage Process Monitoring with Adaptive Chart

We examine a two-stage process with two dependent stages. It is assumed that X is the measurable quality characteristic from the first stage of the process, which follows a normal distribution with a mean of $\mu.X$ and a standard deviation of $\sigma.X$. The process initially starts under statistical control. Also, assume that Y is the measurable quality characteristic from the second stage of the process and follows a similar distribution as X.After the initial sampling, the normality of the obtained samples from both stages is examined using a probability plot. Then, to discover the relationship between the variables X and Y, a scatter plot is used. Monitoring continues if our assumption about normality and the existence of a linear relationship between the variables is correct. In this case, the second stage of the process is influenced by the first stage, and the relationship between X and Y is expressed as follows [2]:

$$Y|X = f(x) + \varepsilon_0$$

ε_0: Error variation when the process is under control and follows a normal distribution. To monitor the X and Y charts, the effect of X on Y must first be eliminated. To achieve this, the model $\hat{Y}|X$, which is a paired observation model (x, y) constructed using the least squares error method, is calculated. The residuals are then obtained as follows[2]:

$$e = Y|X - \hat{Y}\big|X$$

After calculating the residuals, we proceed to monitor the X and e charts instead of the X and Y charts.To calculate the average changes in the mean of the first stage of the process ($\delta_1 \neq 0$) that causes the process mean to shift from μ_x to $\mu_x + \delta_1\sigma_x$, and also the average changes in the mean of e ($\delta_2 \neq 0$) that shifts 0 to $\delta_2\sigma_e$, we use the historical process data [2].

Assume that T_i is the time until the occurrence of the specific event i (causing a justified deviation in the mean), where i = 1, 2. Also, assume that T_i follows an exponential distribution given by the following formula[15]:

$$f(T_i) = \lambda_i \cdot e^{-\lambda_i \cdot T_i}$$

Here, λ_i represents the average rate at which the specific event occurs during the process[15].

$$(t_i) = \lambda_i \exp(-\lambda_i t_i) \ , \qquad t_i > 0 \ , \qquad i = 1,2,$$

$\frac{1}{\lambda_i}$ is the average time during which the i-th stage of the process remains within a statistically controlled region. The analysis of a controlled region based on the intersection of control charts ASSI $Z_{\overline{X}}$ and ASSI ASSI $Z_{\overline{e}}$ can only be valid under the assumption that no shift in the process mean occurs at the initial time in either stage one or two, but rather, such a shift takes place at some future time. Samples with variable sizes are taken from both stages of the controlled process. The standardized sample means $Z_{\overline{X}}$ and $Z_{\overline{e}}$ are expressed as follows[15]:

$$Z_{\overline{X}} = \frac{\overline{X} - \mu_X}{\frac{\sigma_X}{\sqrt{n_j}}} \qquad\qquad Z_{\overline{e}} = \frac{\frac{\overline{e}}{\sigma_\varepsilon}}{\sqrt{n_j}}$$

In control charts with warning limits in the form of $\pm W\overline{X}$ and $\pm W\overline{e}$, and action limits in the form of $\pm K_{\overline{X}}$ and $\pm K_{\overline{e}}$ (where $0 \leq W_{\overline{X}} < K_{\overline{X}}$ and $0 \leq W_{\overline{e}} < K_{\overline{e}}$), for the first and second stages of the process, assume that sample points are plotted on a standard $\overline{X}$ chart and an $\overline{e}$ chart. The search for determining the cause starts when the sample mean points ($Z_{\overline{X}}$) fall outside the range ($-K_{\overline{X}}$, $K_{\overline{X}}$), or when the sample mean points ($Z_{\overline{e}}$) fall outside the range ($-K_{\overline{e}}$, $K_{\overline{e}}$). This search is conducted whenever the control charts ($Z_{\overline{X}}$ and $Z_{\overline{e}}$) generate a signal (indicator). When there is no change in the process mean during either stage one or stage two, any signal generated is considered a false alarm. Conversely, when there is a shift in the sample mean during both stages of the process, the generated signal is regarded as a true alarm. In the case of a non-continuous (intermittent) process, whether the signal is true or false, the process is halted for investigation and elimination of assignable causes, and then restored to a statistically controlled state [15].

$$UCL_{Z_{\overline{X}}} = K_{\overline{X}} \qquad UCL_{Z_{\overline{e}}} = K_{\overline{e}}$$
$$UWL_{Z_{\overline{X}}} = W_{\overline{X}} \qquad UWL_{Z_{\overline{e}}} = W_{\overline{e}}$$
$$CL_{Z_{\overline{X}}} = 0 \qquad CL_{Z_{\overline{e}}} = 0$$
$$LWL_{Z_{\overline{X}}} = -W_{\overline{X}} \qquad LWL_{Z_{\overline{e}}} = -W_{\overline{e}}$$
$$LCL_{Z_{\overline{X}}} = -K_{\overline{X}} \qquad LCL_{Z_{\overline{e}}} = -K_{\overline{e}}$$

The position of the sample mean points obtained from the current sample in each chart determines the sample size and the next sampling time. We have divided the proposed control charts $ASSI_{Z\overline{X}}$ and $ASSI_{Z\overline{e}}$ into the following regions[15]:

$$I_{\overline{X}_1} = [-W_{\overline{X}}, W_{\overline{X}}] \qquad I_{\overline{e}_1} = [-W_{\overline{e}}, W_{\overline{e}}]$$
$$I_{\overline{X}_2} = (-K_{\overline{X}}, -W_{\overline{X}}) \cup (W_{\overline{X}}, K_{\overline{X}}) \qquad I_{\overline{e}_2} = (-K_{\overline{e}}, -W_{\overline{e}}) \cup (W_{\overline{e}}, K_{\overline{e}})$$
$$I_{\overline{X}_3} = (-\infty, -K_{\overline{e}}) \cup (K_{\overline{e}}, \infty) \qquad I_{\overline{e}_3} = (-\infty, -K_{\overline{X}}) \cup (K_{\overline{X}}, \infty)$$

If the sample mean point $Z_{\overline{X}}$ falls within the range $I(\overline{X}_1)$ and the sample mean point $Z_{\overline{e}}$ falls within the range $I(\overline{e}_1)$, then the next sample should be small (n_1) and should be taken after a long-time interval t_3. If the sample mean point $Z_{\overline{X}}$ and $Z\overline{e}$ fall within the ranges $I(\overline{X}_1)$ and I_($\overline{e}_2$) or $I(\overline{X}_2)$ and $I(\overline{e}_1)$, then the next sample should be of average size (n_2) and taken after an average sampling interval t_2. If the sample mean point $Z_{\overline{X}}$ and $Z\overline{e}$ fall within the range $I(\overline{X}_2)$ and I_($\overline{e}_2$), then the next sample should be large (n_3) and taken after a short sampling interval t_1. The relationship between the next sample size, sampling interval (t(j), n(j), j=1,2,3), and the location of the taken sample statistic ($Z_{\overline{e}}$, $Z_{\overline{X}}$) is expressed as follows [15]:

$$\big(t(j), n(j)\big) = \begin{cases} (t_3, n_1) & \text{if } Z_{\overline{X},i-1} \in I_{\overline{X}_1} \cap Z_{\overline{e},i-1} \in I_{\overline{e}_1} \\ (t_2, n_2) & \text{if } Z_{\overline{X},i-1} \in I_{\overline{X}_1} \cap Z_{\overline{e},i-1} \in I_{\overline{e}_2} \\ (t_2, n_2) & \text{if } Z_{\overline{X},i-1} \in I_{\overline{X}_2} \cap Z_{\overline{e},i-1} \in I_{\overline{e}_1} \\ (t_1, n_3) & \text{if } Z_{\overline{X},i-1} \in I_{\overline{X}_2} \cap Z_{\overline{e},i-1} \in I_{\overline{e}_2} \end{cases}$$

The first sample size is randomly selected precisely at the start of the process. If the selected sample has a large size (n_3), then the next sampling should be carried out after a short time interval (t_1). If the selected sample has a medium size (n_2), then sampling should occur after a medium time interval (t_2). If the selected sample has a small size (n_1), then sampling should take place after a long time interval (t_3). During in-control processes, all samples - including the first one - must follow the probability distribution as follows: a probability P_0 of being small, a probability P_1 of being medium, and a probability $1- P_0-P_1$ of being large, where P_0 and P_1 are defined as follows [15]:

$$P_0 = P_r(|Z_{\overline{X}}| < W_{\overline{X}} \| Z_{\overline{X}}| < K_{\overline{X}}) * P_r(|Z_{\overline{e}}| < W_{\overline{e}} \| Z_{\overline{e}}| < K_{\overline{e}})$$
$$P_1 = P_r(|Z_{\overline{X}}| < W_{\overline{X}} \| Z_{\overline{X}}| < K_{\overline{X}}) * (1 - P_r(|Z_{\overline{e}}| < W_{\overline{e}} \| Z_{\overline{e}}| < K_{\overline{e}})) +$$
$$(1 - P_r(|Z_{\overline{X}}| < W_{\overline{X}} \| Z_{\overline{X}}| < K_{\overline{X}})) * P_r(|Z_{\overline{e}}| < W_{\overline{e}} \| Z_{\overline{e}}| < K_{\overline{e}}),$$
$$Z_{\overline{X}} \sim N(0,1) \qquad Z_{\overline{e}} \sim N(0,1)$$

To facilitate the calculation of performance metrics, $\overline{W}_X$, $\overline{K}_X$, $\overline{W}_e$, and $\overline{K}e$ are defined within a specified range such that the probability of sample points falling into the warning zone, while the process is in control, remains the same for both control charts $\overline{Z}_X$ and $\overline{Z}e$. Accordingly [15]:

$$P_r(|Z_{\overline{X}}| < W_{\overline{X}} \| Z_{\overline{X}}| < K_{\overline{X}}) = P_r(|Z_{\overline{e}}| < W_{\overline{e}} \| Z_{\overline{e}}| < K_{\overline{e}})$$
$$K_{\overline{X}} = K_{\overline{e}} = K \qquad W_{\overline{X}} = W_{\overline{e}} = W$$

When $\bar{W}_X = \bar{W}e = 0$ and $n_1 = n_2 = n_3 = n_0$ and $t_1 = t_2 = t_3 = t_0$, the joint control charts $\bar{Z}_X$ and $\bar{Z}e$ utilize a fixed sample size of n_0 and a constant sampling interval t_0. These charts are referred to as FSSI $\bar{Z}_X$ and FSSI $\bar{Z}e$. When $n_1 = n_2 = n_3 = n_0$ and $t_1 < t_0 < t_2 < t_3$, the sample size remains constant at n_0, while the sampling interval varies. Thus, the joint charts ASSI $\bar{Z}X$ and ASSI$\bar{Z}e$ are known as ASI $\bar{Z}_X$ and ASI $\bar{Z}_e$. When $t_1 = t_2 = t_3 = t_0$ and $n_1 < n_2 < n_0 < n_3$, the sampling interval is fixed at t_0 and the sample size varies. Accordingly, the joint charts ASSI $\bar{Z}_X$ and ASSI $\bar{Z}e$ are referred to as ASS $\bar{Z}_X$ and ASS $\bar{Z}e$ [15].

3.2. Comparison of Sample Size and Sampling Interval in Fixed and Proposed Control Charts [15-17]:

Here we compare two different sampling schemes under identical conditions. Specifically, the average sample size and average sampling interval should be calculated while the process is in-control, as follows [15-17]**:**

$$E[n(j)|\delta_1 = 0, \delta_2 = 0, |Z_{\bar{X}}| < K, |Z_{\bar{e}}| < K] = n_0$$
$$E[t(j)|\delta_1 = 0, \delta_2 = 0, |Z_{\bar{X}}| < K, |Z_{\bar{e}}| < K] = t_0$$

As can be observed, the above conditional expectation is computed under the assumption of no shift in the mean, i.e., $\delta_1 = 0$, $\delta_2 = 0$. Based on the above equations, the following formulas are derived[15-17]**:**

$$\begin{gathered} n_1 * P(Z_{\bar{X},i-1} \in I_{\bar{X}_1} | \delta_1 = 0, \delta_2 = 0) * P(Z_{\bar{e},i-1} \in I_{\bar{e}_1} | \delta_1 = 0, \delta_2 = 0) + \\ n_2 * P\left(Z_{\bar{X},i-1} \in I_{\bar{X}_1} | \delta_1 = 0, \delta_2 = 0\right) * P\left(Z_{\bar{e},i-1} \in I_{\bar{e}_2} | \delta_1 = 0, \delta_2 = 0\right) + \\ n_2 * P\left(Z_{\bar{X},i-1} \in I_{\bar{X}_2} | \delta_1 = 0, \delta_2 = 0\right) * P\left(Z_{\bar{e},i-1} \in I_{\bar{e}_1} | \delta_1 = 0, \delta_2 = 0\right) + \\ n_3 * P\left(Z_{\bar{X},i-1} \in I_{\bar{X}_2} | \delta_1 = 0, \delta_2 = 0\right) * P\left(Z_{\bar{e},i-1} \in I_{\bar{e}_2} | \delta_1 = 0, \delta_2 = 0\right) + \\ 0 * P \text{ (false alarms)} = n_0(2\Phi(K) - 1)^2 \end{gathered}$$

Simply put, we have:

$$\begin{gathered} 4\,\Phi(w)^2[n_1 - 2n_2 + n_3] + 4\Phi(w)[-n_1 + 2n_2\Phi(K) + n_2 - 2n_3\Phi(K)] = \\ n_0(2\Phi(K) - 1)^2 - n_1 + 4n_2\Phi(K) - 4n_3(\Phi(K))^2 \end{gathered}$$

Where $\Phi(.)$ represents the standard normal distribution function $\left(\Phi(Z) = \int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} du\right)$, which can be obtained from the standard normal distribution table. The calculation of the warning limits using the values n1, n2, and n3 is as follows [15-17]**:**

$$W = \Phi^{-1}\left(\frac{-4B_1 \pm \sqrt{16B_1^2 - 16A_1C_1}}{8A_1}\right)$$
$$A_1 = n_1 - 2n_2 + n_3$$
$$B_1 = -n_1 + 2n_2\Phi(K) + n_2 - 2n_3\Phi(K)$$
$$C_1 = -[n_0(2\Phi(K) - 1)^2 - n_1 + 4n_2\Phi(K) - 4n_3(\Phi(K))^2]$$

Similarly, the warning limits can be calculated using the values t1, t2, and t3 as follows [15-18]:

$$W=\Phi^{-1}\left(\frac{-4B_2\pm\sqrt{16B_2^2-16A_2C_2}}{8A_2}\right)$$
$$A_2=t_1-2t_2+t_1$$
$$B_2=-t_3+2t_2\Phi(K)+t_2-2t_1\Phi(K)$$
$$C_2=-[t_0(2\Phi(K)-1)^2-t_3+4t_2\Phi(K)-4t_1(\Phi(K))^2]$$

Prabhu, Montgomery, and Runger (1994) stated that the selected sample size for obtaining the warning limit W should be rounded to the nearest integer to prevent approximation errors. herefore, two out of the three parameters t_3, t_2, and t1 must be determined. Since the minimum sampling interval often depends on the type of inspection and sampling method, and the values of t1 must be less than t0, we determine t2 and t1 based on the residuals. Using the obtained W and the determined t2 and t1, the last parameter t3 is obtained according to the following equation [15-18]**:**

$$t_3=\frac{-4(\Phi(w))^2(-2t_2+t_1)-4\Phi(w)(2\Phi(K)t_2+t_2-2\Phi(K)t_1)+t_0(2\Phi(K)-1)^2+4\Phi(K)t_2-4(\Phi(K))^2t_1}{-4(\Phi(w))^2-4\Phi(w)+1}$$

With the parameters specified and the control limits determined, a comparison between the ASSI and FSSI schemes is conducted. To clearly demonstrate this comparison, a numerical example is provided.

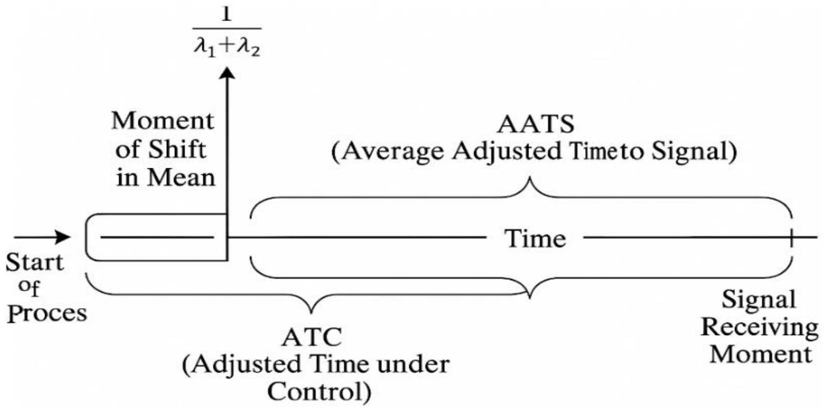

Figure 1. Expressing the performance of adaptive control charts [15-18].

3.3. Statement of the Performance of Adaptive Control Charts:

Typically, as shown in the figure below, from the moment when the process goes out of control (causing fluctuations in the mean), until the moment when an alert appears on the Shewhart control chart, some time will pass. Here, the aim is to minimize this time by using proposed adaptive control charts and leveraging the rules governing Markov chains, thus allowing for a faster identification of the causes of the process being out of control [1-3].

The average time from when the process goes out of control until the signal is received. The average duration from the start of the process until the signal is received.

By default, the following table is considered as the table of possible states of the process. In this table, the probability of transitioning from one state to another is determined according to the rule of the Markov chain random walk, and the probability of moving from one state to another only depends on the current state and does not depend on any of the previous states [4,5].

Table 1. Process Control Data

Sample $Z\bar{e}$ Statistic	**Is the out-of-control signal received from the second stage of the process?**	**Has a shift in the mean occurred in the (e)̅ chart?**	**Sample $Z\_\bar{X}$ Statistic**	**Is the out-of-control signal received from the first stage of the process?**	**Has a shift in the mean occurred in the (X)̅ chart?**	**Component**
I_1	No message	No	I_1	No message	No	1
I_1	No message	Yes	I_1	No message	No	2
I_1	No message	No	I_1	No message	Yes	3
I_1	No message	Yes	I_1	No message	Yes	4
I_1	No message	No	I_2	No message	No	5
I_1	No message	Yes	I_2	No message	No	6
I_1	No message	No	I_2	No message	Yes	7
I_1	No message	Yes	I_2	No message	Yes	8
I_2	No message	No	I_1	No message	No	9
I_2	No message	Yes	I_1	No message	No	10
I_2	No message	No	I_1	No message	Yes	11
I_2	No message	Yes	I_1	No message	Yes	12
I_2	No message	No	I_2	No message	No	13
I_2	No message	Yes	I_2	No message	No	14
I_2	No message	No	I_2	No message	Yes	15
I_2	No message	Yes	I_2	No message	Yes	16

Markov Chain Transition Probabilities and ATC Calculation

These probabilities form a square matrix of order 16 as follows:

P im (t_j,n_j): The probability matrix from the previous component i to the current component m with sample size nj and tj, determined by the previous component i.

i = 1, 2, …, 16 m = 1, 2, …, 16 j = 1, 2, 3

For example, the transition probability from the first component to the sixth component is calculated as follows:

$$P_{1,6}(t_3, n_1) = P[-K < Z_{\bar{X}} < -W \cup W \leq Z_{\bar{X}} < K] * P[|Z_{\bar{e}}| < W|\delta_2] * e^{-\lambda_1 t_3} * (1 - e^{-\lambda_2 t_3})$$

The formula for Average Time to Change (ATC) calculation is as follows [5,6]:

$$ATC = \acute{b}(I - \varphi)^{-1} t$$

$$\left[P_0, 0,0,0, \frac{P_1}{2}, 0,0,0, \frac{P_1}{2}, 0,0,0, P_2, 0, \dots, 0\right] \begin{bmatrix} 1-P_{11} & 0-P_{12} & \cdots & & 0-P_{1\,16} \\ \vdots & \vdots & & & \vdots \\ 0-P_{16\,1} & 1-P_{16\,2} & 0-P_{16\,3} & \dots & 1-P_{16\,16} \end{bmatrix}^{-1} \begin{bmatrix} t_3 \\ t_3 \\ t_3 \\ t_3 \\ t_2 \\ t_2 \\ t_2 \\ t_2 \\ t_2 \\ t_2 \\ t_2 \\ t_2 \\ t_1 \\ t_1 \\ t_1 \\ t_1 \end{bmatrix}$$

Probability Vector and Transition Matrix

b: The initial probability vector for components 1 to 16.

When the first sample has a probability P_0 of being small (or component 1 with probability P_0),

a probability P1/2 of being medium (or components 5 and 9 with probability P1/2), and a probability P_2 of being large (or component 13 with probability P_2).

I: The identity matrix of order 16.

φ: The transition probability matrix, consisting of elements that indicate the transition probabilities as shown in the default table.

P_im(tj, nj): The transition probability from transient state i = 1, 2, …, 16 to transient state m = 1, 2, …, 16.

t: The sampling interval vector for components 1 to 16 after calculating ATC from the following formula AATS [1-3].

We obtain:

$$\text{AATS}=\text{ATC}-\frac{1}{\lambda_1+\lambda_2}$$

4-Research Findings

An Example: A sampling process from a tea bag production machine has been carried out.

In this example, the structure of the proposed control charts ASSI($Z_{\bar{X}}$) and ASSI($Z_{\bar{e}}$) is illustrated and compared with the FSSI ($Z_{\bar{X}}$) and FSSI ($Z_{\bar{e}}$) charts in terms of mean shift detection speed.

Values:

X =Weight of tea in each small tea bag from the first stage of the process.

Y =Weight of a tea bag canister containing 25 tea bags from the second stage of the process.

Each sampling consists of a sample size of 5.

Paired observations are measured as (X_i, Y_i) for i = 1, 2, 3, …, n_j.

The average time under control for the first stage of the process (T_1) follows an exponential distribution with parameter $\lambda_1 = 0.033$.

The average time under control for the second stage of the process (T_2) follows an exponential distribution with parameter $\lambda_2 = 0.043$.

Initially, five samples are taken every hour (n0 = 5, t0 = 1). Based on previous data analysis, both variables X and Y follow a normal distribution.

However, to ensure the normality of the two variables, an initial sampling of 20 samples of size 5 from each stage of the process, at one-hour intervals, was conducted. The normality of the two variables was tested using the Kolmogorov-Smirnov (K-S) test with the help of Minitab software [1-7].

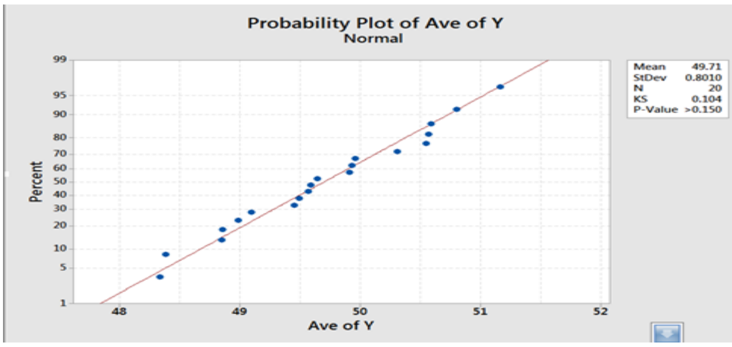

Figure 2. Probability plot of Ave of Y Normal

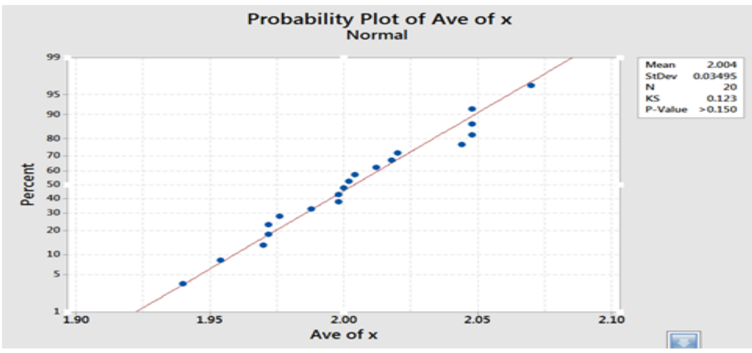

Figure 3. Probability plot of Ave of X Normal

In this test, the null hypothesis states that the distribution of the data X and Y is normal, while the alternative hypothesis states that the data is not normally distributed. Given that the p-value is greater than 0.01, we accept the null hypothesis of normality. The variables X and Y are not independent, indicating that there is a relationship between them. An statistical relationship between two variables is most clearly represented in scatterplots. In this example, the sample points in the scatterplot appear as an elliptical cloud, indicating the presence of a linear relationship between the variables X and Y. The model is expressed as follows [1-7]:

$$\hat{Y}|X = \beta_0 + \beta_1 X$$

The parameters β_0 and β_1 are calculated as follows:

$$\beta_1 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} \qquad \beta_0 = \bar{Y} - \beta_1 \bar{X}$$

$$\hat{Y}|X = 47.10 + 1.30 \times X$$

Therefore, the residual (e) is obtained as follows:

$$e = Y|X - \hat{Y}|X$$

The estimated mean and standard deviation of the variables x and e are given as follows, respectively:

$$\hat{\mu}_X = 2.011 \qquad \hat{\sigma}_X = 0.110 \qquad \hat{\mu}_e = 0 \qquad \hat{\sigma}_e = 1.657$$

These are given as follows when both stages of the process are in control and with a fixed sample size n_0:

$$\bar{X} \sim N(\hat{\mu}_X, (\frac{\hat{\sigma}_X}{\sqrt{n_0}})^2) \qquad \bar{e} \sim N(\hat{\mu}_X, (\frac{\hat{\sigma}_X}{\sqrt{n_0}})^2)$$

Shewhart control charts based on samples collected at fixed intervals are constructed for $\bar{X}$ and $\bar{e}$ as follows:

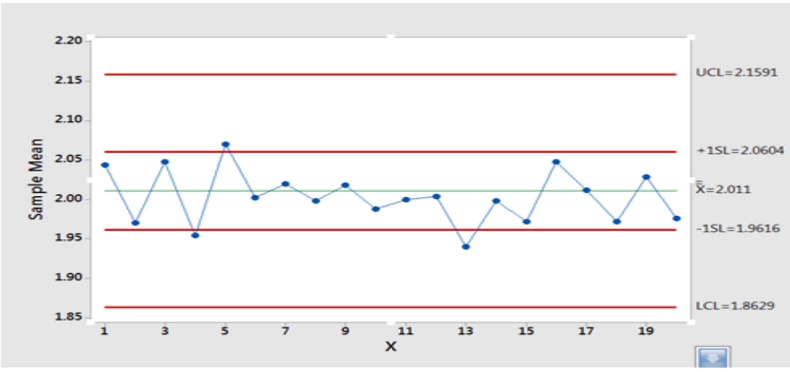

Figure 4. Shewhart control charts based on samples collected at fixed intervals

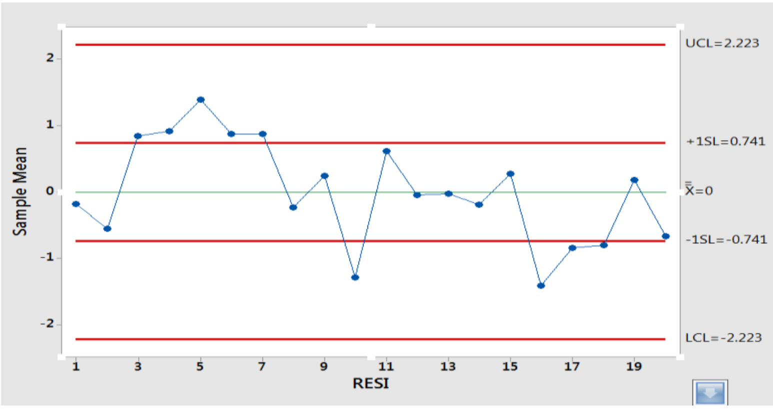

Figure 5. Shewhart control charts based on samples collected at fixed intervals

During or alongside the fixed sampling, a variable sampling scheme has also been conducted to compare the proposed design with the traditional one. For out-of-control processes, the estimated mean shift in the first stage is $\delta_1 \hat{\sigma}_X$, where $\hat{\delta}_1 = 0/25$, and the estimated mean shift in the second stage of the process is $\delta_2 \hat{\sigma}_X$, where $\hat{\delta}_1 = 0/5$. Furthermore, for the first stage of the out-of-control process, we have [1-7]:

$$\overline{X} \sim N(\hat{\mu}_X + \hat{\sigma}_X * 0/25, (\frac{\hat{\sigma}_X}{\sqrt{n_0}})^2)$$

The second stage is out of control

$$\bar{e} \sim N(\hat{\mu}_e + \hat{\sigma}_e * 0/5, (\frac{\hat{\sigma}_e}{\sqrt{n_0}})^2)$$

Using Z̄_X and Z̄_e with Variable Sample Size and Interval

Here, we use $\bar{Z}_X$ and Z̄e with a variable sample size and variable sampling interval according to the following procedure:

Step 1: Assume K=3 (It is better to use K=3 to reduce the Type I error).
Step 2: Determine:
n1=2, n2=3, n3=20
t1=0.01, t2=0.7
Step 3: Calculate W and t_3:
W=0.88, t3=1.74

With the designed parameter settings, the combined control charts Z̄X and Z̄e can be used for monitoring the two-stage process of tea bag production. When a process begins, according to the randomized decision-making method, the initial sample size and the first sampling interval are planned with t = 0.7 hours and a sample size of 3. Three pairs of observations (Xi, Yi) are: (51.13, 2.10), (49.99, 2.00), and (49.33, 1.92). The sample statistics (X̄, ē) from the first sample are (0.44, 2.01). The corresponding ($\bar{Z}_X$, Z̄e) values are (0.46, -0.09). The sample statistic Z̄_e falls in the central region, while Z̄_X falls in the warning region. As a result, the next sample should be taken at time 0.7 + 0.7 = 1.4 hours with a sample size of 3. The three observed data points (Xi, Yi) are (48.19, 2.05), (50.26, 2.21), and (49.52, 2.09) [1-7].

$$Z_{\bar{x}} = \frac{\overline{X} - \mu_X}{\frac{\sigma_X}{\sqrt{n_j}}} \qquad Z_{\bar{e}} = \frac{\frac{\bar{e}}{\sigma_e}}{\sqrt{n_j}}$$

The values of (Z̄X, Z̄e) = (-0.55, 1.67), where both Z̄X and Z̄e fall within the central region, indicate that the next sample will be taken at time 2.78 = 1.04 + 1.74 with a sample size of 2. The process continues similarly, producing values of (-0.09, -0.55), (0.80, 1.17), (-1.27, 3.04), (-0.13, -0.08), and (1.65, -1.01), which are obtained from the sample results for (Z̄X, $\bar{Z}_e$) and plotted on the control charts. As shown in the figure, at time 4.56 after the start of the process, a point outside the control limits is observed on the $\bar{Z}_X$ chart. At this point, the process is stopped to investigate the cause of being out of control [1-7].

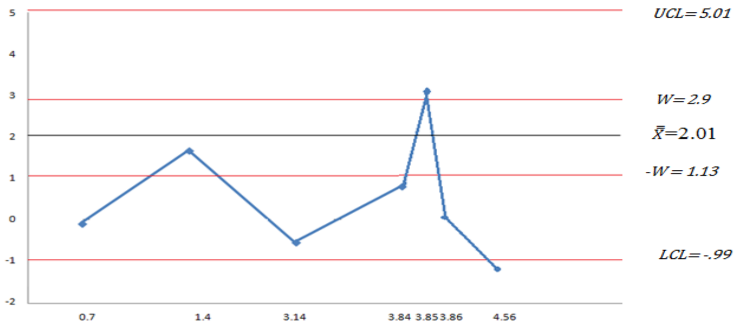

Figure 6. The combined control charts $\bar{Z}X$ and $\bar{Z}e$

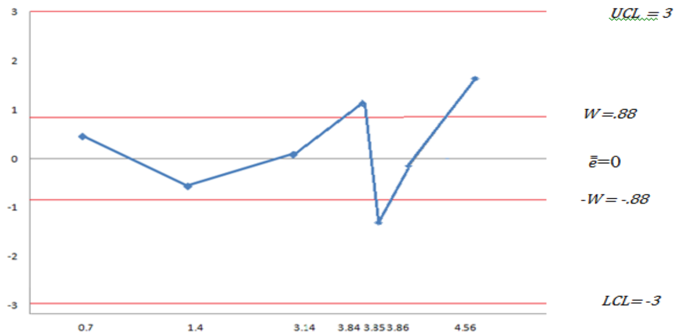

Figure 7. The combined control charts $\bar{Z}X$ and $\bar{Z}e$

5. Conclusion

As the evidence suggests, the feedback from this operation indicated that adaptive control charts detect the out-of-control state of the process more quickly at lower levels of mean shifts. This can be highly beneficial for quality control engineers and, consequently, for manufacturing plants. In the Shewhart control charts shown in Figures (4) and (5), which use fixed sampling and correspond to two working days of machine data, no out-of-control signals were observed. However, according to Figures (6) and (7), on the first of those two days, less than five hours after the process began, a sign of the process being out of control appeared. This allows for an earlier investigation into the cause of the process deviation and the possibility of promptly correcting the defects if necessary. The ASSI scheme improves the sensitivity of the combined control charts $\bar{Z}X$ and $\bar{Z}e$. Based on recent calculations, it appears that for detecting a change in the process mean, the AATS (Average Adjusted Time to Signal) of the combined charts $\bar{Z}_X$ and $\bar{Z}e$ is shorter than that of the fixed scheme [1-7].

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