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Dynamical behavior of a multi-group stochastic SEIR system with symptomatic and asymptomatic individuals

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Abstract. In this paper, a multigroup stochastic SEIR model dealing with both symptomatic and asymptomatic cases is considered. First, we show there exists a unique global positive solution to the system for any given positive initial value. Next, we provide sufficient criteria for the existence of a unique stationary ergodic distribution of the positive solutions. Finally, sufficient criteria for disease extinction are derived.

Keywords: Multi-group SEIR model, asymptomatic individuals, ergodic stationary distribution, extinction.
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1 Introduction

Multigroup stochastic epidemic models are commonly employed in mathematical epidemiology to analyze how infectious diseases spread within diverse populations. Disease transmission can be affected by multiple pathways, such as person-to-person contact, respiratory droplets, or contact with infected bodily fluids. Due to the inherent uncertainty in disease dynamics, accurately predicting and controlling its transmission remains a significant challenge. To address this complexity, numerous epidemic models have been developed to analyze infection dynamics. For example, Husniah et al. [7] proposed an SEIR (susceptible, exposed, infected and recovered) model incorporating convalescent plasma transfusion, demonstrating that this intervention reduces the peak of infection and improves epidemiological outcomes. Additionally, researchers have increasingly integrated stochastic process into deterministic models to account for the effects of random noise of environment on disease spread [2–4, 9, 14, 16].

Incorporating a multi-group structure is a widely used approach to enhance the realism of epidemic models. In such models, individual heterogeneity such as differences in age, geographic location, sex, or social role can be systematically explained by indexing subgroups with distinct parameters [8, 12–14, 16]. For instance, Zhong and Deng [19] developed a stochastic multi-group SEIR system and derived its basic

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reproduction number R_0^s. They established asymptotic stability for the disease-free equilibrium (DFE) when $R_0^s < 1$, and persistence of disease when $R_0^s > 1$ where the system admits a globally asymptotically stable stationary distribution. In a related work, Liu and Jiang [12] derived sufficient criteria for the existence of a stationary ergodic distribution for the model's positive solutions.

In this paper, we extend the stochastic SEIR-type model proposed in [11] to a multi-group framework, formulated as follows:

$$\begin{cases} dS_k = \left(\Lambda_k - \sum_{j=1}^n \beta_{kj}^C S_k I_j^C - \sum_{j=1}^n \beta_{kj}^A S_k I_j^A - \mu_k S_k\right) dt, \\ dE_k = \left(\sum_{j=1}^n \beta_{kj}^C S_k I_j^C + \sum_{j=1}^n \beta_{kj}^A S_k I_j^A - (\mu_k + \theta_k) E_k\right) dt, \\ dI_k^C = \left(p_k \theta_k E_k - (\mu_k + \alpha_k^C + \gamma_k^C) I_k^C\right) dt, \\ dI_k^A = \left((1 - p_k)\theta_k E_k - (\mu_k + \alpha_k^A + \gamma_k^A) I_k^A\right) dt, \\ dR_k = \left(\gamma_k^C I_k^C + \gamma_k^A I_k^A - \mu_k R_k\right) dt, \quad k = 1, 2, \ldots, n \end{cases} \tag{1}$$

where $S_k(t)$, $E_k(t)$, $I_k^C(t)$, $I_k^A(t)$ and $R_k(t)$ denote the number of susceptible, exposed, undetected asymptomatic infected, symptomatic infected and recovered individuals with heterogeneity k at time t, respectively. For indices k and j, the parameters in system (1) are as follows:

- Λ_k: The rate at which new individuals enter the population,
- $\beta_{kj}^C, \beta_{kj}^A$: Transmission rate from S_k to I_j^C and I_j^A respectively,
- μ_k: Natural mortality rate across all population groups,
- p_k: Proportion of infected individuals who develop symptomatic illness,
- $1 - p_k$: Proportion of infected individuals who remain asymptomatic,
- θ_k: latency rate
- α_k^C, α_k^A: Fatality rate due to infection in I_j^C and I_j^A respectively,
- γ_k^C, γ_k^A: recovery rate for I_j^C and I_j^A respectively.

To consider environmental fluctuations we add type of white noises which are proportional to each variable, respectively. Therefore, due to the deterministic model (1), the stochastic multi-group system takes the following form:

$$\begin{cases} dS_k = \left(\Lambda_k - \sum_{j=1}^n \beta_{kj}^C S_k I_j^C - \sum_{j=1}^n \beta_{kj}^A S_k I_j^A - \mu_k S_k\right) dt + \delta_k S_k dB_k(t), \\ dE_k = \left(\sum_{j=1}^n \beta_{kj}^C S_k I_j^C + \sum_{j=1}^n \beta_{kj}^A S_k I_j^A - (\mu_k + \theta_k) E_k\right) dt + \lambda_k E_k dB_k(t), \\ dI_k^C = \left(p_k \theta_k E_k - (\mu_k + \alpha_k^C + \gamma_k^C) I_k^C\right) dt + \xi_k I_k^C dB_k(t), \\ dI_k^A = \left((1 - p_k)\theta_k E_k - (\mu_k + \alpha_k^A + \gamma_k^A) I_k^A\right) dt + \nu_k I_k^A dB_k(t), \\ dR_k = \left(\gamma_k^C I_k^C + \gamma_k^A I_k^A - \mu_k R_k\right) dt \quad k = 1, 2, \ldots, n. \end{cases} \tag{2}$$

Here, $B_k(t)$, $k = 1, \ldots, n$ are standard Brownian motions that are mutually independent with $B_k(0) = 0$ which are defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t \geq 0}$. In probability theory, an event happens almost surely ($\mathbb{P}$-a.s.) if it happens with probability 1. Also, δ_k^2,

λ_k^2, ξ_k^2 and ν_k^2s denote the intensities of white noises $\dot{B}_k(t)$. All the parameters in (1) are assumed to be nonnegative. Also, the groups $S_k(t)$, $E_k(t)$, $I_k^C(t)$, $I_k^A(t)$ and $R_k(t)$ represent the number of individuals and thus are nonnegative.

This paper is organized as follows. In Section 2, we first establish fundamental properties of systems (1) and (2), then prove that model (1) admits a unique global positive solution for any positive initial value. In Section 3, under the persistence scenario, we derive sufficient criteria for the existence of a unique ergodic stationary distribution by constructing an appropriate stochastic Lyapunov function. Finally, Section 4 presents sufficient criteria for disease extinction.

2 Existence and uniqueness of the global positive solution

In this section, we first state some basic properties of systems (1) and (2). Then we show that unique positive solution exists to system (1). Recall that the population size in distinct compartments of the heterogeneity k are shown by S_k, E_k, I_k^C I_k^A and R_k.

Remark 1. *Systems* (1) *and* (2) *have the following properties:*

- *For $S_k^0 = \frac{\Lambda_k}{\mu_k}$, $k = 1, \dots, n$, the DFE point of the systems* (1) *and* (2) *is*
$$P_0 = (S_1^0, 0, 0, 0, 0, S_2^0, 0, \dots, S_n^0, 0, 0, 0).$$

- *We take the infected compartments to be E_k, I_k^C and I_k^A. Then,*
$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sum_{j=1}^n \beta_{kj}^C S_k & 0 \\ \sum_{j=1}^n \beta_{kj}^A S_k & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \mu_k + \theta_k & -p_k\theta_k & -(1-p_k)\theta_k \\ 0 & \mu_k + \alpha_k^C + \gamma_k^C & 0 \\ 0 & 0 & \mu_k + \alpha_k^A + \gamma_k^A \end{pmatrix},$$
and the next generation matrix for system (1) *is*
$$M_0 =: FV^{-1} = \left(\frac{\Lambda_k \beta_{kj}^C p_k \theta_k}{\mu_k(\mu_k+\theta_k)(\mu_k+\alpha_k^C+\gamma_k^C)} + \frac{\Lambda_k \beta_{kj}^A (1-p_k)\theta_k}{\mu_k(\mu_k+\theta_k)(\mu_k+\alpha_k^A+\gamma_k^A)} \right)_{(n\times n)}.$$
Therefore, the basic reproduction number $R_0 = \rho(M_0)$ for system (1) *equals*
$$\sum_{k=1}^n \left(\frac{\Lambda_k \sum_{j=1}^n \beta_{kj}^C p_k \theta_k}{\mu_k(\mu_k+\theta_k)(\mu_k+\alpha_k^C+\gamma_k^C)} + \frac{\Lambda_k \sum_{j=1}^n \beta_{kj}^A (1-p_k)\theta_k}{\mu_k(\mu_k+\theta_k)(\mu_k+\alpha_k^A+\gamma_k^A)} \right),$$
where ρ is the spectral radius of the matrix.

- *Likewise, the stochastic reproduction number equals $R_0^s = \rho(M_0^s)$, where*
$$\begin{aligned} M_0^s = \Bigg(& \frac{\Lambda_k \beta_{kj}^C p_k \theta_k}{(\mu_k + \frac{\delta_k^2}{2})(\mu_k + \theta_k + \frac{\lambda_k^2}{2})(\mu_k + \alpha_k^C + \gamma_k^C + \frac{\xi_k^2}{2})} \\ & + \frac{\Lambda_k \beta_{kj}^A (1-p_k)\theta_k}{(\mu_k + \frac{\delta_k^2}{2})(\mu_k + \theta_k + \frac{\lambda_k^2}{2})(\mu_k + \alpha_k^A + \gamma_k^A + \frac{\nu_k^2}{2})} \Bigg)_{(n\times n)}. \end{aligned} \tag{3}$$

- *Denote by* $B^C = (\beta_{kj^C})_{n\times n}$ *and* $B^A = (\beta_{kj^A})_{n\times n}$ *the matrix of parameters in system* (1). *If both matrices* B^C *and* B^A *are irreducible and* $R_0 < 1$, *then system* (1) *is globally asymptotically stable (GAS) in the positive invariant set*

$$\Omega = \left\{ (S_1, E_1, I_1^C, I_1^A, \ldots, S_n, E_n, I_n^C, I_n^A) \in \mathbb{R}_+^{4n} : S_k \leq \frac{\Lambda_k}{\mu_k}, S_k + E_k + I_k^C + I_k^A \leq \frac{\Lambda_k}{\mu_k} \right\}.$$

- *If* $R_0 > 1$ *and both matrices* B^C *and* B^A *are irreducible, then* P_0 *is unstable and there is an endemic equilibrium*

$$P^* = (S_1^*, E_1^*, I_1^{A*}, I_1^{C*}, R_1^*, \cdots, S_n^*, E_n^*, I_n^{A*}, I_n^{C*}, R_n^*) \in int\, \Omega$$

which is GAS in int Ω *(by int* Ω*, we mean the interior of the set* Ω*).*

Lemma 1. *For* $\mathbb{P}$*-a.s. any initial value* $(S_k(0), E_k(0), I_k^C(0), I_k^A(0), R_k(0)) \in \mathbb{R}_+^{5n}$, *system* (1) *has a unique positive solution* $(S_k(t), E_k(t), I_k^C(t), I_k^A(t), R_k(t))$ *for every* $t \geq 0$, *which remains positive a.s., i.e.*

$$(S_k(t), E_k(t), I_k^C(t), I_k^A(t), R_k(t)) \in \mathbb{R}_+^{5n}, \;\; \mathbb{P}\text{-}a.s.,$$

for all $t \geq 0$. *Moreover, by letting* $N_k = S_k + E_k + I_k^C + I_k^A + R_k$,

$$\limsup_{t\to\infty} N_k(t) = \frac{\Lambda_k}{\mu_k}.$$

Proof. To prove the last statement, note that according to system (1),

$$\frac{N_k}{dt} = \Lambda_k - \mu_k N_k - \alpha_k^C I_k^C - \alpha_k^A I_k^A. \tag{4}$$

Therefore

$$\limsup_{t\to\infty}(N_k(t)) \leq \limsup_{t\to\infty} \frac{N_k(0) + \int_0^t \Lambda_k e^{\mu_k s} ds}{e^{\mu_k t}} = \frac{\Lambda_k}{\mu_k}. \tag{5}$$

To prove the first statement, note that the last equation in (1) is independent from the others and it suffices to prove the lemma for the first four equations. Since system (1) is locally Lipschitz continuous, any $(S_k(0), E_k(0), I_k^C(0), I_k^A(0)) \in \mathbb{R}_+^{4n}$ has a unique local solution on $[0, \kappa_0)$ for some $\kappa_0 > 0$.

To prove the globality of solution, it suffices to show $\kappa_0 = +\infty$ for $\mathbb{P}$-a.s. Choose a large enough value $m_0 \geq 1$ such that $S_k(0), E_k(0), I_k^C(0), I_k^A(0) \in [\frac{1}{m_0}, m_0]$. For any $m > m_0$ and $1 \leq k \leq n$, define

$$\kappa_m = \inf\{t \in [0, \kappa_0) : \min\{S_k(t), E_k(t), I_k^C(t), I_k^A(t)\} \leq \frac{1}{m} \text{ or } \max\{S_k(t), E_k(t), I_k^C(t), I_k^A(t)\} \geq m\}. \tag{6}$$

Since κ_m increases, let $\kappa_\infty = \lim_{k\to\infty} \kappa_m$. Thus, $\kappa_\infty \leq \kappa_0$ $\mathbb{P}$-a.s. We aim to prove that $\kappa_\infty = +\infty$. If $\kappa_\infty \neq +\infty$, then there exist $T \geq 0$ and $\varepsilon \in (0,1)$ such that $\mathbb{P}(\kappa_\infty \leq T) \geq \varepsilon$. Thus there exists $m_1 \geq m_0$ with $\mathbb{P}(\kappa_m \leq T) \geq \varepsilon$ for all $m \geq m_1$. Let $\mathscr{K}_m = \{\kappa_m \leq T\}$. Define a Lyapunov function $V : \mathbb{R}_+^{4n} \to \mathbb{R}_+ \cup \{0\}$ as

$$\begin{aligned} V &= V(S_1, \ldots, S_n, E_1, \ldots, E_n, I_1^C, \ldots, I_n^C, I_1^A \ldots, I_n^A) \\ &= \sum_{k=1}^n (S_k - a - a\ln\frac{S_k}{a}) + \sum_{k=1}^n (E_k - 1 - \ln E_k) \end{aligned}$$

$$+\sum_{k=1}^{n}(I_k^C-1-\ln I_k^C)+\sum_{k=1}^{n}(I_k^A-1-\ln I_k^A),$$

where a is a positive constant to be chosen later. Since $x-1-\ln x\geq 0$ for $x>0$, the function V is nonnegative. By applying Itô's formula to V, the following upper bound is obtained for LV:

$$\begin{aligned}
LV &= \sum_{k=1}^{n}\left(\Lambda_k-\sum_{j=1}^{n}\beta_{kj}^{C}S_kI_j^C-\sum_{j=1}^{n}\beta_{kj}^{A}S_kI_j^A-\mu_kS_k\right)-a\sum_{k=1}^{n}\left(\frac{\Lambda_k}{S_k}-\sum_{j=1}^{n}\beta_{kj}^{C}I_j^C-\sum_{j=1}^{n}\beta_{kj}^{A}I_j^A-\mu_k-\frac{\delta_k^2}{2}\right)\\
&+\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\beta_{kj}^{C}S_kI_j^C+\sum_{j=1}^{n}\beta_{kj}^{A}S_kI_j^A-(\mu_k+\theta_k)E_k\right)\\
&-\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\beta_{kj}^{C}\frac{S_kI_j^C}{E_k}+\sum_{j=1}^{n}\beta_{kj}^{A}\frac{S_kI_j^A}{E_k}-(\mu_k+\theta_k)-\frac{\lambda_k^2}{2}\right)+\sum_{k=1}^{n}\left(p_k\theta_kE_k-(\mu_k+\alpha_k^C+\gamma_k^C)I_k^C\right)\\
&-\sum_{k=1}^{n}\left(p_k\theta_k\frac{E_k}{I_k^C}-(\mu_k+\alpha_k^C+\gamma_k^C)\right)\sum_{k=1}^{n}\frac{\xi_k^2}{2}+\sum_{k=1}^{n}\left((1-p_k)\theta_kE_k-(\mu_k+\alpha_k^A+\gamma_k^A)I_k^A\right)\\
&-\sum_{k=1}^{n}\left((1-p_k)\theta_k\frac{E_k}{I_k^A}-(\mu_k+\alpha_k^A+\gamma_k^A)\right)+\sum_{k=1}^{n}\frac{\nu_k^2}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
LV &\leq \sum_{j=1}^{n}\left[\sum_{k=1}^{n}\left(\beta_{kj}^{C}(a+S_k)-\mu_j-\alpha_j^C-\gamma_j^C\right)I_j^C+\sum_{k=1}^{n}\left(\beta_{kj}^{A}(a+S_k)-\mu_j-\alpha_j^A-\gamma_j^A\right)I_j^A\right]\\
&+\sum_{k=1}^{n}(\Lambda_k+\theta_kE_k+a\mu_k+3\mu_k+\theta_k+\alpha_k^C+\gamma_k^C+\alpha_k^A+\gamma_k^A+\frac{a\delta_k^2+\lambda_k^2+\xi_k^2+\nu_k^2}{2}).
\end{aligned}$$

Choose

$$a=\min_{1\leq j\leq n}\left\{\frac{\mu_j+\alpha_j^C+\gamma_j^C-\sum_{k=1}^{n}\beta_{kj}^{C}}{\sum_{k=1}^{n}\beta_{kj}^{C}},\ \frac{\mu_j+\alpha_j^A+\gamma_j^A-\sum_{k=1}^{n}\beta_{kj}^{A}}{\sum_{k=1}^{n}\beta_{kj}^{A}}\right\},$$

then $\sum_{k=1}^{n}\beta_{kj}^{C}(a+S_k)-\mu_j-\alpha_j^C-\gamma_j^C$ and $\sum_{k=1}^{n}\beta_{kj}^{A}(a+S_k)-\mu_j-\alpha_j^A-\gamma_j^A$ would be negative terms and

$$LV \leq \sum_{k=1}^{n}\left(\Lambda_k+(a+3)\mu_k+\theta_k+\alpha_k^C+\gamma_k^C+\alpha_k^A+\gamma_k^A+\theta_k\frac{\Lambda_k}{\mu_k}+\frac{a\delta_k^2+\lambda_k^2+\xi_k^2+\nu_k^2}{2}\right)=:\hat{N}.$$

Here $\hat{N}$ is positive and independent of the variables S_k, E_k, I_k^C and I_k^A. The expectation of V for random time $(\kappa_m\wedge T)$, has the property

$$\mathbb{E}[V((\kappa_m\wedge T),I(\kappa_m\wedge T))]\leq\mathbb{E}[V(S_k(0),E_k(0),I_k^C(0),I_k^A(0))]+T\hat{N},$$

where $(\kappa_m\wedge T)=\min\{\kappa_m,T\}$. For any $\kappa\in\mathscr{K}_m$, at least one of S_k, E_k, I_k^C or I_k^A does not belong to the set (6). Let

$$\mathscr{N}=\min\left\{m-1-\ln(m),\ \frac{1}{m}-1+\ln(m)\right\}.$$

Therefore

$$V(S_k(\kappa_m), E_k(\kappa_m), I_k^C(\kappa_m), I_k^A(\kappa_m)) \geq \mathscr{N},$$

and also

$$\mathbb{E}[V(S_k(0), E_k(0), I_k^C(0), I_k^A(0))] + T\hat{N} \geq \varepsilon \mathscr{N}.$$

Thus, if $t \to \infty$, then $\infty > \mathbb{E}[V((S_k(0), E_k(0), I_k^C(0), I_k^A(0))] + T\hat{N} = \infty$, which is a contradiction. □

The following remark determines the bounded set which contains the global solution to system (1).

Remark 2. *Let $N_k = S_k + E_k + I_k^C + I_k^A + R_k$ for $k = 1, \ldots, n$. According to system (1),*

$$dN_k = (\Lambda_k - \mu_k N_k - \alpha_k^C I_k^C - \alpha_k^A I_k^A)dt.$$

Therefore,

$$(\Lambda_k - (\mu_k + \alpha_k^C + \alpha_k^A)N_k)dt \leq dN_k \leq (\Lambda_k - \mu_k N_k)dt.$$

Thus,

$$\frac{\Lambda_k}{\mu_k + \alpha_k^C + \alpha_k^A} + e^{-(\mu_k + \alpha_k^C + \alpha_k^A)t}(N_k(0) - \frac{\Lambda_k}{\mu_k + \alpha_k^C + \alpha_k^A}) \leq N_k(t) \leq \frac{\Lambda_k}{\mu_k} + e^{-\mu_k t}(N_k(0) - \frac{\Lambda_k}{\mu_k}).$$

If $\frac{\Lambda_k}{\mu_k + \alpha_k^C + \alpha_k^A} < N_k(0) < \frac{\Lambda_k}{\mu_k}$, then $\frac{\Lambda_k}{\mu_k + \alpha_k^C + \alpha_k^A} < N_k(t) < \frac{\Lambda_k}{\mu_k}$ for $\mathbb{P}$-a.s. which means the set

$$\Gamma^* = \left\{ (S_1, \ldots, R_n) \in \mathbb{R}_+^{5n} : \frac{\Lambda_k}{\mu_k + \alpha_k^C + \alpha_k^A} < N_k < \frac{\Lambda_k}{\mu_k}, k = 1, \ldots, N \right\}$$

is a positively invariant set of system (1).

3 Existence of ergodic stationary distribution

Here, we determine the required criteria that guarantees an ergodic stationary distribution exists and is unique for the positive solutions to system (1).

Lemma 2. *([1]) If A is nonnegative irreducible weight matrix, then $\rho(A)$ is a simple eigenvalue and A has a positive eigenvector $\omega = (\omega_1, \ldots, \omega_n)$ corresponding to $\rho(A)$.*

Lemma 3. *([5]) If for Markov process $X(t)$, there exists a bounded domain $\mathscr{U} \subset \mathbb{R}^d$ with a regular boundary Γ, such that:*

(i) $\exists \mathscr{M} > 0$ *with* $\sum_{i,j=1}^{d} \Lambda_{ij}(x)\omega_i\omega_j \geq \mathscr{M}|\omega^2|$, $x \in \mathscr{U}$, $\omega \in \mathbb{R}^d$,

(ii) there exists a nonnegative C^2-function V such that LV is negative for $\mathbb{R}^d \setminus \mathscr{U}$,

then $X(t)$ has a unique ergodic stationary distribution $\mu(.)$, and for all $x \in \mathscr{U}$ and $f \in L^1(\mu)$

$$\mathbb{P}\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(t))dt = \int_{\mathscr{U}} f(x)\mu(dx) \right\} = 1.$$

Let

$$\mathscr{C} = (\mu_k - \frac{\delta_k^2}{2})(\mu_k + \theta_k + \frac{\lambda_k^2}{2})(\mu_k + \alpha_k^C + \gamma_k^C + \frac{\xi_k^2}{2}),\ \mathscr{A} = (\mu_k - \frac{\delta_k^2}{2})(\mu_k + \theta_k + \frac{\lambda_k^2}{2})(\mu_k + \alpha_k^A + \gamma_k^A + \frac{\nu_k^2}{2}).$$

Theorem 1. *Suppose that* $1-p>0$, *and the matrices* $B^C = (\beta_{kj}^C)_{n\times n}$ *and* $B^A = (\beta_{kj}^C)_{n\times n}$ *are irreducible. If* $R_0^s > 1$ *and*

$$\min_{1\le k\le n}\{\mu_k+\theta_k\}(R_0^s-1) \; > \; -2n\frac{\min_{1\le k,j\le n}\left\{\frac{\omega_k\beta_{k,j}^C}{\mu_k+\theta_k},\ \frac{\omega_k\beta_{k,j}^A}{\mu_k+\theta_k}\right\}}{\max_{1\le k\le n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}}\sum_{k=1}^n\frac{\mu_k}{\Lambda_k}$$
$$+n\frac{\max_{1\le k,j\le n}\left\{\frac{\beta_{kj}^C\omega_k\Lambda_k p_k\theta_k}{\mathscr{C}},\ \frac{\beta_{kj}^A\omega_k\Lambda_k(1-p_k)\theta_k}{\mathscr{A}}\right\}}{\min_{1\le k\le n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}}+\frac{1}{2}\sum_{k=1}^n\lambda_k^2, \tag{7}$$

then there exists a positive solution $(S_1(t),\dots,R_n(t))$ *to system* (2) *which is an ergodic stationary Markov process, where* M_0^s *is as in* (3) *and*

$$R_0^s = \rho(M_0^s). \tag{8}$$

Proof. To prove an ergodic stationary process exists, it suffices to show that the conditions of Lemma 2 are fulfilled. The first condition holds since for any compact subset of $\mathbb{R}^{4n}$, the diffusion matrix of system (2) is positive definite. To verify the second condition, we should construct a nonnegative C^2-function $V:\mathbb{R}_+^{5n}\to\mathbb{R}_+$ and a bounded open set $\mathscr{U}\subset\Gamma^*\subset\mathbb{R}_+^{5n}$ such that $LV\le -1$ for any $(S_1,\dots,R_n)\in\Gamma^*\setminus\mathscr{U}$.

It is assumed that $1-p>0$, and the matrices $B_C = (\beta_{kj}^C)_{n\times n}$ and $B_A = (\beta_{kj}^C)_{n\times n}$ are irreducible. Therefore, the matrix R_0^s is nonnegative and irreducible. Thus according to Lemma 2, $\rho(M_0^s)$ is a simple eigenvalue and M_0^s has a positive eigenvector $\omega = (\omega_1,\dots,\omega_n)$ corresponding to $\rho(M_0^s)$ such that

$$(\omega_1,\dots,\omega_n)\rho(M_0^s) = (\omega_1,\dots,\omega_n)M_0^s. \tag{9}$$

Note that $S_k\ge 1$. Define a C^2-Lyapunov function $\tilde{V}:\Gamma^*\to\mathbb{R}$ as

$$\begin{aligned}\tilde{V} \;&=\; MV_1+V_2+V_3+V_5+V_6+V_7+V_8\\ &\quad+\frac{\max_{1\le k\le n}\{\mu_k+\alpha_k^C,\mu_k+\alpha_k^A\}+n\max_{1\le k,j\le n}\{\beta_{kj}^C,\ \beta_{kj}^A\}}{\min_{1\le k\le n}\{\gamma_k^C,\gamma_k^A\}}V_4,\end{aligned}$$

where $V_1 = -\ln\sum_{k=1}^n\frac{\omega_k}{\mu_k+\theta_k}E_k+R_1$, $V_2 = -\sum_{k=1}^n N_k$, $V_3 = -\sum_{k=1}^n\ln S_k$, $V_4 = -\sum_{k=1}^n R_k$ $V_5 = -\sum_{k=1}^n\ln E_k$, $V_6 = -\sum_{k=1}^n\ln I_k^C$, $V_7 = -\sum_{k=1}^n\ln I_k^A$, $V_8 = -\sum_{k=1}^n\ln R_k$, and $M>0$ is a sufficiently large number such that

$$-M\Bigg(\min_{1\le k\le n}\{\mu_k+\theta_k\}(R_0^s-1)+2n\frac{\min_{1\le k,j\le n}\left\{\frac{\omega_k\beta_{k,j}^C}{\mu_k+\theta_k},\ \frac{\omega_k\beta_{k,j}^A}{\mu_k+\theta_k}\right\}}{\max_{1\le k\le n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}}\sum_{k=1}^n\frac{\mu_k}{\Lambda_k}$$
$$-n\frac{\max_{1\le k,j\le n}\left\{\frac{\beta_{kj}^C\omega_k\Lambda_k p_k\theta_k}{\mathscr{C}},\ \frac{\beta_{kj}^A\omega_k\Lambda_k(1-p_k)\theta_k}{\mathscr{A}}\right\}}{\min_{1\le k\le n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}}+\frac{1}{2}\sum_{k=1}^n\lambda_k^2\Bigg)$$

$$+\frac{1}{2}\sum_{k=1}^{n}(\delta_k^2+\lambda_k^2+\xi_k^2+\nu_k^2)+\sum_{k=1}^{n}(5\mu_k+\alpha_k^C+\gamma_k^C+\alpha_k^A+\gamma_k^A+\theta_k)$$
$$+\left(\frac{\max_{1\le k\le n}\{\mu_k+\alpha_k^C,\mu_k+\alpha_k^A\}+n\max_{1\le k,j\le n}\{\beta_{kj}^C,\ \beta_{kj}^A\}}{\min_{1\le k\le n}\{\gamma_k^C,\gamma_k^A\}}+\frac{\max_{1\le k\le n}\{\mu_k\}}{\min_{1\le k\le n}\{\mu_k\}}-1\right)\times\sum_{1}^{n}\Lambda_k<-2. \tag{10}$$

Let $(\tilde{S}_1,\tilde{E}_1,\tilde{I}_1^C,\tilde{I}_1^A,\tilde{R}_1,\dots,\tilde{S}_n,\tilde{E}_n,\tilde{I}_n^C,\tilde{I}_n^A,\tilde{R}_n)$ be the lower bound of the function $\tilde{V}$. Define $V:\Gamma^*\to\mathbb{R}_+\cup\{0\}$ as

$$\begin{aligned}V(S_1,E_1,I_1^C,I_1^A,R_1,\dots,S_n,E_n,I_1^Cn,I_n^A,R_n)\ =\ &\tilde{V}(S_1,E_1,I_1^C,I_1^A,R_1,\dots,S_n,E_n,I_n^C,I_n^A,R_n)\\&-\tilde{V}(\tilde{S}_1,\tilde{E}_1,\tilde{I}_1^C,\tilde{I}_1^A,\tilde{R}_1,\dots,\tilde{S}_n,\tilde{E}_n,\tilde{I}_n^C,\tilde{I}_n^A,\tilde{R}_n).\end{aligned}$$

Applying Itô's formula to $V_1=-\ln\sum_{k=1}^n\frac{\omega_k}{\mu_k+\theta_k}E_k+R_1$, we have

$$\begin{aligned}
LV_1\ =\ &-\frac{1}{\sum_{k=1}^n\frac{\omega_k}{\mu_k+\theta_k}E_k}\sum_{k=1}^{n}\frac{\omega_k}{\mu_k+\theta_k}\left(\sum_{j=1}^{n}\beta_{kj}^CS_kI_j^C+\sum_{j=1}^{n}\beta_{kj}^AS_kI_j^A-(\mu_k+\theta_k)E_k\right)\\
&+\frac{\sum_{k=1}^n\frac{\omega_k^2}{(\mu_k+\theta_k)^2}}{2(\sum_{k=1}^n\frac{\omega_k}{\mu_k+\theta_k}E_k)^2}\times\lambda_k^2E_k^2+\left(\gamma_1^CI_1^C+\gamma_1^AI_1^A-\mu_kR_1\right)\\
=\ &-\frac{1}{\sum_{k=1}^n\frac{\omega_k}{\mu_k+\theta_k}E_k}\sum_{k=1}^{n}\frac{\omega_k}{\mu_k+\theta_k}\left(\sum_{j=1}^{n}\beta_{kj}^CS_kI_j^C+\sum_{j=1}^{n}\beta_{kj}^AS_kI_j^A-(\mu_k+\theta_k)E_k\right.\\
&+\left[\frac{\sum_{j=1}^k\beta_{kj}^C\Lambda_kp_k\theta_k(\mu_k+\theta_k)}{\mathscr{C}}E_k-\frac{\sum_{j=1}^k\beta_{kj}^C\Lambda_kp_k\theta_k(\mu_k+\theta_k)}{\mathscr{C}}E_k\right]\\
&\left.+\left[\frac{\sum_{j=1}^k\beta_{kj}^A\Lambda_k(1-p_k)\theta_k(\mu_k+\theta_k)}{\mathscr{A}}E_k-\frac{\sum_{j=1}^k\beta_{kj}^A\Lambda_k(1-p_k)\theta_k(\mu_k+\theta_k)}{\mathscr{A}}E_k\right]\right)\\
&+\frac{\sum_{k=1}^n\frac{\omega_k^2}{(\mu_k+\theta_k)^2}\lambda_k^2E_k^2}{2(\sum_{k=1}^n\frac{\omega_k}{\mu_k+\theta_k}E_k)^2}+\left(\gamma_1^CI_1^C+\gamma_1^AI_1^A-\mu_kR_1\right)\\
\le\ &-\frac{1}{\sum_{k=1}^n\frac{\omega_k}{\mu_k+\theta_k}E_k}\sum_{k=1}^{n}\left(-\omega_kE_k+\frac{\sum_{j=1}^k\beta_{kj}^C\omega_k\Lambda_kp_k\theta_k}{\mathscr{C}}E_k+\frac{\sum_{j=1}^n\beta_{kj}^A\omega_k\Lambda_k(1-p_k)\theta_k}{\mathscr{A}}E_k\right)+\frac{1}{2}\sum_{k=1}^{n}\lambda_k^2\\
&-\frac{1}{\sum_{k=1}^n\frac{\omega_k}{\mu_k+\theta_k}E_k}\sum_{k=1}^{n}\frac{\omega_k}{\mu_k+\theta_k}\left(\sum_{j=1}^{n}\beta_{kj}^CI_j^C+\sum_{j=1}^{n}\beta_{kj}^AI_j^A\right)+\frac{1}{\sum_{k=1}^n\frac{\omega_k}{\mu_k+\theta_k}E_k}\sum_{k=1}^{n}\frac{\omega_k}{\mu_k+\theta_k}\\
&\times\left(\frac{\sum_{j=1}^k\beta_{kj}^C\Lambda_kp_k\theta_k(\mu_k+\theta_k)}{\mathscr{C}}+\frac{\sum_{j=1}^k\beta_{kj}^A\Lambda_k(1-p_k)\theta_k(\mu_k+\theta_k)}{\mathscr{A}}\right)E_k+\left(\gamma_1^CI_1^C+\gamma_1^AI_1^A\right)\\
\le\ &-\frac{1}{\sum_{k=1}^n\frac{\omega_k}{\mu_k+\theta_k}E_k}(\omega M_0^sE-\omega E)+\frac{1}{2}\sum_{k=1}^{n}\lambda_k^2+\left(\gamma_1^CI_1^C+\gamma_1^AI_1^A\right)\\
&\sum_{k=1}^{n}\sum_{j=1}^{n}\left(-\frac{\min_{1\le k,j\le n}\left\{\frac{\omega_k\beta_{k,j}^C}{\mu_k+\theta_k},\ \frac{\omega_k\beta_{k,j}^A}{\mu_k+\theta_k}\right\}}{\max_{1\le k\le n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}\sum_{k=1}^nE_k}(I_j^C+I_j^A)+\frac{\max_{1\le k,j\le n}\left\{\frac{\beta_{kj}^C\omega_k\Lambda_kp_k\theta_k}{\mathscr{C}},\ \frac{\beta_{kj}^A\omega_k\Lambda_k(1-p_k)\theta_k}{\mathscr{A}}\right\}}{\min_{1\le k\le n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}\sum_{k=1}^nE_k}E_k\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{1}{\sum_{k=1}^{n}\frac{\omega_k}{\mu_k+\theta_k}E_k}((\rho(M_0^s)-1)\omega E)-\frac{\min_{1\leq k,j\leq n}\left\{\frac{\omega_k\beta^C_{k,j}}{\mu_k+\theta_k},\ \frac{\omega_k\beta^A_{k,j}}{\mu_k+\theta_k}\right\}}{\max_{1\leq k\leq n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}}\times n\times\frac{\sum_{j=1}^{n}(I_j^C+I_j^A)}{\sum_{k=1}^{n}E_k}\\
&\quad+\frac{\max_{1\leq k,j\leq n}\left\{\frac{\beta^C_{kj}\omega_k\Lambda_k p_k\theta_k}{\mathscr{C}},\ \frac{\beta^A_{kj}\omega_k\Lambda_k(1-p_k)\theta_k}{\mathscr{A}}\right\}}{\min_{1\leq k\leq n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}}\times n+\frac{1}{2}\sum_{k=1}^{n}\lambda_k^2+\left(\gamma_1^C I_1^C+\gamma_1^A I_1^A\right)\\
&\leq -\min_{1\leq k\leq n}\{\mu_k+\theta_k\}(R_0^s-1)+\frac{1}{2}\sum_{k=1}^{n}\lambda_k^2+\max\{\gamma_1^C,\gamma_1^A\}(I_1^C+I_1^A)\\
&\quad-2n\frac{\min_{1\leq k,j\leq n}\left\{\frac{\omega_k\beta^C_{k,j}}{\mu_k+\theta_k},\ \frac{\omega_k\beta^A_{k,j}}{\mu_k+\theta_k}\right\}}{\max_{1\leq k\leq n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}}\sum_{k=1}^{n}\frac{\mu_k}{\Lambda_k}+n\frac{\max_{1\leq k,j\leq n}\left\{\frac{\beta^C_{kj}\omega_k\Lambda_k p_k\theta_k}{\mathscr{C}},\ \frac{\beta^A_{kj}\omega_k\Lambda_k(1-p_k)\theta_k}{\mathscr{A}}\right\}}{\min_{1\leq k\leq n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}}.
\end{aligned}
$$

Recall that $V_2=-\sum_{k=1}^{n}N_k$. Let $\mu+\alpha=:\max_{1\leq k\leq n}\{\mu_k+\alpha_k^C,\mu_k+\alpha_k^A\}$. Then

$$
\begin{aligned}
LV_2 &= -\sum_{k=1}^{n}(\Lambda_k-\mu_k(S_k+E_k+I_k^C+I_k^A+R_k)-\alpha_k^C I_k^C-\alpha_k^A I_k^A)\\
&\leq -\sum_{k=1}^{n}\Lambda_k+\max_{1\leq k\leq n}\{\mu_k\}\sum_{k=1}^{n}(S_k+E_k+R_k)+(\mu+\alpha)\sum_{k=1}^{n}(I_k^C+I_k^A)\\
&\leq -\sum_{k=1}^{n}\Lambda_k+\max_{1\leq k\leq n}\{\mu_k\}\sum_{k=1}^{n}\frac{\Lambda_k}{\mu_k}+(\mu+\alpha)\sum_{k=1}^{n}(I_k^C+I_k^A).
\end{aligned}
$$

Let $\beta_{ij}=:\max_{1\leq k,j\leq n}\{\beta^C_{kj},\ \beta^A_{kj}\}$. By applying Itô's formula to $V_3=-\sum_{k=1}^{n}\ln S_k$, we have

$$
\begin{aligned}
LV_3 &= -\sum_{k=1}^{n}\frac{\Lambda_k}{S_k}+\sum_{k=1}^{n}\sum_{j=1}^{n}\beta^C_{kj}I_j^C+\sum_{k=1}^{n}\sum_{j=1}^{n}\beta^A_{kj}I_j^A+\sum_{k=1}^{n}\mu_k+\frac{1}{2}\sum_{k=1}^{n}\delta_k^2\\
&\leq -\sum_{k=1}^{n}\frac{\Lambda_k}{S_k}+\beta_{ij}\times n\times\sum_{j=1}^{n}(I_j^C+I_j^A)+\sum_{k=1}^{n}\mu_k+\frac{1}{2}\sum_{k=1}^{n}\delta_k^2.
\end{aligned}
$$

For $V_4=-\sum_{k=1}^{n}R_k$, we have

$$
LV_4 = -\sum_{k=1}^{n}\left(\gamma_k^C I_k^C+\gamma_k^A I_k^A-\mu_k R_k\right)\leq -\min_{1\leq k\leq n}\{\gamma_k^C,\gamma_k^A\}\sum_{k=1}^{n}(I_k^C+I_k^A)+\sum_{k=1}^{n}\mu_k R_k.
$$

Now

$$
\begin{aligned}
&L(V_2+V_3+\frac{\mu+\alpha+n\beta_{ij}}{\min_{1\leq k\leq n}\{\gamma_k^C,\gamma_k^A\}}V_4)\\
&\quad\leq -\sum_{k=1}^{n}\Lambda_k+\max_{1\leq k\leq n}\{\mu_k\}\sum_{k=1}^{n}\frac{\Lambda_k}{\mu_k}+(\mu+\alpha)\sum_{k=1}^{n}(I_k^C+I_k^A)-\sum_{k=1}^{n}\frac{\Lambda_k}{S_k}+n\beta_{ij}\sum_{j=1}^{n}(I_j^C+I_j^A)+\sum_{k=1}^{n}\mu_k\\
&\qquad+\frac{1}{2}\sum_{k=1}^{n}\delta_k^2-((\mu+\alpha)+n\beta_{ij})(I_j^C+I_j^A)+\frac{\mu+\alpha+n\beta_{ij}}{\min_{1\leq k\leq n}\{\gamma_k^C,\gamma_k^A\}}\sum_{k=1}^{n}\mu_k R_k
\end{aligned}
$$

$$\leq -\sum_{k=1}^{n}\Lambda_k + \max_{1\leq k\leq n}\{\mu_k\}\sum_{k=1}^{n}\frac{\Lambda_k}{\mu_k} - \sum_{k=1}^{n}\frac{\Lambda_k}{S_k} + \sum_{k=1}^{n}\mu_k + \frac{1}{2}\sum_{k=1}^{n}\delta_k^2 + \frac{\mu+\alpha+n\beta_{ij}}{\min_{1\leq k\leq n}\{\gamma_k^C,\gamma_k^A\}}\sum_{k=1}^{n}\Lambda_k.$$

For $V_5 = -\sum_{k=1}^{n}\ln E_k$, LV_5 is as follows:

$$\begin{aligned} LV_5 &= -\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\beta_{kj}^C S_k I_j^C}{E_k} - \sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\beta_{kj}^A S_k I_j^A}{E_k} + \sum_{k=1}^{n}(\mu_k+\theta_k) + \frac{1}{2}\sum_{k=1}^{n}\lambda_k^2 \\ &\leq -\min_{1\leq k,j\leq n}\{\beta_{kj}^C,\ \beta_{kj}^A\}\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{I_j^C+I_j^A}{E_k}S_k + \sum_{k=1}^{n}(\mu_k+\theta_k) + \frac{1}{2}\sum_{k=1}^{n}\lambda_k^2. \end{aligned}$$

For $V_6 = -\sum_{k=1}^{n}\ln I_k^C$, Itô's formula gives

$$\begin{aligned} LV_6 &= -\sum_{k=1}^{n}\frac{p_k\theta_k E_k}{I_k^C} + \sum_{k=1}^{n}(\mu_k+\alpha_k^C+\gamma_k^C) + \frac{1}{2}\sum_{k=1}^{n}\xi_k^2 \\ &\leq -\min_{1\leq k\leq n}\{p_k\theta_k\}\sum_{k=1}^{n}\frac{E_k}{I_k^C} + \sum_{k=1}^{n}(\mu_k+\alpha_k^C+\gamma_k^C) + \frac{1}{2}\sum_{k=1}^{n}\xi_k^2. \end{aligned}$$

If $V_7 = -\sum_{k=1}^{n}\ln I_k^A$, then

$$\begin{aligned} LV_7 &= -\sum_{k=1}^{n}\frac{(1-p_k)\theta_k E_k}{I_k^A} + \sum_{k=1}^{n}(\mu_k+\alpha_k^A+\gamma_k^A) + \frac{1}{2}\sum_{k=1}^{n}\nu_k^2 \\ &\leq -\min_{1\leq k\leq n}\{(1-p_k)\theta_k\}\sum_{k=1}^{n}\frac{E_k}{I_k^A} + \sum_{k=1}^{n}(\mu_k+\alpha_k^A+\gamma_k^A) + \frac{1}{2}\sum_{k=1}^{n}\nu_k^2. \end{aligned}$$

Recall that $V_8 = -\sum_{k=1}^{n}\ln R_k$. Then

$$LV_8 = -\sum_{k=1}^{n}\frac{\gamma_k^C I_k^C}{R_k} - \sum_{k=1}^{n}\frac{\gamma_k^A I_k^A}{R_k} + \sum_{k=1}^{n}\mu_k \leq -\min_{1\leq k\leq n}\{\gamma_k^C,\gamma_k^A\}\sum_{k=1}^{n}\frac{I_k^C+I_k^A}{R_k} + \sum_{k=1}^{n}\mu_k.$$

Therefore

$$\begin{aligned} &L(V) \\ &= L(MV_1+V_2+V_3+V_5+V_6+V_7+V_8+\frac{\mu+\alpha+n\beta_{ij}\}}{\min_{1\leq k\leq n}\{\gamma_k^C,\gamma_k^A\}}V_4) \\ &\leq -M\Bigg(\min_{1\leq k\leq n}\{\mu_k+\theta_k\}(R_0^s-1) + 2n\frac{\min_{1\leq k,j\leq n}\left\{\frac{\omega_k\beta_{k,j}^C}{\mu_k+\theta_k},\ \frac{\omega_k\beta_{k,j}^A}{\mu_k+\theta_k}\right\}}{\max_{1\leq k\leq n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}}\sum_{k=1}^{n}\frac{\mu_k}{\Lambda_k} \\ &\quad -n\frac{\max_{1\leq k,j\leq n}\left\{\frac{\beta_{kj}^C\omega_k\Lambda_k p_k\theta_k}{\mathscr{C}},\ \frac{\beta_{kj}^A\omega_k\Lambda_k(1-p_k)\theta_k}{\mathscr{A}}\right\}}{\min_{1\leq k\leq n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}} - \frac{1}{2}\sum_{k=1}^{n}\lambda_k^2\Bigg) + M\max\{\gamma_1^C,\gamma_1^A\}(I_1^C+I_1^A) \\ &\quad -\sum_{k=1}^{n}\Lambda_k + \max_{1\leq k\leq n}\{\mu_k\}\sum_{k=1}^{n}\frac{\Lambda_k}{\mu_k} - \sum_{k=1}^{n}\frac{\Lambda_k}{S_k} + \sum_{k=1}^{n}\mu_k + \frac{1}{2}\sum_{k=1}^{n}\delta_k^2 + \frac{\mu+\alpha+n\beta_{ij}\}}{\min_{1\leq k\leq n}\{\gamma_k^C,\gamma_k^A\}}\sum_{k=1}^{n}\Lambda_k \end{aligned}$$

$$
\begin{aligned}
&-\beta_{ij}\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{I_j^C+I_j^A}{E_k}S_k+\sum_{k=1}^{n}(\mu_k+\theta_k)+\frac{1}{2}\sum_{k=1}^{n}\lambda_k^2-\min_{1\le k\le n}\{p_k\theta_k\}\sum_{k=1}^{n}\frac{E_k}{I_k^C}+\sum_{k=1}^{n}(\mu_k+\alpha_k^C+\gamma_k^C)\\
&+\frac{1}{2}\sum_{k=1}^{n}\xi_k^2-\min_{1\le k\le n}\{(1-p_k)\theta_k\}\sum_{k=1}^{n}\frac{E_k}{I_k^A}+\sum_{k=1}^{n}(\mu_k+\alpha_k^A+\gamma_k^A)+\frac{1}{2}\sum_{k=1}^{n}\nu_k^2\\
&-\min_{1\le k\le n}\{\gamma_k^C,\gamma_k^A\}\sum_{k=1}^{n}\frac{I_k^C+I_k^A}{R_k}+\sum_{k=1}^{n}\mu_k.\\
\le\ &-M\left(\min_{1\le k\le n}\{\mu_k+\theta_k\}(R_0^s-1)+D\right)+M\max\{\gamma_1^C,\gamma_1^A\}(I_1^C+I_1^A)+G\sum_{k=1}^{n}\Lambda_k\\
&+\sum_{k=1}^{n}(5\mu_k+\alpha_k^C+\gamma_k^C+\alpha_k^A+\gamma_k^A+\theta_k)+\frac{1}{2}\sum_{k=1}^{n}(\delta_k^2+\lambda_k^2+\xi_k^2+\nu_k^2)\\
&-\sum_{k=1}^{n}\frac{\Lambda_k}{S_k}-\min_{1\le k,j\le n}\{\beta_{kj}^C,\ \beta_{kj}^A\}\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{I_j^C+I_j^A}{E_k}S_k-\min_{1\le k\le n}\{p_k\theta_k\}\sum_{k=1}^{n}\frac{E_k}{I_k^C}\\
&-\min_{1\le k\le n}\{(1-p_k)\theta_k\}\sum_{k=1}^{n}\frac{E_k}{I_k^A}-\min_{1\le k\le n}\{\gamma_k^C,\gamma_k^A\}\sum_{k=1}^{n}\frac{I_k^C+I_k^A}{R_k},
\end{aligned}
$$

where

$$
D=2n\frac{\min_{1\le k,j\le n}\left\{\frac{\omega_k\beta_{k,j}^C}{\mu_k+\theta_k},\ \frac{\omega_k\beta_{k,j}^A}{\mu_k+\theta_k}\right\}}{\max_{1\le k\le n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}}\sum_{k=1}^{n}\frac{\mu_k}{\Lambda_k}+\frac{1}{2}\sum_{k=1}^{n}\lambda_k^2-n\frac{\max_{1\le k,j\le n}\left\{\frac{\beta_{kj}^C\omega_k\Lambda_k p_k\theta_k}{\mathscr{C}},\ \frac{\beta_{kj}^A\omega_k\Lambda_k(1-p_k)\theta_k}{\mathscr{A}}\right\}}{\min_{1\le k\le n}\left\{\frac{\omega_k}{\mu_k+\theta_k}\right\}},
$$

$$
G\ =\ \frac{\mu+\alpha+n\beta_{ij}}{\min_{1\le k\le n}\{\gamma_k^C,\gamma_k^A\}}+\frac{\max_{1\le k\le n}\{\mu_k\}}{\min_{1\le k\le n}\{\mu_k\}}-1.
$$

For small $\varepsilon>0$ and $1\le j\le n$, define a set as follows

$$
\begin{aligned}
\mathscr{U}_\varepsilon\ =\ &\left\{(S_1,E_1,I_1^C,I_1^A,R_1,\ldots,S_n,E_n,I_n^C,I_n^A,R_n)\in\Gamma^*:\ S_j>\varepsilon,S_j<\varepsilon^2,E_j>\varepsilon^5,E_j<\varepsilon^6,\right.\\
&\left.I_j^C>\varepsilon,I_j^C<\varepsilon^2,I_j^A>\varepsilon^2,I_j^A<\varepsilon^2R_j>\varepsilon^2,R_j<\varepsilon^3\right\}.
\end{aligned}
$$

Take $p=\frac{1}{2}\sum_{k=1}^{n}(\delta_k^2+\lambda_k^2+\xi_k^2+\nu_k^2)+G\sum_1^n\Lambda_k+\sum_{k=1}^{n}(5\mu_k+\alpha_k^C+\gamma_k^C+\alpha_k^A+\gamma_k^A+\theta_k)$. Choose ε be such that the following conditions hold in $\Gamma^*\setminus\mathscr{U}_\varepsilon$:

i) $2M\max\{\gamma_1^C,\gamma_1^A\}\varepsilon\le 1$,

ii) $-\frac{\Lambda_1}{\varepsilon}+\frac{2M\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\le -1$,

iii) $-\sum_{j=2}^{n}\frac{\Lambda_j}{\varepsilon}+\frac{2M\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\le -1$,

iv) $-\sum_{k=1}^{n}\min_{1\le k\le n}\{\beta_{k1}^C,\ \beta_{k1}^A\}\frac{2}{\varepsilon}+\frac{2M\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\le -1$,

v) $-\sum_{k=1}^{n}\sum_{j=2}^{n}\min_{1\le k,j\le n}\{\beta_{kj}^C,\ \beta_{kj}^A\}\frac{2}{\varepsilon}+\frac{2M\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\le -1$,

vi) $-\frac{p_1\theta_1}{\varepsilon}+2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P\leq -1$,

vii) $-\sum_{j=2}^n \min_{1\leq j\leq n}\{p_j\theta_j\}\frac{1}{\varepsilon}+2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P\leq -1$,

viii) $-\{(1-p_1)\theta_1\}\frac{1}{\varepsilon}+2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P\leq -1$,

ix) $-\sum_{j=2}^n \min_{1\leq j\leq n}\{(1-p_j)\theta_j\}\frac{1}{\varepsilon}+2M\max\{\gamma_1^C,\gamma_1^A\}+P\leq -1$,

x) $-\min\{\gamma_1^C,\gamma_1^A\}\frac{2}{\varepsilon}+\frac{2M\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\leq -1$,

xi) $-\sum_{j=2}^n \min_{1\leq j\leq n}\{\gamma_j^C,\gamma_j^A\}\frac{2}{\varepsilon}+\frac{2M\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\leq -1$,

Divide $\Gamma^*\setminus\mathscr{U}_\varepsilon$ into the following subsets:

- $\mathscr{U}_1=\{(S_1,\dots,R_n)\in\Gamma^*: I_1^C\leq\varepsilon, I_1^A\leq\varepsilon\}$,
- $\mathscr{U}_2=\{(S_1,\dots,R_n)\in\Gamma^*: S_1\leq\varepsilon\}$,
- $\mathscr{U}_3=\{(S_1,\dots,R_n)\in\Gamma^*: S_j\leq\varepsilon,\ j=2,\dots,n\}$,
- $\mathscr{U}_4=\{(S_1,\dots,R_n)\in\Gamma^*: S_1\geq\varepsilon^2, E_1\leq\varepsilon^5, I_1^C\geq\varepsilon^2, I_1^A\geq\varepsilon^2, k=1,\dots,n\}$,
- $\mathscr{U}_5=\{(S_1,\dots,R_n)\in\Gamma^*: S_k\geq\varepsilon^2, E_k\leq\varepsilon^5, I_j^C\geq\varepsilon^2, I_j^A\geq\varepsilon^2,\ j=2,\dots,n,\ k=1,\dots,n\}$,
- $\mathscr{U}_6=\{(S_1,\dots,R_n)\in\Gamma^*: I_1^C\leq\varepsilon, I_1^A\leq\varepsilon\}$,
- $\mathscr{U}_7=\{(S_1,\dots,R_n)\in\Gamma^*: I_j^C\leq\varepsilon, I_j^A\leq\varepsilon,\ j=2,\dots,n\}$,
- $\mathscr{U}_8=\{(S_1,\dots,R_n)\in\Gamma^*: I_1^C\geq\varepsilon, I_1^A\geq\varepsilon, R_1\leq\varepsilon^2\}$,
- $\mathscr{U}_9=\{(S_1,\dots,R_n)\in\Gamma^*: I_j^C\geq\varepsilon, I_j^A\geq\varepsilon, R_j\leq\varepsilon^2,\ j=2,\dots,n\}$,

Now we determine the sign of LV on each of these subsets.

Case 1. For $\mathscr{U}_1$, condition (i) and (10), gives

$$\begin{aligned}LV &\leq -M\left(\min_{1\leq k\leq n}\{\mu_k+\theta_k\}(R_0^s-1)+D\right)+P+M\max\{\gamma_1^C,\gamma_1^A\}(I_1^C+I_1^A)\\ &\leq -M\left(\min_{1\leq k\leq n}\{\mu_k+\theta_k\}(R_0^s-1)+D\right)+P+2M\max\{\gamma_1^C,\gamma_1^A\}\varepsilon\leq -2+1\leq -1.\end{aligned}$$

Case 2. Using condition (ii) for points in $\mathscr{U}_2$, gives

$$LV \leq -\frac{\Lambda_1}{S_1}+2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\leq -\frac{\Lambda_1}{\varepsilon}+2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\leq -1.$$

Case 3. For points in $\mathscr{U}_3$, condition (iii) gives

$$LV \leq -\sum_{j=2}^n\frac{\Lambda_k}{S_k}+2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\leq \frac{-\sum_{j=2}^n\Lambda_j}{\varepsilon}+2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\leq -1.$$

Case 4. Applying condition (iv) for points in $\mathscr{U}_4$, gives

$$\begin{aligned} LV &\leq -\sum_{k=1}^{n} \min_{1\leq k\leq n}\{\beta_{k1}^C,\ \beta_{k1}^A\}\frac{I_1^C+I_1^A}{E_k}S_k + 2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P \\ &\leq -\sum_{k=1}^{n} \min_{1\leq k\leq n}\{\beta_{k1}^C,\ \beta_{k1}^A\}\frac{2}{\varepsilon} + 2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P \leq -1. \end{aligned}$$

Case 5. For region $\mathscr{U}_5$, condition (v) gives

$$\begin{aligned} LV &\leq -\sum_{k=1}^{n}\sum_{j=2}^{n} \min_{1\leq k,j\leq n}\{\beta_{kj}^C,\ \beta_{kj}^A\}\frac{I_j^C+I_j^A}{E_k}S_k + 2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P \\ &\leq -\sum_{k=1}^{n}\sum_{j=2} \min_{1\leq k,j\leq n}\{\beta_{kj}^C,\ \beta_{kj}^A\}\frac{2}{\varepsilon} + 2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P \leq -1. \end{aligned}$$

Case 6. For region $\mathscr{U}_6$, from condition (vi) and the fact that $E_1 \geq 1$, we have

$$LV \leq -p_1\theta_1\frac{E_1}{I_1^C} + 2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P \leq -\frac{p_1\theta_1}{\varepsilon} + 2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P \leq -1.$$

Case 7. For region $\mathscr{U}_7$, condition (vii) and the fact that $E_k \geq 1$, gives

$$\begin{aligned} LV &\leq -\min_{1\leq k\leq n}\{p_k\theta_k\}\sum_{k=2}^{n}\frac{E_k}{I_k^C} + 2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P \\ &\leq -\sum_{k=2}^{n}\min_{1\leq k\leq n}\{p_k\theta_k\}\frac{1}{\varepsilon} + 2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P \leq -1. \end{aligned}$$

Case 8. For points in $\mathscr{U}_6$, from condition (viii) we have

$$LV \leq -(1-p_1)\theta_1\frac{E_1}{I_1^A} + 2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P \leq -\frac{(1-p_1)\theta_1}{\varepsilon} + 2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P \leq -1.$$

Case 9. Applying (ix) on points in $\mathscr{U}_7$, we have

$$\begin{aligned} LV &\leq -\min_{2\leq k\leq n}\{(1-p_k)\theta_k\}\sum_{k=1}^{n}\frac{E_k}{I_k^A} + 2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P \\ &\leq -\sum_{k=2}^{n}\min_{2\leq k\leq n}\{(1-p_k)\theta_k\}\frac{1}{\varepsilon} + 2M\varepsilon\max\{\gamma_1^C,\gamma_1^A\}+P \leq -1. \end{aligned}$$

Case 10. The condition (x) on the set $\mathscr{U}_8$ gives

$$\begin{aligned} LV &\leq -\min\{\gamma_1^C,\gamma_1^A\}\frac{I_1^C+I_1^A}{R_1} + 2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P \\ &\leq -\min\{\gamma_1^C,\gamma_1^A\}\frac{2}{\varepsilon} + 2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P \leq -1. \end{aligned}$$

Case 11. Condition (ix) on $\mathscr{U}_9$ yields

$$\begin{aligned} LV &\leq -\sum_{k=2}^{n} \min_{2\leq k\leq n}\{\gamma_k^C,\gamma_k^A\}\frac{I_k^C+I_k^A}{R_k}+2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P \\ &\leq -\sum_{k=2}^{n} \min_{2\leq k\leq n}\{\gamma_k^C,\gamma_k^A\}\frac{2}{\varepsilon}+2M\frac{\Lambda_1}{\mu_1}\max\{\gamma_1^C,\gamma_1^A\}+P\leq -1. \end{aligned}$$

Thus, for any point $(S_1,\ldots,R_n)\in\Gamma^*\setminus\mathscr{U}_\varepsilon$, one gets $LV\leq -1$. □

4 Extinction of the disease

In this section, we give sufficient criteria which proves extinction of the disease.

Theorem 2. *Let $B=(\beta_{kj})_{n\times n}$ be irreducible and $(S_1(t),I_1(t),\ldots,R_n(t))$ be the solution of system (1) with an initial condition in Γ^*. Set*

$$R_0^e=\max_k\left\{\frac{4\left(\sum_{j=1}^n\beta_{kj}^C+\sum_{j=1}^n\beta_{kj}^A\right)\Lambda_k-4\mu_k^2}{\mu_k(\lambda_k^2\wedge\xi_k^2\wedge\nu_k^2)}\right\}.$$

If $R_0^e<1$, then both I_k and Q_k, $k=1,\ldots,n$ extinct with probability one. It means for $k=1,\cdots,n$,

$$\lim_{t\to\infty}E_k=0,\ \ \lim_{t\to\infty}I_k^C=0,\ \ \lim_{t\to\infty}I_k^A=0\ a.s.$$

Proof. Consider the first equation of model (2). Integrating both sides from 0 to t and deviding by t, we have

$$\frac{S_k(t)-S_k(0)}{t}=\Lambda_k-\sum_{j=1}^n\beta_{kj}^C\frac{\int_0^t S_kI_j^Cdt}{t}-\sum_{j=1}^n\beta_{kj}^A\frac{\int_0^t S_kI_j^Adt}{t}-\mu_k\frac{\int_0^t S_kdt}{t}+\delta_k\frac{\int_0^t S_kdB_k(s)}{t}.$$

Therefore

$$\frac{\int_0^t S_kdt}{t}=\frac{1}{\mu_k}\left(\Lambda_k-\frac{S_k(t)-S_k(0)}{t}-\sum_{j=1}^n\beta_{kj}^C\frac{\int_0^t S_kI_j^Cdt}{t}-\sum_{j=1}^n\beta_{kj}^A\frac{\int_0^t S_kI_j^Adt}{t}+\delta_k\frac{\int_0^t S_kdB_k(s)}{t}\right).$$

Let $\frac{\int_0^t S_kdt}{t}=<S_k>$. Then we have

$$\begin{aligned} \lim_{t\to\infty}<S_k> &= \lim_{t\to\infty}\left(\frac{\Lambda_k}{\mu_k}-\sum_{j=1}^n\frac{\beta_{kj}^C}{\mu_k}<S_kI_j^Cdt>-\sum_{j=1}^n\frac{\beta_{kj}^A}{\mu_k}<S_kI_j^Adt>+\frac{\delta_k}{\mu_k}\frac{\int_0^t S_kdB_k(s)}{t}\right) \\ &\leq \lim_{t\to\infty}\left(\frac{\Lambda_k}{\mu_k}+\frac{\delta_k}{\mu_k}\frac{\int_0^t S_kdB_k(s)}{t}\right). \end{aligned}$$

Take

$$\mathscr{M}_1=\int_0^t S_kdB_k(s),$$

and applying the strong law of large numbers for local martingales [15], we obtain $\lim_{t\to\infty}\frac{\mathscr{M}_1}{t}=0$ and $\lim_{t\to\infty}<S_k>\le\frac{\Lambda_k}{\mu_k}$ a.s. Let $Z_k(t)=E_k(t)+I_k^C(t)+I_k^A(t)$. Then

$$\begin{aligned}
d\ln Z_k &= \frac{1}{E_k+I_k^C+I_k^A}\left(\sum_{j=1}^n \beta_{kj}^C S_k I_j^C+\sum_{j=1}^n \beta_{kj}^A S_k I_j^A-\mu_k(E_k+I_j^C+I_j^A)\right.\\
&\quad\left.-(\alpha_k^C+\gamma_k^C)I_k^C-(\alpha_k^A+\gamma_k^A)I_k^A-\frac{\lambda_k^2E_k^2+\xi_k^2(I_k^C)^2+\nu_k^2(I_k^A)^2}{2(E_k+I_k^C+I_k^A)}\right)dt\\
&\quad+\frac{\lambda_kE_k}{E_k+I_k^C+I_k^A}dB_k+\frac{\xi_kI_k^C}{E_k+I_k^C+I_k^A}dB_k+\frac{\nu_kI_k^A}{E_k+I_k^C+I_k^A}dB_k\\
&\le \left(\sum_{j=1}^n \beta_{kj}^C S_k+\sum_{j=1}^n \beta_{kj}^A S_k-\mu_k-\frac{(\lambda_k^2\wedge\xi_k^2\wedge\nu_k^2)(E_k^2+(I_k^C)^2+(I_k^A)^2)}{4(E_k^2+(I_k^C)^2+(I_k^A)^2)}\right)dt\\
&\quad+\frac{\lambda_kE_k}{E_k+I_k^C+I_k^A}dB_k+\frac{\xi_kI_k^C}{E_k+I_k^C+I_k^A}dB_k+\frac{\nu_kI_k^A}{E_k+I_k^C+I_k^A}dB_k\\
&= \left(\sum_{j=1}^n \beta_{kj}^C S_k+\sum_{j=1}^n \beta_{kj}^A S_k-\mu_k-\frac{(\lambda_k^2\wedge\xi_k^2\wedge\nu_k^2)}{4}\right)dt\\
&\quad+\frac{\lambda_kE_k}{E_k+I_k^C+I_k^A}dB_k+\frac{\xi_kI_k^C}{E_k+I_k^C+I_k^A}dB_k+\frac{\nu_kI_k^A}{E_k+I_k^C+I_k^A}dB_k.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\ln Z(t)-\ln Z(0)}{t} &\le \left(\frac{\sum_{j=1}^n\beta_{kj}^C+\sum_{j=1}^n\beta_{kj}^A}{t}\int_0^t S_k(s)ds-\mu_k-\frac{(\lambda_k^2\wedge\xi_k^2\wedge\nu_k^2)}{4}\right)dt\\
&\quad+\frac{\lambda_k}{t}\int_0^t\frac{E_k}{E_k+I_k^C+I_k^A}dB_k(s)+\frac{\xi_k}{t}\int_0^t\frac{I_k^C}{E_k+I_k^C+I_k^A}dB_k(s)\\
&\quad+\frac{\nu_k}{t}\int_0^t\frac{I_k^A}{E_k+I_k^C+I_k^A}dB_k(s).
\end{aligned}$$

Again by applying the strong law of large numbers for local martingales,

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\ln Z(t)}{t} &\le \left(\sum_{j=1}^n\beta_{kj}^C+\sum_{j=1}^n\beta_{kj}^A\right)\frac{\Lambda_k}{\mu_k}-\mu_k-\frac{(\lambda_k^2\wedge\xi_k^2\wedge\nu_k^2)}{4}\\
&= \frac{(\lambda_k^2\wedge\xi_k^2\wedge\nu_k^2)}{4}\left(\frac{4\left(\sum_{j=1}^n\beta_{kj}^C+\sum_{j=1}^n\beta_{kj}^A\right)\Lambda_k-4\mu_k^2}{\mu_k(\lambda_k^2\wedge\xi_k^2\wedge\nu_k^2)}-1\right)\\
&= \frac{(\lambda_k^2\wedge\xi_k^2\wedge\nu_k^2)}{4}(R_0^e-1)<0\ a.s.,
\end{aligned}$$

where $R_0^e=\max_k\left\{\frac{4\left(\sum_{j=1}^n\beta_{kj}^C+\sum_{j=1}^n\beta_{kj}^A\right)\Lambda_k-4\mu_k^2}{\mu_k(\lambda_k^2\wedge\xi_k^2\wedge\nu_k^2)}\right\}$. Therefore

$$\lim_{t\to\infty}E_k=0,\ \ \lim_{t\to\infty}I_k^C=0,\ \ \lim_{t\to\infty}I_k^A=0.$$

□

Table 1: List of parameters

Parameter	Units	Range	References
Λ_1,Λ_2	day^{-1}	0.05812	www.statista.com
μ_1,μ_2	day^{-1}	$[\frac{83}{3650000},\frac{1}{80.3\times 365}]$	[11]
$\beta_C^{11}=\beta_C^{12}$	day^{-1}	0.4417	[18]
$\beta_C^{21}=\beta_C^{22}$	day^{-1}	0.6532	[18]
$\beta_A^{11}=\beta_A^{12}$	day^{-1}	0.3533	[18]
$\beta_A^{21}=\beta_A^{22}$	day^{-1}	0.2650×0.75	[18]
$p_1=p_2$	day^{-1}	0.5	Assumed
$\theta_1=\theta_2$	day^{-1}	0.5	[10]
$\alpha_{1C}=\alpha_{2C}$	day^{-1}	0.05	Estimated
$\alpha_{1A}=\alpha_{2A}$	day^{-1}	0.04	Estimated
$\gamma_{1C}=\gamma_{2C}$	day^{-1}	0.24	Assumed
$\gamma_{2C}=\gamma_{2C}$	day^{-1}	$\frac{1}{10.5}$	[17]

5 Numerical examples

In this section, we present numerical simulations to validate our theoretical findings. We implement the stochastic system (2) for the case of two subpopulations ($n=2$) using Milstein's higher-order method [6], which yields the following discretization scheme:

$$\begin{cases} S_1^{k+1} = S_1^k + \left(\Lambda_1 - S_1^k \sum_{j=1}^2 (\beta_C^{1j} I_C^j + \beta_A^{1j} I_A^j) - \mu_1 S_1^k\right)\Delta t + S_1^k f(\delta_1,1),\\ S_2^{k+1} = S_2^k + \left(\Lambda_2 - S_2^k \sum_{j=1}^2 (\beta_C^{2j} I_C^j + \beta_A^{2j} I_A^j) - \mu_2 S_2^k\right)\Delta t + S_2^k f(\delta_2,2),\\ E_1^{k+1} = E_1^k + \left(S_1^k \sum_{j=1}^2 (\beta_C^{1j} I_C^j + \beta_A^{1j} I_A^j) - (\mu_1+\theta_1)E_1^k\right)\Delta t + E_1^k f(\lambda_1,3),\\ E_2^{k+1} = E_2^k + \left(S_2^k \sum_{j=1}^2 (\beta_C^{2j} I_C^j + \beta_A^{2j} I_A^j) - (\mu_2+\theta_2)E_2^k\right)\Delta t + E_2^k f(\lambda_2,4),\\ I_{1C}^{k+1} = I_{1C}^k + \left[p_1\theta E_1^k - (\mu_1+\alpha_{1C}+\gamma_{1C})I_{1C}^k\right]\Delta t + I_{1C}^k f(\xi_1,5),\\ I_{2C}^{k+1} = I_{2C}^k + \left[p_2\theta E_2^k - (\mu_2+\alpha_{2C}+\gamma_{2C})I_{2C}^k\right]\Delta t + I_{2C}^k f(\xi_2,6),\\ I_{1A}^{k+1} = I_{1A}^k + \left[(1-p_1)\theta E_1^k - (\mu_1+\alpha_{1A}+\gamma_{1A})I_{1A}^k\right]\Delta t + I_{1A}^k f(\nu_1,7),\\ I_{2A}^{k+1} = I_{2A}^k + \left[(1-p_2)\theta E_2^k - (\mu_2+\alpha_{2A}+\gamma_{2A})I_{2A}^k\right]\Delta t + I_{2A}^k f(\nu_2,8), \end{cases}$$

where $f(x,m) =: (x\sqrt{\Delta t}\varpi_{m,k} + \frac{x^2}{2}(\varpi_{m,k}^2 - 1)\Delta t)$, and the time increment $\Delta t > 0$, δ_i, λ_i, ξ_i and ν_i are mutually independent Gaussian random variables whith the distribution $\mathcal{N}(0,1)$ for $k,i=1,2$. The parameters are read from Table 1.

Example 1. In this example, we confirm the existence of an ergodic stationary distribution numerically under the conditions of Theorem 1. The quasi-endemic equilibrium is the point

$$\begin{aligned} P^* &= (S_1^*, E_1^*, I_1^{A*}, I_1^{C*}, S_1^*, E_1^*, I_1^{A*}, I_1^{C*})\\ &= (0.2669, 0.11622, 0.10018, 0.21481, 0.2669, 0.11622, 0.10018, 0.21481). \end{aligned}$$

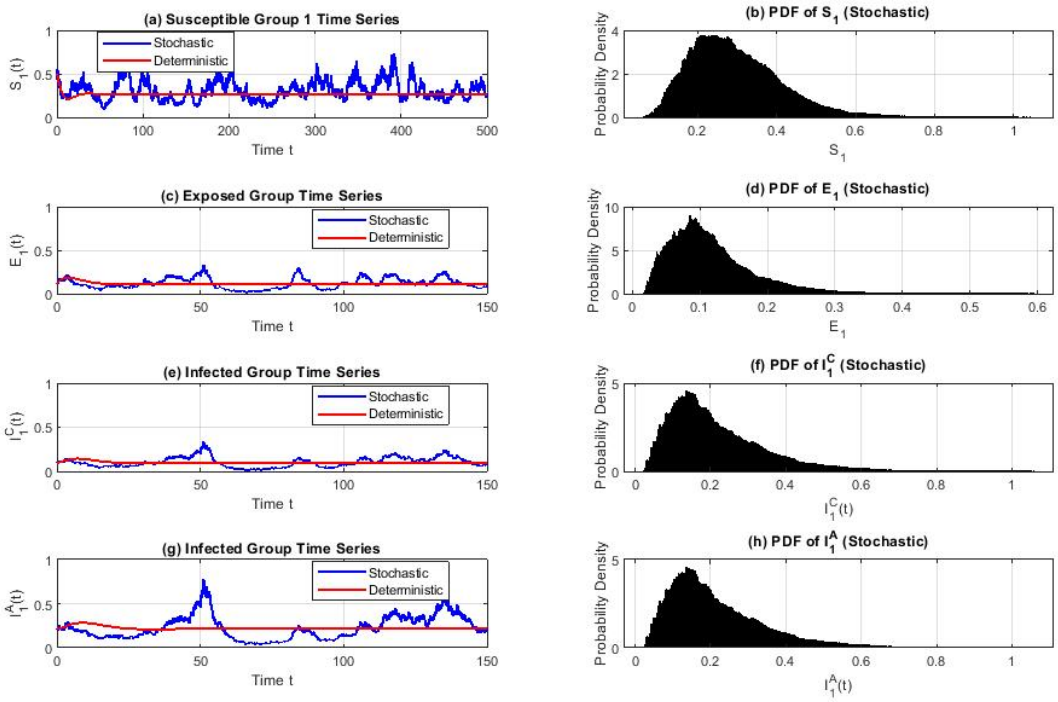

Figure 1: The simulation of solution (S_1, E_1, I_1^A, I_1^C) for deterministic and stochastic systems (1) and (2) when $R_0^s = 18.66060438 > 1$.

By taking $\delta_i = \lambda_i = \xi_i = \nu_i = 0.15$, we obtain

$$R_0^s = 18.66060438 > 1.$$

A direct calculation shows that (7) holds where the eigenvector is $\omega = (\omega_1, \omega_2) = (0.6627, 0.7504)$. Therefore, by Theorem 1, system (2) has a unique ergodic stationary distribution which shows that the disease is persistent in the mean. This fact is depicted in Figures 1 and 2.

Example 2. For $i, j \in \{1, 2\}$, we consider the following parameter changes in system (2):

$$\mu_i = 0.002\ \Lambda_i = 0.01,\ \delta_i = \lambda_i = \xi_i = \nu_i = 0.9,\ \beta_C^{ij} =: 0.01,\ \beta_A^{ij} =: 0.005.$$

Then

$$R_0^e = \max_k \left\{ \frac{4\left(\sum_{j=1}^n \beta_{kj}^C + \sum_{j=1}^n \beta_{kj}^A\right)\Lambda_k - 4\mu_k^2}{\mu_k(\lambda_k^2 \wedge \xi_k^2 \wedge \nu_k^2)} \right\} = 0.7308 < 1.$$

Theorem 2 guarantees that when $R_0^e < 1$, the disease will extinct exponentially which is illustrated in Figure 3.

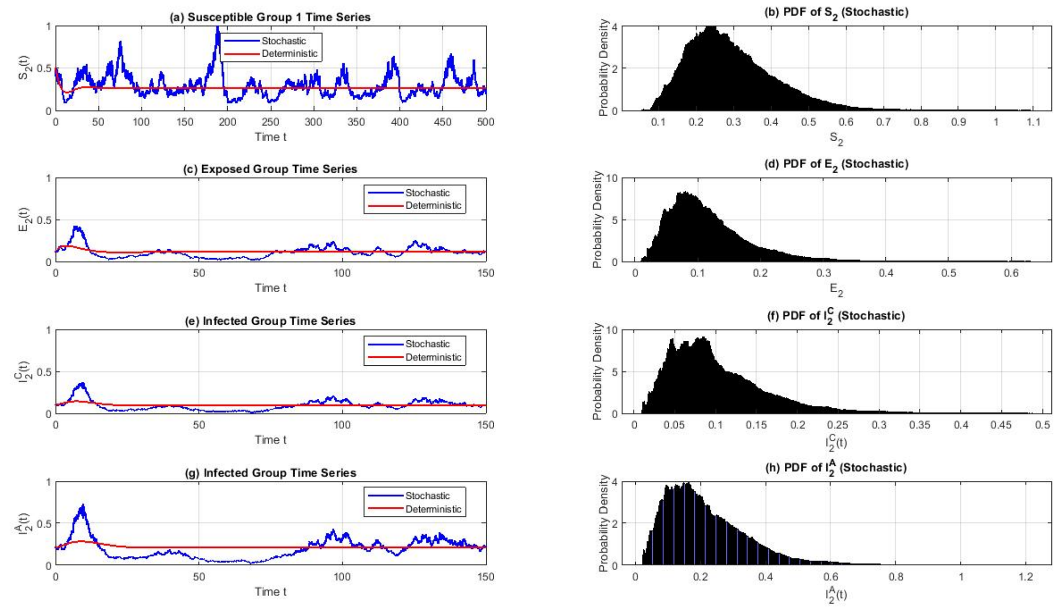

Figure 2: The simulation of solution (S_2, E_2, I_2^A, I_2^C) for deterministic and stochastic systems (1) and (2) when $R_0^s = 18.66060438 > 1$.

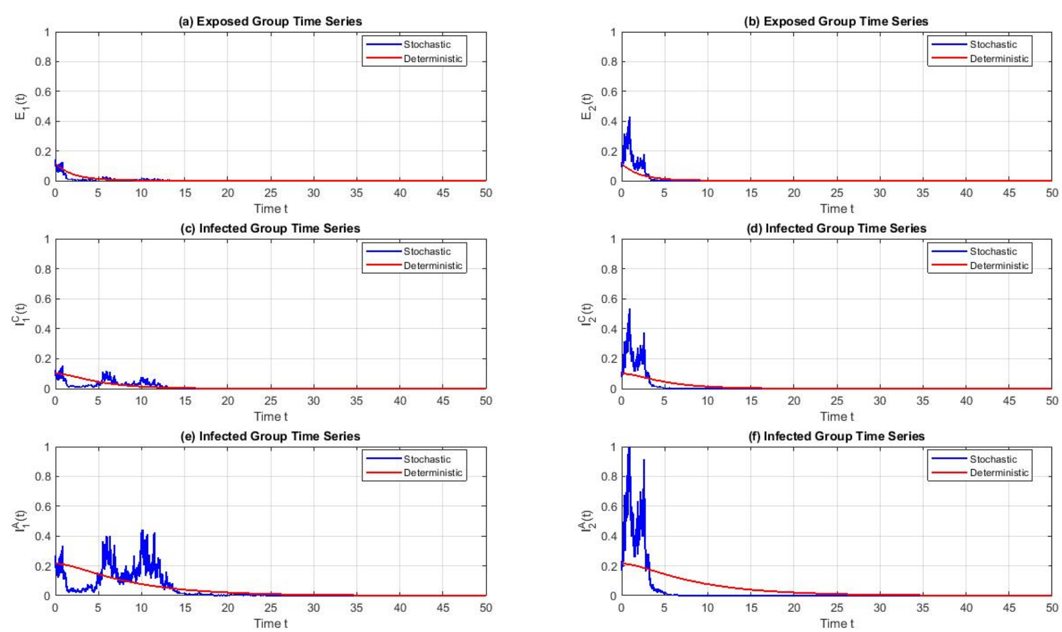

Figure 3: When $R_0^e < 1$, the disease will extinct exponentially.

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