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Generalized distributed-order fractional optimal control problem using Laguerre wavelet method

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Abstract. This research presents a novel numerical method for solving a class of complex optimal control problems characterized by distributed-order derivatives. By effectively employing fractional-order Laguerre wavelets, the study transforms the original continuous-time problem into a discrete set of algebraic equations. This transformation is facilitated by the use of Reimann-Liouville distributed-order operational matrices and a carefully chosen set of Newton-Cotes collocation points. The optimized solution is then determined by applying the Lagrange multiplier method to solve the resulting system of equations. The paper rigorously investigates the convergence properties of this approach, establishing error bounds that provide a measure of its accuracy. Finally, the effectiveness of this method is demonstrated through a series of illustrative examples, showcasing its high precision and applicability to a wide range of generalized distributed-order optimal control problems.

Keywords: Distributed-order, optimal control problem, Laguerre, wavelets, operational matrix.
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1 Introduction

The calculus of variations, dating back to the 17th century, focuses on determining functions that optimize integrals. This fundamental concept evolved into optimal control theory, which incorporates control variables into state equations, enabling the optimization of dynamic systems. This groundbreaking approach was pioneered by Isaacs in the 1950s and further refined by Bellman through the introduction of dynamic programming. Optimal control problems (OCPs) provide a rigorous framework for finding the most effective control strategies for dynamic systems. A fractional order model outperforms an integer model in complex systems with memory effects. Fractional models are typically derived by substituting

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the classical derivative with a non-integer order derivative. Consequently, numerous researchers have focused on analyzing fractional partial differential equations [1–3]. By optimizing a performance criterion while adhering to system dynamics and constraints, OCPs find widespread application in diverse fields, including technological advancements, economics, and scientific research [4]. Researchers can effectively develop efficient control strategies for real-world scenarios by combining analytical and numerical methods to tackle the inherent challenges of solving these complex problems [5, 6]. A distributed-order fractional OCP (DO-FOCP) is a specialized type of OCP that deals with dynamic systems governed by distributed-order fractional derivatives. Unlike traditional fractional-derivatives with a fixed non-integer order, distributed-order fractional-derivatives allow the order of differentiation to vary over a range of values [7]. This characteristic enables more flexible and accurate modeling of complex processes exhibiting multi-scale dynamics or memory effects, as the system's behavior is influenced by a spectrum of fractional derivative orders. Fractional wavelets, derived from multi-resolution analysis, provide powerful tools for signal and image analysis, facilitating efficient time-frequency localization. Their applications extend across natural sciences and engineering, underscoring their significance in modern data analysis [8–10]. Analytical solutions to distributed-order fractional differential equations often appear in complex forms [11–13]. This necessitates the development of efficient numerical methods to address these equations and manage the computational complexity associated with analytical solutions. The study of generalized distributed-order fractional OCPs (GDFOCPs) represents an advanced frontier in fractional calculus. Continued research and development in this area hold the promise of significantly enhancing the effectiveness and applicability of optimal control strategies across various scientific and engineering domains. The basic motivation of this paper is to develop a fractional order Laguerre wavelets method to solve OCPs, Laguerre wavelets utilize Laguerre polynomials as their basis functions. They provide strong interpolation properties and achieve higher accuracy with fewer collocation points. Applications of Laguerre wavelets in numerical approximations can be found in reference [14]. It is observed that the proposed method is fully compatible with the complexity of such problems and is very user-friendly. The error estimates provide clear indications of the impressive accuracy level achieved by the proposed technique, suggesting its strong potential for reliable application.

1.1 Problem statement

Consider GDFOCPs

$$\min \mathfrak{J}(x,u) = \int_0^1 \mathscr{F}\left(x_1(\xi), x_2(\xi), \ldots, x_{s_1}(\xi), u_1(\xi), u_2(\xi), \ldots, u_{s_2}(\xi)\right) d\xi, \tag{1}$$

subject to

$$\begin{aligned} \mathscr{K}_i\left({}_0^C D_\xi^{\rho(\alpha)} x_1(\xi), {}_0^C D_\xi^{\rho(\alpha)} x_2(\xi), \ldots, {}_0^C D_\xi^{\rho(\alpha)} x_{s_1}(\xi)\right) = \mathscr{H}_i(\xi, x_1(\xi), x_2(\xi), \ldots, \\ x_{s_1}(\xi), u_1(\xi), u_2(\xi), \ldots, u_{s_2}(\xi)) \end{aligned} \tag{2}$$

$$x_1(0) = x_{10}, x_2(0) = x_{20}, \ldots, x_{s_1}(0) = x_{s_1 0}, \tag{3}$$

where $1 \leq i \leq s_1$, $s_1, s_2 \in \mathbb{N}$, x and u, respectively, are the state and control variables and ${}_0^C D_\xi^{\rho(\alpha)}$ is the distributed-order fractional derivative with respect to ξ. $\mathscr{F}$ and $\mathscr{H}$ are supposed to be continuously differentiable functions and the variable ξ represents time.

2 Preliminaries

In this section, some definitions for distributed-order $\rho(\alpha)$ are given, where $\rho(0)=0$ and $0<\int_0^1 \rho(\alpha)d\alpha<\infty$.

Definition 1 ([7]). *The Caputo derivative of function $f(\xi)$ of the distributed-order $\rho(\alpha)$ is given by*

$$ {}_0^C D_\xi^{\rho(\alpha)} f(\xi) = \int_0^1 \rho(\alpha)\ {}_0^C D_\xi^{\alpha} f(\xi) d\alpha, \tag{4}$$

where

$$ {}_0^C D_\xi^{\alpha} f(\xi) = \frac{1}{\Gamma(1-\alpha)} \int_0^\xi (\xi-\tau)^{-\alpha} \frac{df(\tau)}{d\tau} d\tau,\ \ \alpha \in (0,1).$$

Definition 2 ([7]). *The Riemann-Liouville integral of function $f(\xi)$ of the distributed-order $\rho(\alpha)$ is given by*

$$ {}_0 I_\xi^{\rho(\alpha)} f(\xi) = \int_0^1 \rho(\alpha) {}_0 I_\xi^{\alpha} f(\xi) d\alpha, \tag{5}$$

where

$$ {}_0 I_\xi^{\alpha} f(\xi) = \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi-\tau)^{\alpha-1} f(\tau) d\tau,\ \ \alpha \in (0,1).$$

Lemma 1. *The relation between distributed-order Caputo derivative and distributed-order Riemann-Liouville integral is given by*

$$ {}_0 I_\xi^{\rho(\alpha)} \left({}_0^C D_\xi^{\rho(\alpha)} f(\xi) \right) = f(\xi) - f(0). \tag{6}$$

Proof. By using the property of Caputo type definition, we can get

$$\begin{aligned} {}_0 I_\xi^{\rho(\alpha)} \left({}_0^C D_\xi^{\rho(\alpha)} f(\xi) \right) =& {}_0 I_\xi^{\rho(\alpha)} {}_0 I_\xi^{1-\rho(\alpha)} \frac{df(\xi)}{d\xi} \\ =& \int_0^\xi \frac{df(\vartheta)}{d\vartheta} d\vartheta \\ =& f(\xi) - f(0). \end{aligned}$$

□

3 Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(\xi) = |a|^{\frac{-1}{2}} \Psi\left(\frac{\xi - b}{a}\right), \quad a, b \in \mathbb{R}, a \neq 0.$$

3.1 Fractional order Laguerre wavelets (FLWs)

The fractional order Laguerre wavelets [14] are defined on [0,1) as

$$\psi^{\beta}_{n,m}(\xi) = \begin{cases} \frac{2^{\frac{k}{2}}}{m!} L_m(2^k \xi^{\beta} - 2n + 1), & \left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\beta}} \leq \xi < \left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\beta}}, \\ 0, & \text{otherwise}, \end{cases}$$

where $k \in \mathbb{N}, n = 1, 2, \ldots, 2^{k-1}, m = 0, 1, \ldots, M-1$ and $L_m(\xi)$ is Laguerre polynomials (LPs) on the interval $[0,1]$ given by

$$L_m(\xi) = \sum_{k=0}^{m} (-1)^k \frac{m!}{(m-k)!(k!)^2} \xi^k.$$

The LPs are given by

$$\begin{aligned} L_0(\xi) =& 1, \\ L_1(\xi) =& -\xi + 1, \\ L_2(\xi) =& \frac{1}{2}(\xi^2 - 4\xi + 2). \end{aligned}$$

When $M = 4, k = 1$ and $\beta = 0.9$

$$\psi^{0.9}_{1,0}(\xi) = \begin{cases} 1.4142135623731, & 0 \leq \xi < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi^{0.9}_{1,1}(\xi) = \begin{cases} 2.82842712474619 - 2.82842712474619\xi^{0.9}, & 0 \leq \xi < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi^{0.9}_{2,1}(\xi) = \begin{cases} -2.82842712474619\xi^{0.9} + 0.353553390593274(2\xi^{0.9} - 1)^2 & \\ \quad +2.12132034355964, & 0 \leq \xi < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi^{0.9}_{2,2}(\xi) = \begin{cases} -1.4142135623731\xi^{0.9} - 0.0392837100659193(2\xi^{0.9} - 1)^3 & \\ \quad +0.353553390593274(2\xi^{0.9} - 1)^2 + 0.942809041582063, & 0 \leq \xi < 1, \\ 0, & \text{otherwise}. \end{cases}$$

4 Function approximation

Any function $g(\xi) \in L^2[0,1]$ can be approximated by using FLWs

$$g(\xi) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi^{\beta}_{n,m}(\xi) = \overline{C}^T \Psi^{\beta}(\xi). \tag{7}$$

where $\overline{C} = [\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_{\widehat{n}}]^T$, $\Psi^{\beta}(\xi) = [\psi^{\beta}_1(\xi), \psi^{\beta}_2(\xi), \ldots, \psi^{\beta}_{\widehat{n}}(\xi)]^T$ and $\widehat{n} = 2^{k-1}M$. Now taking the inner product of $\Psi^{\beta}(\xi)$ both sides in Eq. (7) we get

$$\overline{C}^T = < g(\xi), \Psi^{\beta}(\xi) > \overline{D}^{-1}(\beta) = \left(\int_0^1 g(\xi)(\Psi^{\beta}(\xi))^T d\xi \right) \overline{D}^{-1}(\beta),$$

where $\overline{D}$ is a square matrix of order $\widehat{n}\times\widehat{n}$ and given by

$$\overline{D}(\beta)=\int_0^1 \Psi^{\beta}(\xi)(\Psi^{\beta}(\xi))^T d\xi,$$

When $M=4$ and $k=1$ and $\beta=0.9$

$$\overline{D}(0.9)=\begin{bmatrix} 2 & 1.89473684 & 1.05639098 & 0.44044797\\ 1.89473684 & 2.43609023 & 1.63320463 & 0.73962521\\ 1.05639098 & 1.63320463 & 1.19247922 & 0.56121192\\ 0.44044797 & 0.73962521 & 0.56121192 & 0.26962843 \end{bmatrix}.$$

4.1 Product of operational matrices of FLWs

The approximation of product of two FLWs [10] can be given as

$$\Psi^{\beta}(\xi)(\Psi^{\beta}(\xi))^T\overline{C}^T \approx C\Psi^{\beta}(\xi),$$
$$C=<\Psi^{\beta}(\xi)(\Psi^{\beta}(\xi))^T\overline{C}^T,\Psi^{\beta}(\xi)>=\left(\int_0^1 \Psi^{\beta}(\xi)(\Psi^{\beta}(\xi))^T\overline{C}^T(\Psi^{\beta}(\xi))^T\right)\overline{D}^{-1}(\beta).$$

where C is a product operational matrix of order $\widehat{n}$.

4.2 Operational matrix for FLWs

We can approximate the distributed-order Riemann-Liouville integral [15] of FLWs $\Psi^{\beta}(\xi)$ as

$$I_{\xi}^{\rho(\alpha)}(\Psi^{\beta}(\xi))\approx \mathscr{P}^{\rho(\alpha)}\Psi^{\beta}(\xi), \tag{8}$$

where $\mathscr{P}^{\rho(\alpha)}$ is called the operational matrix of distributed-order Riemann-Liouville integral of order $\widehat{n}\times\widehat{n}$ and defined as

$$\begin{aligned}\mathscr{P}^{\rho(\alpha)}&=<I_{\xi}^{\rho(\alpha)}(\Psi^{\beta}(\xi)),\Psi^{\beta}(\xi)>\overline{D}^{-1}(\beta)\\ &=\left(\int_0^1 I_{\xi}^{\rho(\alpha)}(\Psi^{\beta}(\xi))(\Psi^{\beta}(\xi))^T d\xi\right)\overline{D}^{-1}(\beta).\end{aligned}$$

When $M=4$,$k=1$, $\beta=0.9$, and $\rho(\alpha)=\delta(\alpha-1)$,

$$\mathscr{P}^{\delta(\alpha-1)}=\begin{bmatrix} 0.95245 & -0.36752 & -0.43758 & 0.57604\\ -0.50234 & 1.14739 & -2.48064 & 1.62510\\ 0.305594 & 0.60768 & -0.83396 & -0.07435\\ 0.20512 & 0.00130 & 0.28686 & -0.75564 \end{bmatrix}.$$

When $M=4$,$k=1$, $\beta=0.9$, and $\rho(\alpha)=N^{(0.8,0.2)}(\alpha)$,

$$\mathscr{P}^{N^{(0.8,0.2)}(\alpha)}=\begin{bmatrix} 0.90129 & -0.45419 & 0.25236 & -0.415784\\ 0.59444 & 0.30626 & 0.02984 & -1.23953\\ 0.46958 & -0.34317 & 1.61929 & -2.68492\\ 0.30872 & -0.52845 & 1.57083 & -2.06907 \end{bmatrix}.$$

5 Numerical scheme

In this section, we will discuss the numerical scheme for DO-FOCP given in Eqs. (1)-(3). The distributed-order Caputo derivative of the state functions using FLWs $\Psi^{\beta}(\xi)$ can be approximated as

$$^{C}D_{\xi}^{\rho(\alpha)}x_i(\xi) \approx \overline{C}_i^T \Psi^{\beta}(\xi),\ 1 \leq i \leq s_1. \tag{9}$$

We can approximate the control functions as

$$u_i(\xi) \approx \overline{U}_i^T \Psi^{\beta}(\xi),\ 1 \leq i \leq s_2, \tag{10}$$

where $\overline{C}_i^T = [\overline{c}_{i1}, \overline{c}_{i2}, \dots, \overline{c}_{i\widehat{n}}]$ and $\overline{U}_i^T = [\overline{u}_{i1}, \overline{u}_{i2}, \dots, \overline{u}_{i\widehat{n}}]$ are unknowns. Similarly, x_{i0} can be approximated as

$$x_{i0} \approx d_i^T \Psi^{\beta}(\xi). \tag{11}$$

Using Eq. (6) and distributed-order Riemann-Liouville integral operational matrix, $x_i(\xi)$ can be approximated as

$$\begin{aligned}
{}_0I_{\xi}^{\rho(\alpha)}\left({}_0^CD_{\xi}^{\rho(\alpha)}x_i(\xi)\right) &= x_i(\xi) - x_i(0) \\
x_i(\xi) &= {}_0I_{\xi}^{\rho(\alpha)}\left({}_0^CD_{\xi}^{\rho(\alpha)}x_i(\xi)\right) + x_{i0} \\
&\approx (\overline{C}_i^T \mathscr{P}^{\rho(\alpha)} + d_i^T)\Psi^{\beta}(\xi) \\
&= \overline{C}_{i1}^T \Psi^{\beta}(\xi),
\end{aligned}$$

where $\overline{C}_{i1}^T = \overline{C}_i^T \mathscr{P}^{\rho(\alpha)} + d_i^T$. By using approximated state and control functions, the cost functions can be approximated as

$$\begin{aligned}
\mathfrak{J} &\approx \int_0^1 \mathscr{F}\left(\overline{C}_{11}^T\Psi^{\beta}(\xi), \overline{C}_{21}^T\Psi^{\beta}(\xi), \dots, \overline{C}_{s_11}^T\Psi^{\beta}(\xi), \overline{U}_1^T\Psi^{\beta}(\xi), \overline{U}_2^T\Psi^{\beta}(\xi), \dots, \overline{U}_{s_2}^T\Psi^{\beta}(\xi)\right) d\xi \\
&= \int_0^1 \mathscr{F}\left(\overline{C}_1^T\Psi^{\beta}(\xi), \overline{C}_2^T\Psi^{\beta}(\xi), \dots, \overline{C}_{s_1}^T\Psi^{\beta}(\xi), \overline{U}_1^T\Psi^{\beta}(\xi), \overline{U}_2^T\Psi^{\beta}(\xi), \dots, \overline{U}_{s_2}^T\Psi^{\beta}(\xi)\right) d\xi \\
&= \mathfrak{J}\left[\overline{C}_1^T, \overline{C}_2^T, \dots, \overline{C}_{s_1}^T, \overline{U}_1^T, \overline{U}_2^T, \dots, \overline{U}_{s_2}^T\right] \\
&= \mathfrak{J}_1.
\end{aligned}$$

The dynamical system can be approximated as

$$\begin{aligned}
\mathscr{K}_i\left(\overline{C}_1^T\Psi^{\beta}(\xi), \overline{C}_2^T\Psi^{\beta}(\xi), \dots, \overline{C}_{s_1}^T\Psi^{\beta}(\xi)\right) &= \mathscr{H}_i(\xi, \overline{C}_{11}^T\Psi^{\beta}(\xi), \overline{C}_{21}^T\Psi^{\beta}(\xi), \dots, \overline{C}_{s_11}^T\Psi^{\beta}(\xi), \\
&\qquad \overline{U}_1^T\Psi^{\beta}(\xi), \overline{U}_2^T\Psi^{\beta}(\xi), \dots, \overline{U}_{s_2}^T\Psi^{\beta}(\xi)), \\
\mathscr{K}_i\left(\overline{C}_1^T\Psi^{\beta}(\xi), \overline{C}_2^T\Psi^{\beta}(\xi), \dots, \overline{C}_{s_1}^T\Psi^{\beta}(\xi)\right) - \mathscr{H}_i(\xi, &\overline{C}_{11}^T\Psi^{\beta}(\xi), \overline{C}_{21}^T\Psi^{\beta}(\xi), \dots, \overline{C}_{s_11}^T\Psi^{\beta}(\xi), \\
&\qquad \overline{U}_1^T\Psi^{\beta}(\xi), \overline{U}_2^T\Psi^{\beta}(\xi), \dots, \overline{U}_{s_2}^T\Psi^{\beta}(\xi)) = 0, \\
\mathscr{K}_i\left(\overline{C}_1^T\Psi^{\beta}(\xi), \overline{C}_2^T\Psi^{\beta}(\xi), \dots, \overline{C}_{s_1}^T\Psi^{\beta}(\xi)\right) - \mathscr{H}_i(\xi, &\overline{C}_1^T\Psi^{\beta}(\xi), \overline{C}_2^T\Psi^{\beta}(\xi), \dots, \overline{C}_{s_1}^T\Psi^{\beta}(\xi), \\
&\qquad \overline{U}_1^T\Psi^{\beta}(\xi), \overline{U}_2^T\Psi^{\beta}(\xi), \dots, \overline{U}_{s_2}^T\Psi^{\beta}(\xi)) = 0.
\end{aligned}$$

Now consider

$$\begin{aligned}\mathscr{R}_i\left(\xi,\overline{C}_1^T,\overline{C}_2^T,\ldots,\overline{C}_{s_1}^T,\overline{U}_1^T,\overline{U}_2^T,\ldots,\overline{U}_{s_2}^T\right) =&\mathscr{K}_i\left(\overline{C}_1^T\Psi^\beta(\xi),\overline{C}_2^T\Psi^\beta(\xi),\ldots,\overline{C}_{s_1}^T\Psi^\beta(\xi)\right)\\ &-\mathscr{H}_i\left(\xi,\overline{C}_{11}^T\Psi^\beta(\xi),\overline{C}_{21}^T\Psi^\beta(\xi),\ldots,\overline{C}_{s_1 1}^T\Psi^\beta(\xi),\right.\\ &\left.\overline{U}_1^T\Psi^\beta(\xi),\overline{U}_2^T\Psi^\beta(\xi),\ldots,\overline{U}_{s_2}^T\Psi^\beta(\xi)\right). \end{aligned} \tag{12}$$

Collocating Eq. (12) at $2^{k-1}M$ points ξ_i give

$$\mathscr{R}_i\left(\xi,\overline{C}_1^T,\overline{C}_2^T,\ldots,\overline{C}_{s_1}^T,\overline{U}_1^T,\overline{U}_2^T,\ldots,\overline{U}_{s_2}^T\right) = \left[\mathscr{R}_i\left(\overline{C}_1^T,\overline{C}_2^T,\ldots,\overline{C}_{s_1}^T,\overline{U}_1^T,\overline{U}_2^T,\ldots,\overline{U}_{s_2}^T\right)\right]_{2^{k-1}M\times 1}. \tag{13}$$

The Newton-cotes points ξ_i are given by

$$\xi_i = \frac{2i-1}{2^k M},\ 1\le i\le 2^{k-1}M.$$

Eq. (13) generates a system of $2^{k-1}M$ algebraic equations. After substituting the collocation points we get,

$$\begin{aligned}\mathfrak{J}^* =&\mathfrak{J}^*\left[\overline{C}_1^T,\overline{C}_2^T,\ldots,\overline{C}_{s_1}^T,\overline{U}_1^T,\overline{U}_2^T,\ldots,\overline{U}_{s_2}^T,\lambda_1^{*T},\lambda_2^{*T},\ldots,\lambda_s^{*T}\right]\\ \approx&\mathfrak{J}_1+\sum_{i=0}^{s_1}\lambda_i^*\mathscr{R}_i\left(\overline{C}_1^T,\overline{C}_2^T,\ldots,\overline{C}_{s_1}^T,\overline{U}_1^T,\overline{U}_2^T,\ldots,\overline{U}_{s_2}^T\right),\end{aligned}$$

where $\lambda_i^* = [\lambda_{i1},\lambda_{i2},\ldots,\lambda_{i\hat{n}}]^T$ are the unknown Lagrange multipliers. The necessary conditions for the extremum are $\nabla\mathfrak{J}^*=0$ *i.e.*,

$$\begin{aligned}\frac{\partial\mathfrak{J}^*}{\partial\overline{C}_1}=0,\frac{\partial\mathfrak{J}^*}{\partial\overline{C}_2}=0,\ldots,\frac{\partial\mathfrak{J}^*}{\partial\overline{C}_{s_1}}=0,\frac{\partial\mathfrak{J}^*}{\partial\overline{U}_1}=0,\frac{\partial\mathfrak{J}^*}{\partial\overline{U}_2}=0,\ldots,\frac{\partial\mathfrak{J}^*}{\partial\overline{U}_{s_2}}=0,\\ \frac{\partial\mathfrak{J}^*}{\partial\lambda_1^*}=0,\frac{\partial\mathfrak{J}^*}{\partial\lambda_2^*}=0,\ldots,\frac{\partial\mathfrak{J}^*}{\partial\lambda_s^*}=0.\end{aligned}$$

The above system of algebraic equations has been solved by Newton's method with tolerance 10^{-10}. By determining the unknown coefficients, we obtain the approximated state functions, control functions and the cost function $\mathfrak{J}$.

6 Convergence analysis

Lemma 2 ([16]). *Assume $x\in H^s[0,1], s\ge 0$ and $\overline{x}$ be the best approximation of x. Then*

$$||x-\overline{x}||_{L^2[0,1]}\le c(M-1)^{-s}(2^{k-1})^{-s}||x^s||_{L^2[0,1]},$$

and for $q\ge 1$, we derive

$$||x-\overline{x}||_{H^s[0,1]}\le c(M-1)^{2q-\frac{1}{2}-s}(2^{k-1})^{q-s}||x^s||_{L^2[0,1]},$$

where c depends on s.

Lemma 3 ([7]). *Assume $x \in H^s[0,1]$ and $s \geq 0$, then*

$$||_0 I_\xi^\alpha x -_0 I_\xi^\alpha \bar{x}||_{L^2[0,1]} \leq \frac{c(M-1)^{-s}(2^{k-1})^{-s}}{\Gamma(\alpha+1)}||x^s||_{L^2[0,1]},$$

and for $q \geq 1$, we derive

$$||_0 I_\xi^\alpha x -_0 I_\xi^\alpha \bar{x}||_{L^2[0,1]} \leq \frac{c(M-1)^{2q-\frac{1}{2}-s}(2^{k-1})^{q-s}}{\Gamma(\alpha+1)}||x^s||_{L^2[0,1]}.$$

Proof. By Young's inequality, we have

$$\begin{aligned}
||x \times y||_{L^p[0,1]} &\leq ||x||_{L^1[0,1]}||y||_{L^p[0,1]}, 1 \leq p \leq \infty. \\
||_0 I_t^\alpha x -_0 I_t^\alpha \bar{x}||_{L^2[0,1]} &\leq \frac{1}{\alpha\Gamma\alpha}||x^{\alpha-1}||_{L^1[0,1]}||x-\bar{x}||_{L^2[0,1]} \\
&\leq \frac{1}{\alpha\Gamma\alpha}c(M-1)^{-s}(2^{k-1})^{-s}||x^s||_{L^2[0,1]} \\
&\leq \frac{c(M-1)^{2q-\frac{1}{2}-s}(2^{k-1})^{q-s}}{\Gamma(\alpha+1)}||x^s||_{L^2[0,1]},
\end{aligned}$$

and for $q \geq 1$

$$\begin{aligned}
||_0 I_t^\alpha x -_0 I_t^\alpha \bar{x}||_{L^2[0,1]} &\leq \frac{1}{\alpha\Gamma\alpha}||x^{\alpha-1}||_{L^1[0,1]}||x-\bar{x}||_{H^s[0,1]} \\
&\leq \frac{1}{\alpha\Gamma\alpha}c(M-1)^{2q-\frac{1}{2}-s}(2^{k-1})^{-s}||x^s||_{L^2[0,1]} \\
&\leq \frac{c(M-1)^{2q-\frac{1}{2}-s}(2^{k-1})^{q-s}}{\Gamma(\alpha+1)}||x^s||_{L^2[0,1]}.
\end{aligned}$$

□

Theorem 1. *Suppose $x \in H^s[0,1]$ and $s \geq 0$, then*

$$||_0 I_\xi^{\rho(\alpha)} x -_0 I_\xi^{\rho(\alpha)} \bar{x}||_{L^2[0,1]} \leq \Im c(M-1)^{-s}(2^{k-1})^{-s}||x^s||_{L^2[0,1]},$$

and for $q \geq 1$, we derive

$$||_0 I_\xi^{\rho(\alpha)} x -_0 I_\xi^{\rho(\alpha)} \bar{x}||_{L^2[0,1]} \leq \Im c(M-1)^{2q-\frac{1}{2}-s}(2^{k-1})^{q-s}||x^s||_{L^2[0,1]},$$

where $||\rho(\alpha)||_{L^2[0,1]} \leq \Im$.

Proof. We have

$$\begin{aligned}
||_0 I_\xi^\alpha x -_0 I_\xi^\alpha \bar{x}||_{L^2[0,1]} &= ||\int_0^1 \rho(\alpha)(_0 I_\xi^\alpha x -_0 I_\xi^\alpha \bar{x})d\alpha||_{L^2[0,1]} \\
&\leq \int_0^1 ||\rho(\alpha)(_0 I_\xi^\alpha x -_0 I_\xi^\alpha \bar{x})||_{L^2[0,1]}d\alpha
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 ||\rho(\alpha)||_{L^2[0,1]}||({}_0I_\xi^\alpha x - {}_0I_\xi^\alpha \overline{x})||_{L^2[0,1]}d\alpha \\
&\leq \int_0^1 \Im \frac{c(M-1)^{-s}(2^{k-1})^{-s}}{\Gamma(\alpha+1)}||x^s||_{L^2[0,1]}d\alpha \\
&\leq \Im c(M-1)^{-s}(2^{k-1})^{-s}||x^s||_{L^2[0,1]}.
\end{aligned}
$$

For $q \geq 1$

$$
\begin{aligned}
||{}_0I_\xi^\alpha x - {}_0I_\xi^\alpha \overline{x}||_{L^2[0,1]} &= ||\int_0^1 \rho(\alpha)({}_0I_\xi^\alpha x - {}_0I_t^\alpha \overline{x})d\alpha||_{L^2[0,1]} \\
&\leq \int_0^1 ||\rho(\alpha)({}_0I_\xi^\alpha x - {}_0I_\xi^\alpha \overline{x})||_{L^2[0,1]}d\alpha \\
&\leq \int_0^1 ||\rho(\alpha)||_{L^2[0,1]}||({}_0I_\xi^\alpha x - {}_0I_\xi^\alpha \overline{x})||_{L^2[0,1]}d\alpha \\
&\leq \int_0^1 \Im \frac{c(M-1)^{2q-\frac{1}{2}-s}(2^{k-1})^{q-s}}{\Gamma(\alpha+1)}||x^s||_{L^2[0,1]}d\alpha \\
&\leq \Im c(M-1)^{2q-\frac{1}{2}-s}(2^{k-1})^{q-s}||x^s||_{L^2[0,1]}.
\end{aligned}
$$

□

Theorem 2. *Let Λ consisting of all functions $(x_1(\xi), x_2(\xi), \dots, x_{s_1}(\xi), u_1(\xi), u_2(\xi), \dots, u_{s_2}(\xi)$ that satisfy the constraints given by Eq. (2) and $\Lambda_{\widehat{n}}$ be a subset of Λ consisting of all functions*

$$
\left(\sum_{j=1}^{\widehat{n}} x_{1j}^* \psi^\beta{}_j(\xi), \sum_{j=1}^{\widehat{n}} x_{2j}^* \psi_j^\beta(\xi), \dots, \sum_{j=1}^{\widehat{n}} x_{s_1 j}^* \psi_j^\beta(\xi),\ \sum_{j=1}^{\widehat{n}} u_{1j}^* \psi_j^\beta(\xi), \sum_{j=1}^{\widehat{n}} u_{2j}^* \psi_j^\beta(\xi), \dots, \sum_{j=1}^{\widehat{n}} u_{s_2 j}^* \psi_j^\beta(\xi)\right).
$$

Also, let $\mathscr{F}$ be a Lipschitz continuous function, then there exist real number Q^* and $L > 0$ such that

$$
|\inf_{\Lambda_{\widehat{n}}} \Im - \inf_{\Lambda} \Im| \leq Q^* L c(M-1)^{-s}(2^{k-1})^{-s},
$$

and for $q \geq 1$

$$
|\inf_{\Lambda_{\widehat{n}}} \Im - \inf_{\Lambda} \Im| \leq Q^* L c(M-1)^{2q-\frac{1}{2}-s}(2^{k-1})^{q-s}.
$$

Proof. Given that $\mathscr{F}$ is a Lipschitz continuous function with respect to state and control functions, and it has a Lipschitz constant L, we have

$$
\begin{aligned}
&|\mathscr{F}(\xi, x_1(\xi), x_2(\xi), \dots, x_{s_1}(\xi), u_1(\xi), u_2(\xi), \dots, u_{s_2}(\xi)) \\
&\quad - \mathscr{F}(\xi, \overline{x}_1(\xi), \overline{x}_2(\xi), \dots, \overline{x}_{s_1}(\xi), \overline{u}_1(\xi), \overline{u}_2(\xi), \dots, \overline{u}_{s_2}(\xi))| \\
&\leq L(||x_1 - \overline{x}_1||_2 + ||x_2 \overline{x}_2||_2 + \dots + ||x_{s_1} - \overline{x}_{s_1}||_2 \\
&\quad + ||u_1 - \overline{u}_1||_2 + ||u_2 - \overline{u}_2||_2 + \dots + ||u_{s_2} - \overline{u}_{s_2}||_2),
\end{aligned}
$$

where $\xi \in [0,1]$, and $\overline{x}_1, \overline{x}_2, \dots, \overline{x}_{s_1}, \overline{u}_1, \overline{u}_2, \dots, \overline{u}_{s_1}, x_1, x_2, \dots, x_{s_1}, u_1, u_2, \dots, u_{s_2}$, belong to $L^2[0,1]$. Since $\Lambda_{\widehat{n}} \subseteq \Lambda$, then $\inf_{\Lambda_{\widehat{n}}} \Im \geq \inf_{\Lambda} \Im$. Suppose that $(x_1(\xi), x_2(\xi), \dots, x_{s_1}(\xi),\ u_1(\xi), u_2(\xi), \dots, u_{s_2}(\xi))$ is the exact solution for the state functions and control function, and

$$\left(\sum_{j=1}^{\widehat{n}} x_{1j}^{*}\psi^{\beta}{}_{j}(\xi), \sum_{j=1}^{\widehat{n}} x_{2j}^{*}\psi_{j}^{\beta}(\xi), \ldots, \sum_{j=1}^{\widehat{n}} x_{s_1 j}^{*}\psi_{j}^{\beta}(\xi), \sum_{j=1}^{\widehat{n}} u_{1j}^{*}\psi_{j}^{\beta}(\xi),\ \sum_{j=1}^{\widehat{n}} u_{2j}^{*}\psi_{j}^{\beta}(\xi), \ldots, \sum_{j=1}^{\widehat{n}} u_{s_2 j}^{*}\psi_{j}^{\beta}(\xi)\right)$$

be the approximate solution by using FLWs basis. Then

$$\begin{aligned}
|\inf_{\Lambda_{\widehat{n}}}\mathfrak{J} - \inf_{\Lambda}\mathfrak{J}| \leq & |\mathfrak{J}\left(\sum_{j=1}^{\widehat{n}} x_{1j}^{*}\psi_{j}^{\beta}(\xi), \sum_{j=1}^{\widehat{n}} x_{2j}^{*}\psi_{j}^{\beta}(\xi), \ldots, \sum_{j=1}^{\widehat{n}} x_{s_1 j}^{*}\psi_{j}^{\beta}(\xi), \sum_{j=1}^{\widehat{n}} u_{1j}^{*}\psi_{j}^{\beta}(\xi)\right., \\
& \left.\sum_{j=1}^{\widehat{n}} u_{2j}^{*}\psi_{j}^{\beta}(\xi), \ldots, \sum_{j=1}^{\widehat{n}} u_{s_2 j}^{*}\psi_{j}^{\beta}(\xi)\right) - \mathfrak{J}(x_1(\xi), x_2(\xi), \ldots, x_{s_1}(\xi), \\
& u_1(\xi), u_2(\xi), \ldots, u_{s_2}(\xi))| \\
= & |\int_0^1 \mathscr{F}\left(\sum_{j=1}^{\widehat{n}} x_{1j}^{*}\psi_{j}^{\beta}(\xi), \sum_{j=1}^{\widehat{n}} x_{2j}^{*}\psi_{j}^{\beta}(\xi), \ldots, \sum_{j=1}^{\widehat{n}} x_{s_1 j}^{*}\psi_{j}^{\beta}(\xi), \sum_{j=1}^{\widehat{n}} u_{1j}^{*}\psi_{j}^{\beta}(\xi),\right. \\
& \left.\sum_{j=1}^{\widehat{n}} u_{2j}^{*}\psi_{j}^{\beta}(\xi) \ldots, \sum_{j=1}^{\widehat{n}} u_{s_2 j}^{*}\psi_{j}^{\beta}(\xi)\right) d\xi - \int_0^1 \mathscr{F}(x_1(\xi), x_2(\xi), \ldots, \\
& x_{s_1}(\xi), u_1(\xi), u_2(\xi), \ldots, u_{s_2}(\xi))\, d\xi| \\
\leq & \int_0^1 |\mathscr{F}\left(\sum_{j=1}^{\widehat{n}} x_{1j}^{*}\psi_{j}^{\beta}(\xi), x_2(\xi), \ldots, x_{s_1}(\xi), u_1(\xi), u_2(\xi), \ldots, u_{s_2}(\xi)\right) \\
& - \mathscr{F}(x_1(\xi), x_2(\xi), \ldots, x_{s_1}(\xi), u_1(\xi), u_2(\xi), \ldots, u_{s_2}(\xi))| d\xi \\
& + \int_0^1 \left(\sum_{j=1}^{\widehat{n}} x_{1j}^{*}\psi_{j}^{\beta}(\xi), \sum_{j=1}^{\widehat{n}} x_{2j}^{*}\psi_{j}^{\beta}(\xi), \ldots, x_{s_1}(\xi), u_1(\xi), u_2(\xi), \ldots, u_{s_2}(\xi)\right) \\
& - \mathscr{F}\left(\sum_{j=1}^{\widehat{n}} x_{1j}^{*}\psi_{j}^{\beta}(\xi), x_2(\xi), \ldots, x_{s_1}(\xi),, u_1(\xi), u_2(\xi), \ldots, u_{s_2}(\xi)\right) | d\xi + \ldots \\
& + \int_0^1 \mathscr{F}\left(\sum_{j=1}^{\widehat{n}} x_{1j}^{*}\psi_{j}^{\beta}(\xi), \sum_{j=1}^{\widehat{n}} x_{2j}^{*}\psi_{j}^{\beta}(\xi), \ldots, \sum_{j=1}^{\widehat{n}} x_{s_{1j}}^{*}\psi_{j}^{\beta}(t), u_2(\xi), \ldots, u_{s_2}(\xi)\right) \\
& - \mathscr{F}\left(\sum_{j=1}^{\widehat{n}} x_{1j}^{*}\psi_{j}^{\beta}(\xi), \sum_{j=1}^{\widehat{n}} x_{2j}^{*}\psi_{j}^{\beta}(\xi), \ldots, \sum_{j=1}^{\widehat{n}} x_{(s_1-j)l}^{*}\psi_{j}^{\beta}(\xi), x_{s_1}(\xi),\right. \\
& \left. u_1(\xi), u_2(\xi), \ldots, u_{s_2}(\xi)\right) d\xi \\
& + \int_0^1 |\mathscr{F}\left(\sum_{j=1}^{\widehat{n}} x_{1j}^{*}\psi_{j}^{\beta}(\xi), \sum_{j=1}^{\widehat{n}} x_{2j}^{*}\psi_{j}^{\beta}(\xi), \ldots, \sum_{j=1}^{\widehat{n}} x_{s_{1j}}^{*}\psi_{j}^{\beta}(\xi), u_1(\xi)\right., \\
& \left.\sum_{j=1}^{\widehat{n}} u_{1j}^{*}\psi_{j}^{\beta}(\xi), u_2(\xi), \ldots, u_{s_2}(\xi)\right) - \mathscr{F}\left(\sum_{j=1}^{\widehat{n}} x_{1j}^{*}\psi_{j}^{\beta}(\xi), \sum_{j=1}^{\widehat{n}} x_{2j}^{*}\psi_{j}^{\beta}(\xi), \ldots,\right.
\end{aligned}$$

$$
\begin{aligned}
&\sum_{j=1}^{\widehat{n}} x^*_{s_1 j}\psi^{\beta}_j(\xi), \sum_{j=1}^{\widehat{n}} u^*_{1j}\psi^{\beta}_j(\xi), u_1(\xi), u_2(\xi), \dots, u_{s_2}(\xi)\Bigg)|d\xi \\
&+\int_0^1 |\mathscr{F}\Bigg(\sum_{j=1}^{\widehat{n}} x^*_{1j}\psi^{\beta}_j(\xi), \sum_{j=1}^{\widehat{n}} x^*_{2j}\psi^{\beta}_j(\xi), \dots, \sum_{j=1}^{\widehat{n}} x^*_{s_1 j}\psi^{\beta}_j(\xi), \sum_{j=1}^{\widehat{n}} u^*_{1j}\psi^{\beta}_j(\xi), \\
&\sum_{j=1}^{\widehat{n}} u^*_{2j}\psi^{\beta}_j(\xi), \dots, u_{s_2}(\xi)\Bigg) - \mathscr{F}\Bigg(\sum_{j=1}^{\widehat{n}} x^*_{1j}\psi^{\beta}_j(\xi), \sum_{j=1}^{\widehat{n}} x^*_{2j}\psi^{\beta}_j(\xi), \dots, \\
&\sum_{j=1}^{\widehat{n}} x^*_{s_1 j}\psi^{\beta}_j(\xi), \sum_{j=1}^{\widehat{n}} u^*_{1j}\psi^{\beta}_j(\xi), u_2(\xi), \dots, u_{s_2}(\xi)\Bigg)|d\xi + \dots \\
&+\int_0^1 |\mathscr{F}\Bigg(\sum_{j=1}^{\widehat{n}} x^*_{1j}\psi^{\beta}_j(\xi), \sum_{j=1}^{\widehat{n}} x^*_{2j}\psi^{\beta}_j(\xi), \dots, \sum_{j=1}^{\widehat{n}} x^*_{s_1 j}\psi^{\beta}_j(\xi), \sum_{j=1}^{\widehat{n}} u^*_{1j}\psi^{\beta}_j(\xi), \\
&\sum_{j=1}^{\widehat{n}} u^*_{2j}\psi^{\beta}_j(\xi), \dots, \sum_{j=1}^{\widehat{n}} u^*_{s_2 j}\psi^{\beta}_j(\xi)\Bigg) - \mathscr{F}\Bigg(\sum_{j=1}^{\widehat{n}} x^*_{1j}\psi^{\beta}_j(\xi), \sum_{j=1}^{\widehat{n}} x^*_{2j}\psi^{\beta}_j(\xi), \dots, \\
&\sum_{j=1}^{\widehat{n}} x^*_{s_1 j}\psi^{\beta}_j(\xi), \sum_{j=1}^{\widehat{n}} u^*_{1j}\psi^{\beta}_j(\xi), u_2(\xi), \dots, \sum_{j=1}^{\widehat{n}} u^*_{(s_2-j)}\psi^{\beta}_j(\xi), u_{s_2}(\xi)\Bigg)|d\xi
\end{aligned}
$$

$$
\begin{aligned}
|\inf_{\Lambda_{\widehat{n}}}\mathfrak{J} - \inf_{\Lambda}\mathfrak{J}| \le &\int_0^1 L\Bigg(||x_1 - \sum_{j=1}^{\widehat{n}} x^*_{1j}\psi^{\beta}_j(\xi)||_2 + ||x_2 - \sum_{j=1}^{\widehat{n}} x^*_{2j}\psi^{\beta}_j(\xi)||_2 + \dots \\
&+||x_{s_1} - \sum_{j=1}^{\widehat{n}} x^*_{sj}\psi^{\beta}_j(\xi)||_2 + ||u_1 - \sum_{j=1}^{\widehat{n}} u^*_{1j}\psi^{\beta}_j(\xi)||_2 + \dots \\
&+ ||u_2 - \sum_{j=1}^{\widehat{n}} u^*_{2j}\psi^{\beta}_j(\xi)||_2 + ||u_{s_2} - \sum_{j=1}^{\widehat{n}} u^*_{s_2 j}\psi^{\beta}_j(\xi)||_2\Bigg)d\xi \\
\le &L\Big(c(M-1)^{-s}(2^{k-1})^{-s}||x^s_1||_{L^2[0,1]} + c(M-1)^{-s}(2^{k-1})^{-s}||x^s_2||_{L^2[0,1]} + \dots \\
&+c(M-1)^{-s}(2^{k-1})^{-s}||x^s_{s_1}||_{L^2[0,1]} + c(M-1)^{-s}(2^{k-1})^{-s}||u^s_1||_{L^2[0,1]} \\
&+c(M-1)^{-s}(2^{k-1})^{-s}||u^s_2||_{L^2[0,1]} + \dots + c(M-1)^{-s}(2^{k-1})^{-s}||u^s_{s_2}||_{L^2[0,1]}\Big) \\
\le &L\Big(Q_1 c(M-1)^{-s}(2^{k-1})^{-s} + Q_2 c(M-1)^{-s}(2^{k-1})^{-s} + \dots \\
&+Q_{s_1} c(M-1)^{-s}(2^{k-1})^{-s} + Q_{s_1+1} c(M-1)^{-s}(2^{k-1})^{-s} + \dots \\
&+Q_{s_1+2} c(M-1)^{-s}(2^{k-1})^{-s} + Q_{s_1+s_2} c(M-1)^{-s}(2^{k-1})^{-s}\Big) \\
= &(Q_1 + Q_2 + \dots + Q_{s_1+s_2})Lc(M-1)^{-s}(2^{k-1})^{-s},
\end{aligned}
$$

where $||x^s_1||_{L^2[0,1]} \le Q_1$, $||x^s_2||_{L^2[0,1]} \le Q_2$, $||x^s_{s_1}||_{L^2[0,1]} \le Q_{s_1}$ and $||u^s_1||_{L^2[0,1]} \le Q_{s_1+1}$, $||u^s_2||_{L^2[0,1]} \le Q_{s_1+2}$, $\dots, ||u^s_{s_2}||_{L^2[0,1]} \le Q_{s_1+s_2}$.

$$|\inf_{\Lambda_{\widehat{n}}}\mathfrak{J} - \inf_{\Lambda}\mathfrak{J}| \le Q^* Lc(M-1)^{-s}(2^{k-1})^{-s},$$

for $q \geq 1$, using Lemma 2, we have

$$|\inf_{\Lambda_{\widehat{n}}} \mathfrak{J} - \inf_{\Lambda} \mathfrak{J}| \leq Q^* Lc(M-1)^{2q-\frac{1}{2}-s}(2^{k-1})^{q-s},$$

where $Q^* = Q_1 + Q_2 + \cdots + Q_{s_1+s_2}$. □

Theorem 3. *The approximate solutions* $x_i(\xi) \approx \overline{C}_i^T \Psi^\beta(\xi), 1 \leq i \leq s_1$ *and* $u_i(\xi) \approx \overline{U}_i^T \Psi^\beta(\xi), 1 \leq i \leq s_2$, *approaches to the exact solution, respectively as* $\widehat{n}$ *tends to* ∞, *where* 2^{k-1} *is finite and* $M \to \infty$.

Proof. The proof is the same as [17, Theorem 8.1]. □

7 Numerical examples and results

Example 1 ([7, 18])**.** Let us consider GDFOCPs

$$\begin{aligned}
&\mathfrak{J} = \frac{1}{2}\int_0^1 \left(x^2(\xi) + u^2(\xi)\right) d\xi, \\
&{}^C D_\xi^{\rho(\alpha)} x(\xi) = -x(\xi) + u(\xi), \\
&x(0) = 1.
\end{aligned}$$

Its exact solution when $\rho(\alpha) = 1$ is

$$\begin{aligned}
&x(\xi) = \theta \sinh(\sqrt{2}\xi) + \cosh(\sqrt{2}\xi), \\
&u(\xi) = (\theta + \sqrt{2})\sinh(\sqrt{2}\xi) + (\sqrt{2}\theta + 1)\cosh(\sqrt{2}\xi), \\
&\mathfrak{J} = 0.1929092980931.
\end{aligned}$$

where $\theta \approx -0.98$. Here we consider the following cases for the distribution function

a) $\rho(\alpha) = \delta(\alpha - \nu)$, b) $\rho(\alpha) = N^{(0.8,0.2)}(\alpha)$, c) $\rho(\alpha) = E(\alpha, 1.5)$, d) $\rho(\alpha) = W(\alpha, 1, 5)$,

where $\delta(\alpha - \nu)$, $N^{(\mu,\sigma)}(x)$, $E(x, \lambda)$ and $W(x, a, b)$ represent the Dirac delta function, Normal distribution, Exponential distribution and Weibull distribution, respectively that are given as

$$\begin{aligned}
&\int_{-\infty}^{\infty} \delta(\alpha - \nu) f(\alpha) d\alpha = f(\nu), \\
&N^{(\mu,\sigma)}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \\
&E(x, \lambda) = \lambda e^{-\lambda x}, x \geq 0, \\
&W(x, a, b) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^b},\ x \geq 0,
\end{aligned}$$

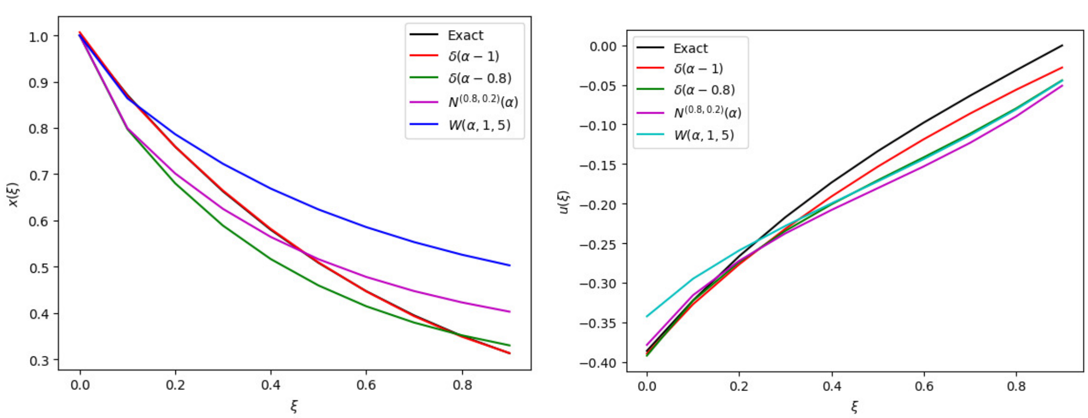
where $a, b, \lambda > 0$ and ν are scale, shape, rate parameters, and real number respectively, where $a, b, \lambda > 0$ and ν are scale, shape, rate parameters, and real number, respectively. The above example has solved using the proposed FLWs method. Table 1 shows the estimated cost value compared with existing methods. Table 2 gives the estimated values of $\mathfrak{J}$, $x(\xi)$ and $u(\xi)$. Figure 1 shows the exact and FLWs results of $x(\xi)$ and $u(\xi)$. From Table 1, we can conclude that our result is better than the existing one in the literature.

Table 1: Estimated values of $\mathfrak{J}$ by using our proposed method and other existing methods for Example 1.

	$\rho(\alpha)$						
Method	$\delta(\alpha-1)$	$\delta(\alpha-0.99)$	$\delta(\alpha-0.9)$	$\delta(\alpha-0.8)$	$N^{(0.8,0.2)}(\alpha)$	$E(\alpha,1.5)$	$W(\alpha,1,5)$
VIM [19]	0.19291	0.19153	0.17953	0.16711	-	-	-
ADM [20]	0.19291	0.19155	0.17962	0.16740	-	-	-
BWM [21] ($M=5,k=2$)	0.18310	0.18185	0.16922	0.15351	-	-	-
LCM [7]	0.19291	-	0.17969	0.16734	-	-	-
FWM [15] ($M=4,k=1$)	0.19291	0.19153	0.17952	0.16705	-	-	-
FWM [15] ($M=6,k=1$)	0.19291	0.19153	0.17952	0.16707	-	-	-
FWM [15] ($M=4,k=2$)	0.19291	0.19153	0.17952	0.16707	-	-	-
FWM [15] ($M=6,k=2$)	0.19291	0.19153	0.17953	0.16707	-	-	-
FCWM [18] ($M=4,k=1$)	0.19291	0.19153	0.17953	0.16709	-	-	-
FCWM [18] ($M=6,k=1$)	0.19291	0.19153	0.17952	0.16707	-	-	-
FCWM [18] ($M=4,k=2$)	0.19291	0.19152	0.17952	0.16707	-	-	-
FCWM [18] ($M=6,k=2$)	0.19291	0.19152	0.17952	0.16707	-	-	-
Present method ($M=4,k=1$)	0.19291	0.19156	0.17989	0.16820	0.18920	0.22954	0.24199
Present method ($M=6,k=1$)	0.19291	0.19154	0.17974	0.16778	0.18965	0.22183	0.24066
Present method ($M=4,k=2$)	0.19291	0.19153	0.17962	0.16740	0.18835	0.21264	0.23960
Present method ($M=6,k=2$)	0.19291	0.19153	0.17957	0.16725	0.18773	0.20776	0.23915
Exact value	0.19291	-	-	-	-	–	-

Table 2: FLWs results of $\mathfrak{J}, x(\xi)$ and $u(\xi)$ for Example 1.

		$\rho(\alpha),\beta=0.9$					
ξ		Exact	$\delta(\alpha-1)$	$\delta(\alpha-0.8)$	$N^{(0.8,0.2)}(\alpha)$	$E(\alpha,1.5)$	$W(\alpha,1,5)$
0.20	$x(\xi)$	0.75937	0.75940	0.68026	0.70152	0.71218	0.78637
	$u(\xi)$	-0.27701	-0.27672	-0.27378	-0.27221	-0.25030	-0.25911
0.40	$x(\xi)$	0.57990	0.58174	0.51659	0.56441	0.64279	0.66892
	$u(\xi)$	-0.19040	-0.19104	-0.20095	-0.20805	-0.20872	-0.19974
0.60	$x(\xi)$	0.44713	0.44664	0.41453	0.47790	0.60130	0.58560
	$u(\xi)$	-0.11913	-0.11878	-0.14159	-0.15307	-0.17223	-0.14383
0.80	$x(\xi)$	0.35036	0.34884	0.35173	0.42289	0.57595	0.52605
	$u(\xi)$	-0.05745	-0.05640	-0.07995	-0.09004	-0.11392	-0.08122
	$\mathfrak{J}$	0.19291	0.19291	0.16820	0.18920	0.22954	0.24199



(a) (b)

Figure 1: FLWs results of $x(\xi)$ and $u(\xi)$ for considered cases for Example 1.

Table 3: FLWs results of $\mathfrak{J}$, $x_1(\xi)$, $x_2(\xi)$ and $u(\xi)$ for Example 2.

				$\rho(\alpha), \beta = 0.9$			
ξ		Exact	$\delta(\alpha-1)$	$\delta(\alpha-0.8)$	$N^{(0.8,0.2)}(\alpha)$	$E(\alpha,1.5)$	$W(\alpha,1,5)$
0.20	$x_1(\xi)$	0.88919	0.88834	0.82800	0.84311	0.84873	0.90602
	$x_2(\xi)$	0.67032	0.67086	0.58577	0.61075	0.62354	0.7031
	$u(\xi)$	-0.38077	-0.38062	-0.36879	-0.35529	-0.30787	-0.32679
0.40	$x_1(\xi)$	0.76790	0.76878	0.69823	0.73902	0.79804	0.83311
	$x_2(\xi)$	0.44933	0.45170	0.40095	0.45218	0.53969	0.55372
	$u(\xi)$	-0.28122	-0.28236	-0.28774	-0.28591	-0.26457	-0.26334
0.60	$x_1(\xi)$	0.65343	0.65363	0.59545	0.65810	0.76393	0.77020
	$x_2(\xi)$	0.30119	0.30007	0.29816	0.35990	0.48990	0.45310
	$u(\xi)$	-0.18597	-0.18582	-0.20902	-0.21599	-0.22137	-0.19520
0.80	$x_1(\xi)$	0.55467	0.55349	0.52226	0.60046	0.74365	0.71961
	$x_2(\xi)$	0.20190	0.19991	0.23740	0.30088	0.48990	0.38169
	$u(\xi)$	-0.09339	-0.09209	-0.11940	-0.12861	-0.14741	-0.11232
	$\mathfrak{J}$	0.43197	0.43197	0.37882	0.42090	0.49769	0.52455

Example 2 ([10])**.** Let us consider GDFOCPs

$$\begin{aligned}
&\mathfrak{J} = \frac{1}{2}\int_0^1 \left(x_1^2(\xi)+x_2^2(\xi)+u^2(\xi)\right)d\xi,\\
&{}^C D_\xi^{\rho(\alpha)} x_1(\xi) = -x_1(\xi)+x_2(\xi)+u(\xi),\\
&{}^C D_\xi^{\rho(\alpha)} x_2(\xi) = -2x_2(\xi),\\
&x_1(0)=1, x_2(0)=1.
\end{aligned}$$

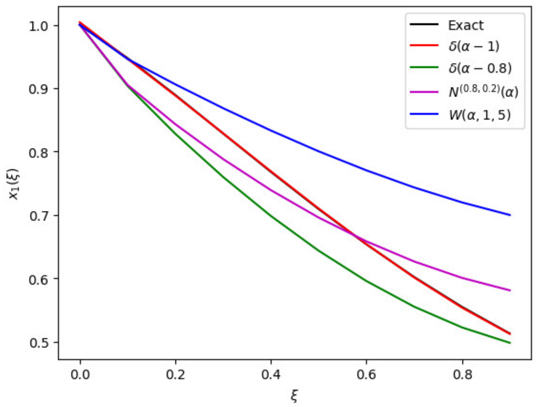
The exact solution when $\rho(\alpha)=1$ is

$$\begin{aligned}
&x_1(\xi) = \frac{-3}{2}e^{-2\xi}+2.48164e^{-\sqrt{2}\xi}+0.018352e^{\sqrt{2}\xi},\\
&x_2(\xi) = e^{-2\xi},\\
&u(\xi) = \frac{1}{2}-1.02793e^{-\sqrt{2}\xi}+0.0443056e^{\sqrt{2}\xi},\\
&\mathfrak{J} = 0.43197.
\end{aligned}$$

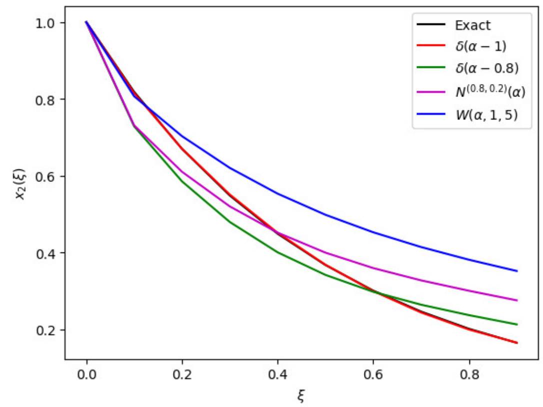
The above example is solved by the proposed FLWs method. Table 3 gives the estimated values of $\mathfrak{J}$, $x_1(\xi)$, $x_2(\xi)$ and $u(\xi)$. Figure 2 shows the exact and FLWs results of $x_1(\xi)$, $x_2(\xi)$ and $u(\xi)$. From Table 3 and Figure 2, we can conclude that there is a good understanding with the estimated results.

Example 3. Let us consider GDFOCPs

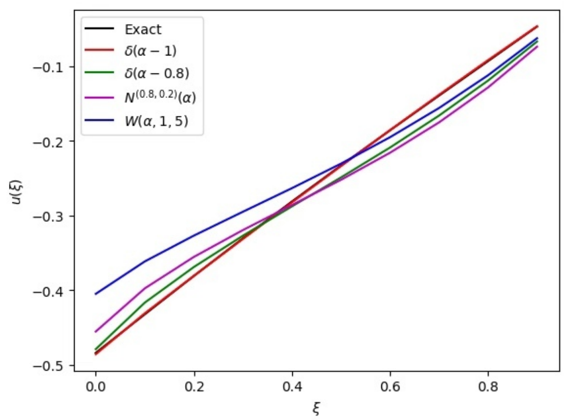
$$\begin{aligned}
&\mathfrak{J} = \frac{1}{2}\int_0^1 \left(x_1^2(\xi)+x_2^2(\xi)-x_3^2(\xi)+u^2(\xi)\right)d\xi,\\
&2\,{}^C D_\xi^{\rho(\alpha)} x_1(\xi) = (2-\xi)x_1(\xi)+u(\xi),\\
&2\,{}^C D_\xi^{\rho(\alpha)} x_2(\xi) = 3x_1(\xi)+x_3(\xi)-u(\xi),\\
&2\,{}^C D_\xi^{\rho(\alpha)} x_3(\xi) = \xi x_1(\xi)+x_2(\xi),\\
&x_1(0)=1,\ x_2(0)=1,\ x_3(0)=1.
\end{aligned}$$



(a)



(b)



(c)

Figure 2: FLWs results of $x_1(\xi)$, $x_2(\xi)$ and $u(\xi)$ for considered cases for Example 2.

The above example is solved by using the proposed FLWs method. Table 4 gives the estimated values of $\mathfrak{J}$, $x_1(\xi)$, $x_2(\xi)$, $x_3(\xi)$ and $u(\xi)$. Figure 3 shows the exact and FLWs results of $x_1(\xi)$, $x_2(\xi)$, $x_3(\xi)$ and $u(\xi)$. From Table 4 and Figure 3, we can conclude that there is a good understanding with the estimated results.

Example 4 ([22])**.** Let us consider GDFOCPs

$$\begin{aligned}
&\mathfrak{J} = \frac{1}{2}\int_0^1 \left(x_1^2(\xi) + x_2^2(\xi) + u_1^2(\xi) + u_2^2(\xi)\right) d\xi,\\
&2\ {}^{C}D_{\xi}^{\rho(\alpha)} x_1(\xi) = -x_1(\xi) + x_2(\xi) + u_1(\xi),\\
&2\ {}^{C}D_{\xi}^{\rho(\alpha)} x_2(\xi) = -2x_2(\xi) + u_2(\xi),\\
&x_1(0) = 1, \quad x_2(0) = 1.
\end{aligned}$$

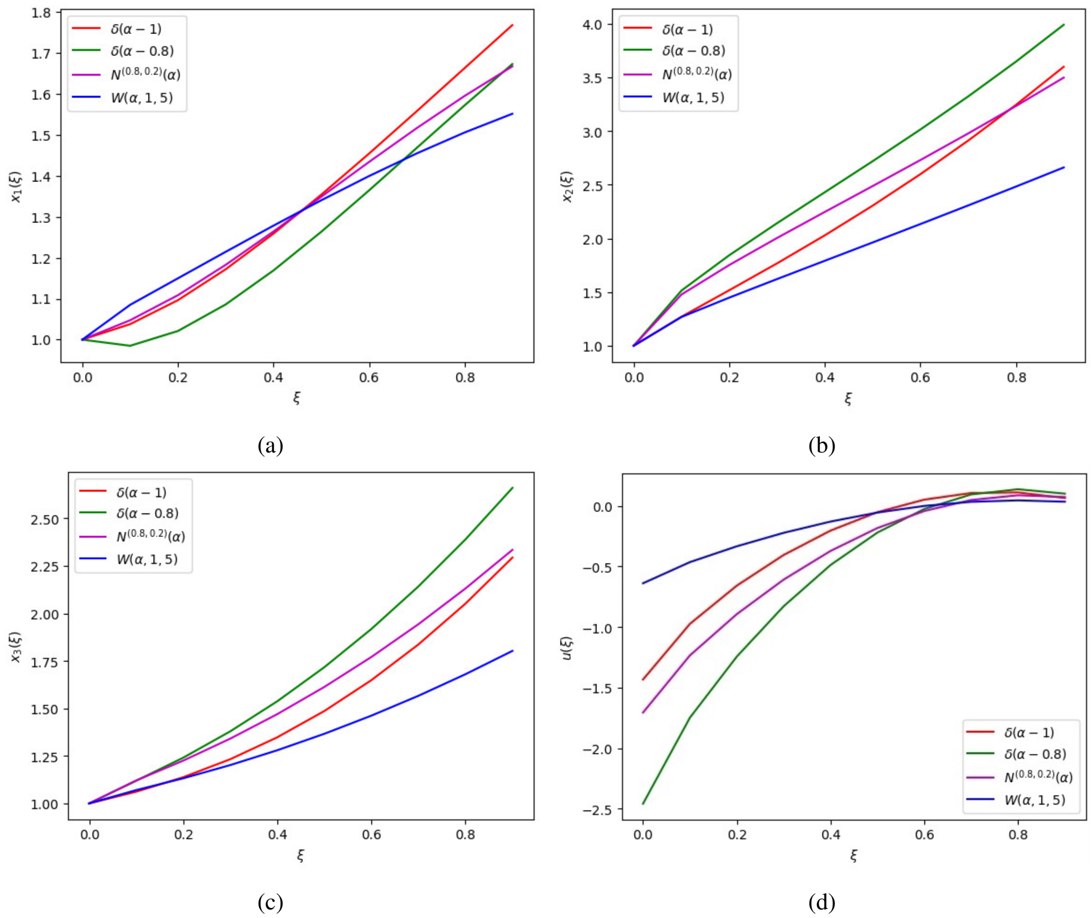


Figure 3: FLWs results of $x_1(\xi)$, $x_2(\xi)$, $x_3(\xi)$ and $u(\xi)$ for considered cases for Example 3.

The above example has solved by the proposed FLWs method. Table 5 gives the estimated values of $\mathfrak{J}$, $x_1(\xi)$, $x_2(\xi)$, $u_1(\xi)$ and $u_2(\xi)$. Figure 4 shows the exact and FLWs results of $x_1(\xi)$, $x_2(\xi)$, $u_1(\xi)$ and $u_2(\xi)$. From Table 5 and Figure 4, we can conclude that there is a good understanding with the estimated results.

8 Conclusion

This paper presents a new numerical method based on fractional Laguerre wavelets for solving generalized fractional optimal control problems (GDFOCPs) whose governing equations are described by distributed -order differential equations. The method transforms the GDFOCPs into a system of algebraic equations using distributed-order Riemann-Liouville operational matrices and Newton-Cotes collocation points. The unknown coefficients are solved to obtain the optimized value of cost function. The paper discusses the error bounds and convergence analysis of the proposed numerical scheme. The examples demonstrate the simplicity, efficiency, accuracy, and high performance of the method. Compared to

Table 4: FLWs results of $\mathfrak{J}, x_1(\xi), x_2(\xi), x_3(\xi)$ and $u(\xi)$ for Example 3.

		$\rho(\alpha), \beta = 0.8$				
ξ		$\delta(\alpha-1)$	$\delta(\alpha-0.8)$	$N^{(0.8,0.2)}(\alpha)$	$E(\alpha,1.5)$	$W(\alpha,1,5)$
0.20	$x_1(\xi)$	1.09631	1.02105	1.10843	1.21970	1.14966
	$x_2(\xi)$	1.51749	1.84350	1.75230	1.76202	1.44856
	$x_3(\xi)$	1.13798	1.24069	1.22508	1.25535	1.13217
	$u(\xi)$	-0.65851	-1.24463	-0.89368	-0.50946	-0.33347
0.40	$x_1(\xi)$	1.25875	1.16882	1.26403	1.32753	1.27822
	$x_2(\xi)$	2.02893	2.43067	2.24703	2.04223	1.79182
	$x_3(\xi)$	1.34806	1.53759	1.46990	1.41160	1.27982
	$u(\xi)$	-0.20380	-0.48797	-0.37182	-0.23247	-0.12916
0.60	$x_1(\xi)$	1.45408	1.36424	1.43367	1.41033	1.39926
	$x_2(\xi)$	2.59878	3.01523	2.73034	2.27826	2.13386
	$x_3(\xi)$	1.64861	1.91797	1.77055	1.57313	1.46186
	$u(\xi)$	0.05190	-0.02524	-0.04186	-0.05995	0.00008
0.80	$x_1(\xi)$	1.66277	1.57188	1.59484	1.46764	1.50561
	$x_2(\xi)$	3.24476	3.64745	3.23372	2.49591	2.48300
	$x_3(\xi)$	2.05131	2.38926	2.13100	1.74338	1.67987
	$u(\xi)$	0.11129	0.13774	0.08797	0.02368	0.04603
	$\mathfrak{J}$	2.92003	3.63720	3.01159	2.16167	1.96012

Table 5: FLWs results of $\mathfrak{J}, x_1(\xi), x_2(\xi), u_1(\xi)$ and $u_2(\xi)$ for Example 4.

		$\rho(\alpha), \beta = 0.8$				
ξ		$\delta(\alpha-1)$	$\delta(\alpha-0.8)$	$N^{(0.8,0.2)}(\alpha)$	$E(\alpha,1.5)$	$W(\alpha,1,5)$
0.20	$x_1(\xi)$	0.88513	0.81979	0.83507	0.83921	0.90213
	$x_2(\xi)$	0.62539	0.52864	0.55769	0.57618	0.66549
	$u_1(\xi)$	-0.37308	-0.35907	-0.34772	-0.30314	-0.32312
	$u_2(\xi)$	-0.23058	-0.23635	-0.23782	-0.22317	-0.22595
0.40	$x_1(\xi)$	0.75723	0.68164	0.72435	0.78607	0.82445
	$x_2(\xi)$	0.39385	0.33942	0.39502	0.49323	0.50839
	$u_1(\xi)$	-0.27430	-0.27837	-0.27913	-0.26226	-0.26008
	$u_2(\xi)$	-0.14058	-0.16022	-0.17039	-0.18068	-0.16334
0.60	$x_1(\xi)$	0.63464	0.0.57192	0.63782	0.75044	0.75714
	$x_2(\xi)$	0.23979	0.0.23796	0.30334	0.44605	0.40503
	$u_1(\xi)$	-0.17742	-0.19890	-0.20799	-0.21697	-0.19070
	$u_2(\xi)$	-0.07367	-0.10270	-0.11591	-0.14243	-0.10885
0.80	$x_1(\xi)$	0.52953	0.49592	0.57769	0.72959	0.70383
	$x_2(\xi)$	0.14968	0.18821	0.25297	0.41594	0.33861
	$u_1(\xi)$	-0.08651	-0.11178	-0.12209	-0.14245	-0.10850
	$u_2(\xi)$	-0.02789	-0.05217	-0.06243	-0.08926	-0.05648
	$\mathfrak{J}$	0.41716	0.36209	0.40391	0.48142	0.50999

previous methods, the proposed GDFOCP technique is highly accurate and easy to implement.

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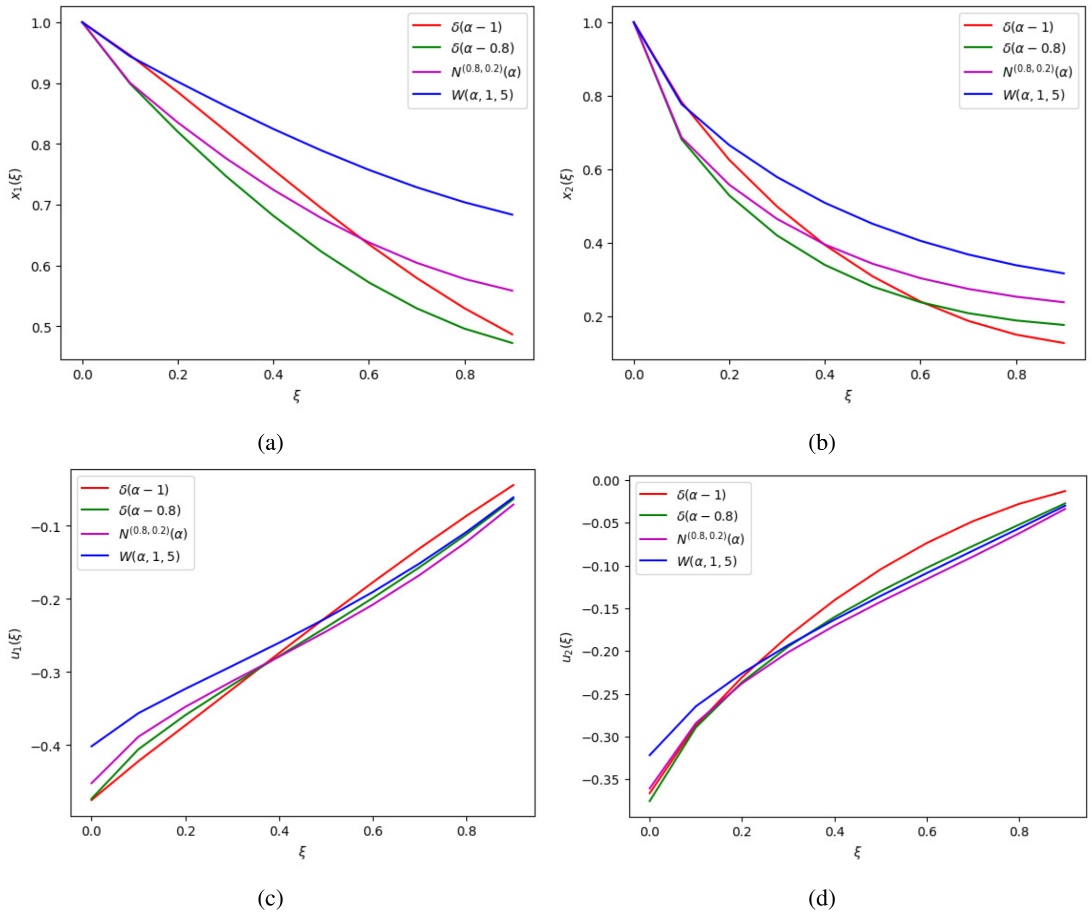

Figure 4: FLWs results of $x_1(\xi)$, $x_2(\xi)$, $u_1(\xi)$ and $u_2(\xi)$ for considered cases for Example 4.

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