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Atangana-Baleanu-Caputo (ABC), Caputo-Fabrizio (CF), and Caputo fractional derivative approaches in fuzzy time fractional cancer tumor growth Models

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Abstract. This article introduces a new approach for solving a time-fractional cancer tumor model using Caputo, Caputo-Fabrizio (CF), and Atangana-Baleanu-Caputo (ABC) fractional derivatives, accounting for varying net-killing rates of cancer cells in an uncertain environment. The model with the Caputo derivative is initially tackled using an explicit finite difference method (EFDM) with a fully time-dependent net killing rate. Approximate solutions for two different net death rates are obtained using the Sumudu transformation (ST) combined with the Adomian decomposition method (ADM), providing more accurate approximations than the EFDM. The model's behavior is analyzed with 2D and 3D visualizations. Convergence and error analysis of the method for the Caputo fractional derivative have been performed. The ADM provides reliable approximations for fractional models with fuzzy parameters, outperforming the EFDM by achieving lower absolute errors. The results exhibit symmetric lower and upper approximations around zero, effectively capturing the fuzzy nature of the solution. All methods converge to zero at higher cuts in fuzzy triangular numbers, i.e. $v = 1$.

Keywords: Adomian decomposition method (ADM), Sumudu transformation (ST), fuzzy set theory, time-fractional cancer tumor models
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1 Introduction

Effective cancer treatment necessitates understanding tumor growth and therapy responses, influenced by the tumor microenvironment and genetic changes. Predictive models guide personalized therapies,

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improving outcomes and reducing recurrence. Tumor progression analysis involves studying cell division rates, angiogenesis, and cancer cell evasion of apoptosis. Key treatment goals include disrupting these processes, preventing tumor growth, and enhancing the immune system's ability to combat cancer. Mathematical modeling, such as time-fractional tumor models, simulates the response of cancer cells to treatments, optimizing dosage and combination therapies. Understanding tumor heterogeneity can lead to more individualized cancer care strategies.

With experimental methods, statistical models like expectation maximization assess cancer tumor development and treatment response [20]. Burgess et al. explored diffusion coefficients and growth rates in spherical tumors, considering therapy-dependent death and proliferation rates [8]. Fractal dimensions and fractional derivatives measure complex shapes, nonlinear growth dynamics, and heterogeneous cell distribution, aiding in treatment planning [9].

Advanced techniques such as fractional calculus and fuzzy logic in cancer modeling have been studied. Fractional calculus (FC) models systems with memory and long-range dependencies, while fuzzy logic handles uncertainty. Fractional-order models offer accurate depictions of tumor growth and treatment responses. El-Sayed et al. [25] analyzed fractional-order models in cancer systems. Iomin [17] modeled tumor superdiffusion with fractional derivatives. Iyiola and Zaman [18] have performed a detailed analysis of fractional order derivatives in the tumor model. Vieira et al. [31] reviewed FC in cancer modeling. Padder et al. [23] discussed the Caputo fractional derivative in stability analysis and cancer modeling.

Fuzzy logic effectively tackles uncertainty, prevalent in various domains in medical fields like diagnosis, treatment planning, and monitoring. Fractional diffusion equations with intrinsic uncertainty have been studied by L.L. Huang et al. [16]. Tang et al. [27] investigated the negative impacts of treatment through FC in a breast cancer model, whereas Debbouche et al. [10] investigated chaos in fractional-order models of tumor growth. Uar and zdemir [28] developed fractional cancer-immune system models using Caputo derivatives. Yager and Filev [32] explored fuzzy logic in cancer dynamics simulation. Integrating fractional differential equations with fuzzy logic investigates drug administration dynamics and enzyme models. Innovative operators like the fuzzy fractional operator, combining fuzzy and fractional concepts, have been introduced by Abdollahi et al. [1]. Further, the human liver model [30] and the fractional smoking model [29] were recently studied in fuzzy sense by Lalchand Verma et al.

Solving cancer models, analytically or numerically, remains challenging. Havard et al. [19] proposed an analytical fuzzy transformation strategy for cancer models, employing Caputo Hukuhara's partial differentiability. Zureigat et al. [21] used explicit EFDM for the fuzzy fractional tumor model, considering numerous fuzzy instances. Numerical methods and Legendre polynomial-based operational matrices solve fuzzy fractional differential equations, offering efficient solutions for drug transduction models and other applications. Gandhi et al. [15] investigated chemotherapy effects with the reduced differential transform method, and Saadeh et al. [26] used Laplace transforms and the residual power series method to find the solution to the tumor model. This article studies a time-fractional cancer tumor model using Caputo, CF, and ABC fractional derivatives. We will analyze the varying net-killing rates of cancer cells in an uncertain environment using an analytical approach. First, the Caputo derivative model with a fully time-dependent net killing rate and an explicit finite difference method has been taken into consideration. This has inspired us to reevaluate this model, which takes into account various net killing rates, and also to solve this model with a new approach. Therefore, we approximate solutions for two different net death rates using the ST [5, 6] combined with the ADM [12], which provides more accurate approximations than EFDM.

2 Preliminaries on fuzzy set theory

Definition 1. *The function $\mu_{\tilde{P}}$, defined on the real numbers $\mathbb{R}$, assumes values in the interval $[0,1]$. A fuzzy set $\tilde{P}$ is described by the set of ordered pairs $(k,\mu_{\tilde{P}}(k))$ for all $k\in\mathbb{R}$, as detailed in [14]:*

$$\tilde{P}=\{(k,\mu_{\tilde{P}}(k)) \mid k\in\mathbb{R},\mu_{\tilde{P}}(k)\in[0,1]\}, \tag{1}$$

where the membership function for the fuzzy set $\tilde{P}$ at the point k is represented by $\mu_{\tilde{P}}(k)$. This membership function represents the degree of membership of k in $\tilde{P}$.

Definition 2. *A function $\tilde{P}:\mathbb{R}\to[0,1]$ defines a fuzzy number if it satisfies the following constraint [14], where, $\tilde{P}$ is normal, i.e. $\exists$ a point $k_0\in\mathbb{R}$ such that $\tilde{P}(k_0)=1$, $\tilde{P}(k)$ is a convex fuzzy set, $\tilde{P}(k)$ is upper semi-continuous on $\mathbb{R}$, and the closure of the set $\{k\in\mathbb{R}\mid\tilde{P}(k)>0\}$ is compact.*

Definition 3. *Fuzzy triangular numbers, denoted by $\tilde{P}(l_1,l_2,l_3)$, are defined as fuzzy numbers with a membership function given by [14]:*

$$\mu(k;l_1,l_2,l_3)=\begin{cases}0, & \text{if } k\le l_1,\\ \frac{k-l_1}{l_2-l_1}, & \text{if } l_1\le k\le l_2,\\ \frac{l_3-k}{l_3-l_2}, & \text{if } l_2\le k\le l_3,\\ 0, & \text{if } k\ge l_3.\end{cases} \tag{2}$$

Here, $\mu(k;l_1,l_2,l_3)$ is the membership function of $\tilde{P}$, representing the degree of membership of k. The v-level sets (or v-cuts) of $\tilde{P}$ are given by:

$$\tilde{P}_v=[l_1+(l_2-l_1)v,l_3-(l_3-l_2)v],\quad v\in[0,1]. \tag{3}$$

Furthermore, the v-cut of the fuzzy number $\tilde{P}$ is defined as:

$$\tilde{P}_v=\{k\in\mathbb{R}\mid\mu(k;l_1,l_2,l_3)\ge v\},\quad v\in[0,1]. \tag{4}$$

Definition 4. *If $\tilde{P}$ is a fuzzy number with its parametric form $\tilde{\mu}_v=[\underline{\mu}_v,\overline{\mu}_v]$, then the double parametric form can be expressed in crisp terms using the equation [14]:*

$$\tilde{\mu}(v,\alpha)=\alpha(\overline{\mu}_v-\underline{\mu}_v)+\underline{\mu}_v,\quad v,\alpha\in[0,1].$$

When $\alpha=0$, we have $\tilde{\mu}(v,0)=\underline{\mu}_v$, which represents the lower bound fuzzy number. Similarly, when $\alpha=1$, $\tilde{\mu}(v,1)=\overline{\mu}_v$, representing the upper bound fuzzy number.

Definition 5. *Let $\tilde{U}=[\underline{U},\overline{U}]$ and $\tilde{V}=[\underline{V},\overline{V}]$ be two fuzzy numbers, and let $x\in\mathbb{R}$ be a scalar. The following properties hold for fuzzy numbers [24]:*

- $\tilde{U}+\tilde{V}=(\underline{U}+\underline{V},\overline{U}+\overline{V})$ *and* $\tilde{U}-\tilde{V}=(\underline{U}-\underline{V},\overline{U}-\overline{V})$
- *For scalar multiplication:*

$$x\tilde{V}=\left\{\begin{cases}[x\underline{V},x\overline{V}], & \text{if}, x\ge 0\\ [x\overline{V},x\underline{V}], & \text{if}, x<0\end{cases}\right\}.$$

Definition 6. *[11] Let $\chi : S \times S \to \mathbb{R}$ be a fuzzy mapping and $c, d \in S$ be fuzzy numbers. The H-distance is defined as:*

$$\chi(c(\alpha), d(\alpha)) = \max\left\{ \sup_{\alpha\in[0,1]} |\underline{c}(\alpha) - \underline{d}(\alpha)|, \sup_{\alpha\in[0,1]} |\overline{c}(\alpha) - \overline{d}(\alpha)| \right\}. \tag{5}$$

Definition 7. *[2] Let there exist a function $h : [u_1, u_2] \to S$ that is fuzzy continuous at a point $a_0 \in [u_1, u_2]$. For each $\varepsilon > 0$, there exists an $\alpha > 0$ such that*

$$\chi(h(a), h(a_0)) < \varepsilon \quad \textit{whenever} \quad |a - a_0| < \alpha.$$

Definition 8. *If $\exists\, \beta \in S$ such that for any two fuzzy numbers, $w_1, w_2 \in S$, satisfying $w_2 = w_1 + \beta$, then β is known as the H-difference of w_1 and w_2, denoted by $w_1 \ominus w_2$. Further details regarding the H-difference and derivatives can be found in [3].*

Definition 9. *In the Caputo sense, the fuzzy-valued function $\tilde{u}(k)$ of order $\rho \in \mathbb{R}^+$ has a $g\mathscr{H}$-fractional derivative given by [22]:*

$$ {}^{C}_{g\mathscr{H}}D^{\rho}_{k}\tilde{u}(k) = \begin{cases} \dfrac{1}{\Gamma(n-\rho)} \int_0^k (k-p)^{n-\rho-1} \frac{\partial^\rho \tilde{u}(p)}{\partial p^\rho} dp, & \textit{if } n-1 < \rho < n, \\ \frac{d^\rho \tilde{u}(p)}{dp^\rho}, & \textit{if } \rho = n. \end{cases} \tag{6}$$

Definition 10. *Assume $f \in \mathscr{H}^1(a,b)$, for $\rho \in (0,1]$ Then, in the Caputo meaning, the AB fractional derivative is defined as [7]:*

$$ {}^{ABC}_{\ a}D^{\rho}_{t}(f(t)) = \frac{B(\rho)}{1-\rho} \int_a^t f'(x) E_\rho \left(-\rho \frac{(t-x)^\rho}{1-\rho} \right) dx. \tag{7}$$

In the case when $B(0) = B(1) = 1$, $B(\rho)$ is a normalization function.

Definition 11. *Assume $f \in \mathscr{H}^1(a,b)$, for $\rho \in (0,1]$ Then, in the Caputo interpretation, the CF fractional derivative is defined as [7]:*

$$ {}^{CF}_{\ a}D^{\rho}_{t}(f(t)) = \frac{B(\rho)}{1-\rho} \int_a^t f'(x) e^{\frac{-\rho}{1-\rho}(t-x)} dx. \tag{8}$$

In the case when $B(0) = B(1) = 1$, $B(\rho)$ is a normalization function.

Definition 12. *The ST coupled with $g\mathscr{H}$-Caputos time-fractional derivative ${}^{C}_{g\mathscr{H}}D^{\rho}_{t}$ is given as [7]:*

$$\mathscr{S}[{}^{C}_{g\mathscr{H}}D^{\rho}_{t}\tilde{u}(x,t)] = \left\{ \frac{\mathscr{S}\{\tilde{u}(x,t)\}}{s^\rho} - \sum_{k=0}^{n-1} \frac{\tilde{u}^k(x,0)}{s^{\rho-k}} \right\}, n-1 < \rho \leq n. \tag{9}$$

Definition 13. *The ST of the CF derivative ($G(s)$) is given as [6]:*

$$\mathscr{S}[{}^{CF}_{\ a}D^{\alpha}_{t}(f(t))] = \frac{B(\alpha)}{1-\alpha(1-s)}(G(s) - f(0)). \tag{10}$$

Definition 14. *The ST of the ABC derivative ($G(s)$) is given as [6]:*

$$\mathscr{S}[{}^{ABC}_{\ a}D^{\alpha}_{t}(f(t))] = \frac{B(\alpha)}{1-\alpha+\alpha s^\alpha}(G(s) - f(0)). \tag{11}$$

Definition 15. *The ST of t^m is given below [6]:*

$$\mathscr{S}\left(\frac{t^m}{m!}\right) = u^m, \quad m = 0, 1, 2, \ldots \textit{and } \mathscr{S}\left(\frac{t^\beta}{\Gamma(\beta+1)}\right) = u^\beta, \beta > -1. \tag{12}$$

3 Methodology overview

Let us take the time-fractional fuzzy cancer tumor differential equation below to demonstrate the methodology:

$$\frac{\partial^{\alpha}\tilde{U}(\zeta,\tau,\alpha)}{\partial\tau^{\alpha}} = \frac{\partial^{2}\tilde{U}(\zeta,\tau)}{\partial\zeta^{2}} - \tilde{K}(\zeta,\tau)\tilde{U}(\zeta,\tau), \quad 0<\alpha\leq 1, \ \ (\zeta,\tau)\in\omega=[0,X]\times[0,T], \tag{13}$$

equipped with boundary conditions as:

$$\tilde{U}(\zeta,0)=\tilde{g}(\zeta), \quad \tilde{U}(0,\tau)=\tilde{m}(0,\tau), \quad \tilde{U}(X,\tau)=\tilde{n}(l,\tau), \tag{14}$$

where, $\tilde{U}(\zeta,\tau,\alpha)$ is the fuzzy concentration of the tumor cells at time t and a fractional order α, and $\tilde{K}(\zeta,\tau)$ is the fuzzy net death rate of the tumor cells of the crisp variable τ and ζ. The time-fractional derivative of order α [4] is $\frac{\partial^{\alpha}\tilde{U}(\zeta,\tau,\alpha)}{\partial\tau^{\alpha}}$ in the Caputo sense and $\frac{\partial^{2}\tilde{U}(\zeta,\tau)}{\partial\zeta^{2}}$ denotes the fuzzy partial Hukuhara derivative concerning x. Furthermore, as stated in [13], the fuzzy functions $\tilde{K}(\zeta,t)$ and $\tilde{g}(\zeta)$ are determined as:

$$\tilde{K}(\zeta,\tau)=\tilde{v}_1 s_1(\zeta,\tau) \ \text{ and } \ \tilde{g}(\zeta)=\tilde{v}_2 s_2(\zeta), \tag{15}$$

where the fuzzy convex numbers are represented by the variables $\tilde{v}_1$ and $\tilde{v}_2$, and the crisp functions $s_1(\zeta,\tau)$ and $s_2(\zeta,\tau)$ of the crisp variables ζ and τ.
Let us do defuzzification of the given equation for all $v\in[0,1]$ in the interval form as [13]:

$$\begin{aligned}\left[\frac{\partial^{\alpha}\underline{U}(\zeta,\tau,\alpha,v)}{\partial\tau^{\alpha}},\frac{\partial^{\alpha}\overline{U}(\zeta,\tau,\alpha,v)}{\partial\tau^{\alpha}}\right] &= \left[\frac{\partial^{2}\underline{U}(\zeta,\tau;v)}{\partial\zeta^{2}},\frac{\partial^{2}\overline{U}(\zeta,\tau,v)}{\partial\zeta^{2}}\right] \\ &\quad -[\underline{K}(\zeta,\tau,v),\overline{K}(\zeta,\tau,v)][\underline{U}(\zeta,\tau,v),\overline{U}(\zeta,\tau,v)],\end{aligned} \tag{16}$$

equipped with the following fuzzy conditions:

$$\begin{aligned}\left[\underline{U}(\zeta,0;v),\overline{U}(\zeta,0;v)\right] &= \left[\underline{g}(\zeta;v),\overline{g}(\zeta;v)\right], \\ \left[\underline{U}(0,\tau;v),\overline{U}(0,\tau;v)\right] &= \left[\underline{m}(0,\tau;v),\underline{m}(0,\tau;v)\right], \\ \left[\underline{U}(X,\tau;v),\overline{U}(X,\tau;v)\right] &= \left[\underline{n}(l,\tau;v),\overline{n}(l,\tau;v)\right].\end{aligned}$$

Using β, a double parametric form with $\beta\in[0,1]$, we obtain

$$\begin{aligned}&\left\{\beta\left(\frac{\partial^{\alpha}\overline{U}(\zeta,\tau,\alpha,v)}{\partial\tau^{\alpha}}-\frac{\partial^{\alpha}\underline{U}(\zeta,\tau,\alpha,v)}{\partial\tau^{\alpha}}\right)+\frac{\partial^{\alpha}\underline{U}(\zeta,\tau,\alpha,v)}{\partial\tau^{\alpha}}\right\} \\ &\quad = \left\{\beta\left(\frac{\partial^{2}\overline{U}(\zeta,\tau,v)}{\partial\zeta^{2}}-\frac{\partial^{2}\underline{U}(\zeta,\tau;v)}{\partial\zeta^{2}}\right)+\frac{\partial^{2}\underline{U}(\zeta,\tau;v)}{\partial\zeta^{2}}\right\}-[(\beta(\overline{K}(\zeta,\tau,v) \\ &\qquad -\underline{K}(\zeta,\tau,v))+\underline{K}(\zeta,\tau,v))(\beta(\overline{U}(\zeta,\tau,v)-\underline{U}(\zeta,\tau,v))+\underline{U}(\zeta,\tau,v))].\end{aligned} \tag{17}$$

Depending on fuzzy starting and boundary conditions, we have

$$\begin{aligned}\beta(\overline{U}(\zeta,0;v)-\underline{U}(\zeta,0;v))+\underline{U}(\zeta,0;v) &= \beta(\overline{g}(\zeta;v)-\underline{g}(\zeta;v))+\underline{g}(\zeta;v), \\ \beta(\overline{U}(0,\tau;v)-\underline{U}(0,\tau;v))+\underline{U}(0,\tau;v) &= \beta(\underline{m}(0,\tau;v)-\underline{m}(0,\tau;v))+\underline{m}(0,\tau;v), \\ \beta(\overline{U}(X,\tau;v)-\underline{U}(X,\tau;v))+\underline{U}(X,\tau;v) &= \beta(\overline{n}(l,\tau;v)-\underline{n}(l,\tau;v))+\underline{n}(l,\tau;v).\end{aligned}$$

By performing the ST to equation (17), we obtain

$$\frac{1}{s^\rho}\mathscr{S}\left\{\beta\left(\frac{\partial^\alpha\overline{U}(\zeta,\tau,\alpha,v)}{\partial\tau^\alpha}-\frac{\partial^\alpha\underline{U}(\zeta,\tau,\alpha,v)}{\partial\tau^\alpha}\right)+\frac{\partial^\alpha\underline{U}(\zeta,\tau,\alpha,v)}{\partial\tau^\alpha}\right\}-\frac{1}{s^\rho}\left\{\beta(\overline{U}(\zeta,0;v)-\underline{U}(\zeta,0;v))+\underline{U}(\zeta,0;v)\right\}$$
$$=\mathscr{S}\left\{\beta\left(\frac{\partial^2\overline{U}(\zeta,\tau,v)}{\partial\zeta^2}-\frac{\partial^2\underline{U}(\zeta,\tau;v)}{\partial\zeta^2}\right)+\frac{\partial^2\underline{U}(\zeta,\tau;v)}{\partial\zeta^2}\right\}$$
$$-\mathscr{S}\left\{[\beta(\overline{K}(\zeta,\tau,v)-\underline{K}(\zeta,\tau,v))+\underline{K}(\zeta,\tau,v)][\beta(\overline{U}(\zeta,\tau,v)-\underline{U}(\zeta,\tau,v))+\underline{U}(\zeta,\tau,v)]\right\},$$

i.e.

$$\begin{aligned}&\left\{\beta\left(\overline{U}(\zeta,s,\alpha,v)-\underline{U}(\zeta,s,\alpha,v)\right)+\underline{U}(\zeta,s,\alpha,v)\right\}\\&\quad=\left\{\beta(\underline{U}(\zeta,0;v)-\underline{U}(\zeta,0;v))+\underline{U}(\zeta,0;v)\right\}+s^\rho\mathscr{S}\left\{\beta\left(\frac{\partial^2\overline{U}(\zeta,\tau,v)}{\partial\zeta^2}-\frac{\partial^2\underline{U}(\zeta,\tau;v)}{\partial\zeta^2}\right)\right.\\&\quad\left.+\frac{\partial^2\underline{U}(\zeta,\tau;v)}{\partial\zeta^2}\right\}-s^\rho\mathscr{S}N(\tilde{U}).\end{aligned}\tag{18}$$

By performing the inverse ST to the above equation i.e. (18), we obtain

$$\begin{aligned}&\left\{\beta\left(\overline{U}(\zeta,\tau,\alpha,v)-\underline{U}(\zeta,\tau,\alpha,v)\right)+\underline{U}(\zeta,\tau,\alpha,v)\right\}\\&\quad=\left\{\beta(\overline{U}(\zeta,0;v)-\underline{U}(\zeta,0;v))+\underline{U}(\zeta,0;v)\right\}+\mathscr{S}^{-1}\left\{s^\rho\mathscr{S}\left\{\beta\left(\frac{\partial^2\overline{U}(\zeta,\tau,v)}{\partial\zeta^2}\right.\right.\right.\\&\quad\left.\left.\left.-\frac{\partial^2\underline{U}(\zeta,\tau;v)}{\partial\zeta^2}\right)+\frac{\partial^2\underline{U}(x,\tau;v)}{\partial\zeta^2}\right\}\right\}-\mathscr{S}^{-1}\{s^\rho\mathscr{S}N(\tilde{u})\}.\end{aligned}\tag{19}$$

Decompose the solution into series form as:

$$\begin{aligned}\tilde{U}(\zeta,\tau;v,\beta)&=\beta(\overline{U}(\zeta,\tau,\alpha,v)-\underline{U}(\zeta,\tau,\alpha,v))+\underline{U}(\zeta,\tau,\alpha,v)\\&=\sum_{n=0}^{\infty}\beta(\overline{U}_n(\zeta,\tau;v,\beta)-\overline{u}_n(\zeta,\tau;v,\beta))+\overline{U}_n(\zeta,\tau;v,\beta)\\&=\sum_{n=0}^{\infty}\tilde{U}_n\end{aligned}\tag{20}$$

and decompose the $N\tilde{U}(\zeta,\tau)$, which is a nonlinear term, by Adomian polynomials (AP) [12] as stated below:

$$N\tilde{U}(\zeta,\tau)=\sum_{m=0}^{\infty}\tilde{A}_m,$$

$\tilde{A}_0=\mathscr{N}\tilde{U}_0$ and where, $\tilde{A}_m$ is called AP computed as:

$$\tilde{A}_m=\frac{1}{m!}\frac{d^m}{d\lambda^m}\mathscr{N}(\sum_{l=0}^{m}\lambda^l\tilde{U}_l)|_{\lambda=0}.\tag{21}$$

The following recurrence formula has been obtained:

$$\begin{aligned}
\tilde{U}_0 &= \{\beta(\overline{U}_0(\zeta,\tau;v)-\underline{U}_0(\zeta,\tau;v))+\underline{U}_0(\zeta,\tau;v)\},\\
\left\{\beta\left(\frac{\partial^\alpha\overline{U}_n(\zeta,\tau,\alpha,v)}{\partial\tau^\alpha}-\frac{\partial^\alpha\underline{U}_n(\zeta,\tau,\alpha,v)}{\partial\tau^\alpha}\right)+\frac{\partial^\alpha\underline{U}_n(\zeta,\tau,\alpha,v)}{\partial\tau^\alpha}\right\}&\\
&= \mathscr{S}^{-1}\left\{s^\rho\mathscr{S}\left\{\beta\left(\frac{\partial^2\overline{U}_n(\zeta,\tau,v)}{\partial\zeta^2}-\frac{\partial^2\underline{U}_n(\zeta,t;v)}{\partial\zeta^2}\right)+\frac{\partial^2\underline{U}_n(\zeta,\tau;v)}{\partial\zeta^2}\right\}\right\}\\
&\quad -\mathscr{S}^{-1}\{s^\rho\mathscr{S}\{\beta(\overline{A}_{n-1}-\underline{A}_{n-1})+\underline{A}_{n-1}\}\},\quad n\geq 1.
\end{aligned}$$

Hence to solve the above equation, we obtain the general series solution as follows:

$$\tilde{U}(\zeta,\tau;v,\beta)=\tilde{U}_0(\zeta,\tau;v,\beta)+\tilde{U}_1(\zeta,\tau;v,\beta)+\tilde{U}_2(\zeta,\tau;v,\beta)+\tilde{U}_3(\zeta,\tau;v,\beta)+\cdots, \tag{22}$$

where the upper bound solution is given by $\beta=1$ and the lower bound solution by $\beta=0$.

4 Modeling with CF fractional derivative

Let us reconsider the model possessing the CF fractional derivative as follows:

$$\frac{\partial^\alpha\tilde{U}(\zeta,\tau,\alpha)}{\partial\tau^\alpha}=\frac{\partial^2\tilde{U}(\zeta,\tau)}{\partial\zeta^2}-\tilde{K}(\zeta,\tau)\tilde{u}(\zeta,\tau),\quad 0<\alpha\leq 1,\ \ (\zeta,\tau)\in\omega=[0,X]\times[0,T], \tag{23}$$

equipped with boundary conditions:

$$\tilde{U}(\zeta,0)=\tilde{g}(\zeta),\quad \tilde{U}(0,t)=\tilde{m}(0,\tau),\quad \tilde{U}(X,\tau)=\tilde{n}(l,t). \tag{24}$$

Instead of considering the Caputo fractional derivative, we prefer the CF derivative. Here, $\frac{\partial^\alpha\tilde{U}(\zeta,\tau,\alpha)}{\partial\tau^\alpha}$ is the derivative of CF. Therefore, on considering the CF derivative, the revised iteration scheme is as follows:

$$\begin{aligned}
\tilde{U}_0 &= \{\beta(\overline{U}_0(\zeta,\tau;v)-\underline{U}_0(\zeta,\tau;v))+\underline{U}_0(\zeta,\tau;v)\}\\
\left\{\beta\left(\frac{\partial^\alpha\overline{U}_n(\zeta,\tau,\alpha,v)}{\partial\tau^\alpha}-\frac{\partial^\alpha\underline{U}_n(\zeta,\tau,\alpha,v)}{\partial\tau^\alpha}\right)+\frac{\partial^\alpha\underline{U}_n(\zeta,\tau,\alpha,v)}{\partial t^\alpha}\right\}&\\
&= \mathscr{S}^{-1}\left\{\frac{B(\alpha)}{1-\alpha(1-s)}\mathscr{S}\left\{\beta\left(\frac{\partial^2\overline{U}_n(\zeta,\tau,v)}{\partial\zeta^2}-\frac{\partial^2\underline{U}_n(\zeta,\tau;v)}{\partial\zeta^2}\right)+\frac{\partial^2\underline{U}_n(\zeta,\tau;v)}{\partial\zeta^2}\right\}\right\}\\
&\quad -\mathscr{S}^{-1}\left\{\frac{B(\alpha)}{1-\alpha(1-s)}\mathscr{S}\{\beta(\overline{A}_{n-1}-\underline{A}_{n-1})+\underline{A}_{n-1}\}\right\},\ \ n\geq 1.
\end{aligned}$$

Similarly, we can obtain the general series solution as follows:

$$\tilde{U}(x,\tau;v,\beta)=\tilde{U}_0(\zeta,\tau;v,\beta)+\tilde{U}_1(\zeta,\tau;v,\beta)+\tilde{U}_2(\zeta,\tau;v,\beta)+\tilde{U}_3(\zeta,\tau;v,\beta)+\cdots. \tag{25}$$

5 Modeling with ABC fractional derivative

Now, we consider the model possessing the ABC fractional derivative as follows:

$$\frac{\partial^{\alpha}\tilde{U}(\zeta,\tau,\alpha)}{\partial\tau^{\alpha}}=\frac{\partial^{2}\tilde{U}(\zeta,\tau)}{\partial\zeta^{2}}-\tilde{K}(\zeta,\tau)\tilde{U}(\zeta,\tau),0<\alpha\leq 1,\ (\zeta,\tau)\in\omega=[0,X]\times[0,T], \tag{26}$$

equipped with: $\tilde{U}(\zeta,0)=\tilde{g}(\zeta)$, $\tilde{U}(0,\tau)=\tilde{m}(0,\tau)$, $\tilde{U}(X,\tau)=\tilde{n}(l,\tau)$. Instead of considering the Caputo fractional derivative, we prefer to use the ABC derivative. Here, $\frac{\partial^{\alpha}\tilde{U}(\zeta,\tau,\alpha)}{\partial\tau^{\alpha}}$ is the fractional derivative of ABC. Therefore, considering the ABC derivative, the revised iteration scheme is as follows:

$$\begin{aligned}
\tilde{U}_0 &= h\quad\{\beta(\overline{U}_0(\zeta,\tau;v)-\underline{U}_0(\zeta,\tau;v))+\underline{U}_0(\zeta,\tau;v)\},\\
&\left\{\beta\left(\frac{\partial^{\alpha}\overline{U}_n(\zeta,\tau,\alpha,v)}{\partial\tau^{\alpha}}-\frac{\partial^{\alpha}\underline{U}_n(\zeta,\tau,\alpha,v)}{\partial\tau^{\alpha}}\right)+\frac{\partial^{\alpha}\underline{U}_n(\zeta,\tau,\alpha,v)}{\partial\tau^{\alpha}}\right\}\\
&= \mathscr{S}^{-1}\left\{\frac{B(\alpha)}{1-\alpha+\alpha s^{\alpha}}\mathscr{S}\left\{\beta\left(\frac{\partial^{2}\overline{U}_n(\zeta,\tau,v)}{\partial\zeta^{2}}-\frac{\partial^{2}\underline{U}_n(\zeta,\tau;v)}{\partial\zeta^{2}}\right)+\frac{\partial^{2}\underline{U}_n(\zeta,\tau;v)}{\partial\zeta^{2}}\right\}\right\}\\
&\quad-\mathscr{S}^{-1}\left\{\frac{B(\alpha)}{1-\alpha+\alpha s^{\alpha}}\mathscr{S}\{\beta(\overline{A}_{n-1}-\underline{A}_{n-1})+\underline{A}_{n-1}\}\right\},\quad n\geq 1.
\end{aligned} \tag{27}$$

Similarly, we can obtain the general series solution as follows:

$$\tilde{U}(\zeta,\tau;v,\beta)=\tilde{U}_0(\zeta,\tau;v,\beta)+\tilde{U}_1(\zeta,\tau;v,\beta)+\tilde{U}_2(\zeta,\tau;v,\beta)+\tilde{U}_3(\zeta,\tau;v,\beta)+\cdots. \tag{28}$$

6 Analysis of convergence and errors

Theorem 1. *Let us consider that there are two functions $\tilde{u}_n(\zeta,\tau;v,\beta)$ and $\tilde{u}(\zeta,\tau;v,\beta)$ in the Banach space $(\mathscr{B}[0,\mathscr{T}],\|.\|)$. If $0<k<1$, consequently, the solution of equation (13) converges to the series solution (20).*

Proof. Let us prove that the sequence of partial sums $\mathscr{S}_m$ of series (20) is a Cauchy sequence in Banach space $(\mathscr{B}[0,\mathscr{T}],\|.\|)$, as follows:

$$\begin{aligned}
\|\mathscr{S}_{m+1}(\zeta,\tau;v,\beta)-\mathscr{S}_m(\zeta,\tau;v,\beta)\| &= \|\tilde{u}_{m+1}(\zeta,\tau;v,\beta)\|\\
&\leq k\|\tilde{u}_m(\zeta,\tau;v,\beta)\|\\
&\leq k^2\|\tilde{u}_{m-1}(\zeta,\tau;v,\beta)\|\\
&\vdots\\
&\leq k^{m+1}\|\tilde{u}_0(\zeta,\tau;v,\beta)\|.
\end{aligned} \tag{29}$$

Now, by choosing arbitrarily partial sums $\mathscr{S}_m$ and $\mathscr{S}_n$ corresponding to two natural numbers m,n where $m\geq n$ and utilizing the triangular inequality, we get

$$\begin{aligned}
\|\mathscr{S}_m-\mathscr{S}_n\| &= \|(\mathscr{S}_m(x,\tau;v,\beta)-\mathscr{S}_{m-1}(\zeta,\tau;v,\beta))+(\mathscr{S}_{m-1}(\zeta,t;v,\beta)-\mathscr{S}_{m-2}(\zeta,\tau;v,\beta))+\cdots\\
&\quad+(\mathscr{S}_{n+1}(\zeta,\tau;v,\beta)-\mathscr{S}_n(\zeta,\tau;v,\beta))\|\\
&\leq \|(\mathscr{S}_m(\zeta,\tau;v,\beta)-\mathscr{S}_{m-1}(\zeta,\tau;v,\beta))\|+\|(\mathscr{S}_{m-1}(\zeta,\tau;v,\beta)-\mathscr{S}_{m-2}(\zeta,\tau;v,\beta))\|+\cdots\\
&\quad+\|(\mathscr{S}_{n+1}(\zeta,\tau;v,\beta)-\mathscr{S}_n(\zeta,\tau;v,\beta))\|
\end{aligned} \tag{30}$$

Using (29) we get

$$\begin{aligned}\|\mathscr{S}_m - \mathscr{S}_n\| &\leq k^m \|(\tilde{u}_0(\zeta,\tau;v,\beta)\| + k^{m-1}\|\tilde{u}_0(\zeta,\tau;v,\beta)\| + \cdots + k^{n+1}\|\tilde{u}_0(\zeta,\tau;v,\beta)\| \\ &\leq (k^m + k^{m-1} + \cdots + k^{n+1})\|\tilde{u}_0(\zeta,\tau;v,\beta)\| \\ &\leq k^{n+1}(k^{m-n-1} + k^{m-n-2} + \cdots + k + 1)\|\tilde{u}_0(\zeta,\tau;v,\beta)\| \\ &\leq k^{n+1}\left(\frac{1-k^{m-1}}{1-k}\right)\|\tilde{u}_0(\zeta,\tau;v,\beta)\|. \end{aligned} \tag{31}$$

For $0 < k < 1$ we have $1 - k^{m-1} < 1$. Therefore

$$\|\mathscr{S}_m - \mathscr{S}_n\| \leq \frac{k^{n+1}}{1-k} \max_{t\in[0,T]} |\tilde{u}_0(\zeta,t;v,\beta)|. \tag{32}$$

As $\tilde{u}_0(\zeta,\tau;v,\beta)$ is bounded, thus $\lim_{n,m\to 0}\|\mathscr{S}_m - \mathscr{S}_n\| = 0$. As a result, the sequence $\mathscr{S}_m$ converges since it is a Cauchy sequence. □

Theorem 2. *For the solution (20) the error bounds are:*

$$|\tilde{u}(\zeta,\tau;v,\beta) - \sum_{i=0}^{n} \tilde{u}_i(\zeta,\tau;v,\beta)| \leq \frac{k^{n+1}}{1-k} max_{\tau\in[0,T]}\|\tilde{u}_0(\zeta,\tau;v,\beta)\| \tag{33}$$

Proof. From (32) we have

$$|\tilde{u}(\zeta,\tau;v,\beta) - \mathscr{S}_n(\zeta,\tau;v,\beta)| \leq k^{n+1}\left(\frac{1-k^{m-1}}{1-k}\right)\|\tilde{u}_0(\zeta,\tau;v,\beta)\|. \tag{34}$$

Since $k \in (0,1)$ and $1 - k^{m-1} < 1$, we have

$$|\tilde{u}(\zeta,\tau;v,\beta) - \sum_{i=0}^{n} \tilde{u}_i(\zeta,\tau;v,\beta)| \leq \frac{k^{n+1}}{1-k} max_{\tau\in[0,T]}\|\tilde{u}_0(\zeta,\tau;v,\beta)\|. \tag{35}$$

□

7 Application to numerical problems

Example 1. Consider the following fuzzy tumor model [33] in which the net killing rate is time-dependent only

$${}_0^C D_t^{\alpha} \tilde{u}(x,t,\alpha) = \frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} - t^2 \tilde{u}(x,t) \quad t \geq 1,\ 0 < \alpha \leq 1, \ \text{ with } \ \tilde{u}(x,0) = \tilde{\phi}(v,\beta)e^{kx}, \tag{36}$$

where $\tilde{\phi}(v,\beta) = \beta((1-v)-(v-1)) + (v-1) \quad v,\beta \in [0,1]$. Also, exact solution of this problem is given by [26]:

$$\tilde{u}(x,t,\alpha) = \tilde{\phi}(v,\beta)e^{kx} + k^2\tilde{\phi}(v,\beta)e^{kx}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + k^4\tilde{\phi}(v,\beta)e^{kx}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

The few terms are calculated below:

$$
\begin{aligned}
\tilde{u}_0(x,t,\alpha) &= \tilde{u}(x,0) = \tilde{\phi}(v,\beta)e^{kx} \\
\tilde{u}_1(x,t,\alpha) &= k^2\tilde{\phi}(v,\beta)e^{kx}\frac{1}{\Gamma(\alpha+1)}t^{\alpha} - \tilde{\phi}(v,\beta)e^{kx}\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{2+\alpha}, \\
\tilde{u}_2(x,t,\alpha) &= \tilde{\phi}(v,\beta)e^{kx}\Bigg[-k^2\left(\frac{\Gamma(3)}{\Gamma(2\alpha+3)} + \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)\Gamma(2\alpha+3)}\right)t^{2+2\alpha} \\
&\qquad + k^4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + k\frac{t^{4+2\alpha}\Gamma(3)\Gamma(\alpha+5)}{\Gamma(\alpha+3)\Gamma(2\alpha+5)}\Bigg].
\end{aligned}
$$

Subsequently, the following is the complete approximate series solution (up to three terms):

$$
\begin{aligned}
\tilde{u}(x,t,\alpha) &= \sum_{i=0}^{2}\tilde{u}_i(x,t,\alpha) \\
&= \tilde{\phi}(v,\beta)e^{kx} + k^2\tilde{\phi}(v,\beta)e^{kx}\frac{1}{\Gamma(\alpha+1)}t^{\alpha} - \tilde{\phi}(v,\beta)e^{kx}\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{2+\alpha} \\
&\quad + k^4\tilde{\phi}(v,\beta)e^{kx}\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} - k^2\tilde{\phi}(v,\beta)e^{kx}\left(\frac{\Gamma(3)}{\Gamma(2\alpha+3)} + \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)\Gamma(2\alpha+3)}\right)t^{2+2\alpha} \\
&\quad + k\tilde{\phi}(v,\beta)e^{kx}\frac{\Gamma(3)\Gamma(\alpha+5)}{\Gamma(\alpha+3)\Gamma(2\alpha+5)}t^{4+2\alpha}.
\end{aligned} \tag{37}
$$

7.1 Solving the CF fractional derivative model

Now, considering the problem (36) equipped with the CF derivative as:

$$
{}^{CF}_{0}D^{\alpha}_{t}\tilde{u}(x,t,\alpha) = \frac{\partial^2\tilde{u}(x,t)}{\partial x^2} - t^2\tilde{u}(x,t), \quad t \geq 1,\ 0 < \alpha \leq 1.
$$

In a similar way as explained in Section 4, the first two terms are calculated below:

$$
\tilde{u}_0(x,t,\alpha) = \tilde{u}(x,0) = \tilde{\phi}(v,\beta)e^{kx}, \tag{38}
$$

$$
\tilde{u}_1(x,t,\alpha) = \frac{k^2\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)}(1-\alpha(1-t)) - \frac{\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)}\left(t^2 - \alpha(t^2 - \frac{t^3}{3})\right), \tag{39}
$$

$$
\begin{aligned}
\tilde{u}_2(x,t,\alpha) &= \frac{k^4\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}(1-2\alpha+2\alpha t+\alpha^2+\frac{\alpha^2t^2}{2}-2\alpha^2 t) - \frac{k^2\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2} \\
&\quad \times\left(2t^2 - 2\alpha t^2 + \frac{4\alpha t^3}{3} - 2\alpha t^2 + \frac{4\alpha t^3}{3!} + 2\alpha^2t^2 - \frac{4\alpha^2t^3}{3} - \frac{4\alpha^3t^3}{3!} + \frac{3!\alpha^2t^4}{4!} + \frac{3!\alpha^2t^5}{3\times 5!}\right) \\
&\quad + \frac{\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}\left(t^4 - 2\alpha t^4 + \alpha^2t^4 + \frac{\alpha t^5}{3} - \frac{\alpha^2t^5}{3} + \frac{4!\alpha t^5}{5!} - \frac{4!\alpha^2t^5}{5!} + \frac{5!\alpha^2t^6}{3\times 6!}\right).
\end{aligned} \tag{40}
$$

Thus, a three-term approximate solution is given as:

$$\begin{aligned}\tilde{u}(x,t,\alpha) &= \tilde{\phi}(v,\beta)e^{kx} + \frac{k^2\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)}(1-\alpha(1-t)) - \frac{\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)}\left(t^2 - \alpha(t^2 - \frac{t^3}{3})\right)\\ &+\frac{k^4\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}(1-2\alpha+2\alpha t+\alpha^2+\frac{\alpha^2t^2}{2}-2\alpha^2 t)\\ &-\frac{k^2\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}\left(2t^2-2\alpha t^2+\frac{4\alpha t^3}{3}-2\alpha t^2+\frac{4\alpha t^3}{3!}+2\alpha^2t^2-\frac{4\alpha^2t^3}{3}-\frac{4\alpha^3t^3}{3!}+\frac{3!\alpha^2 2t^4}{4!}\right)\\ &+\frac{\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}\left(t^4-2\alpha t^4+\alpha^2t^4+\frac{\alpha t^5}{3}-\frac{\alpha^2t^5}{3}+\frac{4!\alpha t^5}{5!}-\frac{4!\alpha^2t^5}{5!}+\frac{5!\alpha^2t^6}{3\times 6!}\right).\end{aligned} \tag{41}$$

7.2 Solving the ABC fractional derivative model

Now considering the problem (36) equipped with the ABC derivative as

$$ {}^{ABC}_{0}D_t^{\alpha}\tilde{u}(\zeta,t,\alpha) = \frac{\partial^2\tilde{u}(x,t)}{\partial x^2} - t^2\tilde{u}(x,t) \quad t\geq 1,\ 0<\alpha\leq 1,$$

In a similar way as explained in the Section 5, the first two terms are calculated below:

$$\begin{aligned}\tilde{u}_0(x,t,\alpha) &= \tilde{u}(x,0) = \tilde{\phi}(v,\beta)e^{kx},\\ \tilde{u}_1(x,t,\alpha) &= \frac{k^2\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)}\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right) - 2\frac{\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)}\left(\frac{(1-\alpha)t^2}{2}+\frac{\alpha t^{\alpha+2}}{\Gamma(\alpha+3)}\right),\\ \tilde{u}_2(x,t,\alpha) &= \frac{k^4\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}\left[(1-\alpha)^2+2\alpha(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\alpha^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\\ &\quad -\frac{2k^2\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}\left[(1-\alpha)^2t^2+\left(\frac{2\alpha(1-\alpha)}{\Gamma(\alpha+3)}+\frac{\alpha(1-\alpha)}{2\Gamma(\alpha+1)}+\frac{\alpha(1-\alpha)}{\Gamma(\alpha+3)}\right)t^{\alpha+2}\right.\\ &\quad \left.+\left(\frac{\alpha^2}{\Gamma(2\alpha+3)}+\frac{\alpha^2\Gamma(\alpha+3)}{2\Gamma(\alpha+1)\Gamma(2\alpha+3)}\right)t^{2\alpha+2}\right]+\frac{2\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}\left[\frac{(1-\alpha)^2t^4}{2}\right.\\ &\quad \left.+\left(\frac{\alpha(1-\alpha)\Gamma(\alpha+4)}{\Gamma(\alpha+3)\Gamma(\alpha+5)}+\frac{\alpha(1-\alpha)\Gamma(5)}{2\Gamma(\alpha+5)}\right)t^{4+\alpha}+\frac{\alpha^2\Gamma(\alpha+4)}{\Gamma(\alpha+3)\Gamma(2\alpha+5)}t^{4+2\alpha}\right].\end{aligned}$$

Thus, a three-term approximate solution is given as:

$$\begin{aligned}\tilde{u}(x,t,\alpha) &= \tilde{\phi}(v,\beta)e^{kx} + \frac{k^2\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)}\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right) - 2\frac{\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)}\left(\frac{(1-\alpha)t^2}{2}+\frac{\alpha t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\\ &+\frac{k^4\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}\left[(1-\alpha)^2+2\alpha(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\alpha^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\\ &-\frac{2k^2\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}\left[(1-\alpha)^2t^2+\left(\frac{2\alpha(1-\alpha)}{\Gamma(\alpha+3)}+\frac{\alpha(1-\alpha)}{2\Gamma(\alpha+1)}+\frac{\alpha(1-\alpha)}{\Gamma(\alpha+3)}\right)t^{\alpha+2}\right.\\ &\left.+\left(\frac{\alpha^2}{\Gamma(2\alpha+3)}+\frac{\alpha^2\Gamma(\alpha+3)}{2\Gamma(\alpha+1)\Gamma(2\alpha+3)}\right)t^{2\alpha+2}\right]+\frac{2\tilde{\phi}(v,\beta)e^{kx}}{B(\alpha)^2}\left[\frac{(1-\alpha)^2t^4}{2}\right.\end{aligned}$$

$$+\left(\frac{\alpha(1-\alpha)\Gamma(\alpha+4)}{\Gamma(\alpha+3)\Gamma(\alpha+5)}+\frac{\alpha(1-\alpha)\Gamma(5)}{2\Gamma(\alpha+5)}\right)t^{4+\alpha}+\frac{\alpha^2\Gamma(\alpha+4)}{\Gamma(\alpha+3)\Gamma(2\alpha+5)}t^{4+2\alpha}\Bigg].$$

Example 2. Take into consideration the following tumor model, where the net mortality rate is dependent on the cell concentration:

$${}_0^C D_t^\alpha \tilde{u}(x,t,\alpha)=\frac{\partial^2\tilde{u}(x,t)}{\partial x^2}-\frac{2}{x}\frac{\partial\tilde{u}(x,t)}{\partial x}-\tilde{u}^2(x,t)\quad t>0,\ 0<x\leq 1\ ,0<\alpha\leq 1, \tag{42}$$

equipped with $\tilde{u}(x,0)=\tilde{\phi}(v,\beta)x^p$, where, $\tilde{\phi}(v,\beta)=\beta((1-v)-(v-1))+(v-1),\quad v,\beta\in[0,1]$. Then the terms are as follows:

$$\tilde{u}_0(x,t,\alpha) = \tilde{u}(x,0)=\tilde{\phi}(v,\beta)x^p, \tag{43}$$

$$\tilde{u}_1(x,t,\alpha) = \tilde{\phi}(v,\beta)p(p-3)x^{p-2}\frac{t^\alpha}{\Gamma(\alpha+1)}-\tilde{\phi}^2(v,\beta)x^{2p}\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{44}$$

$$\begin{aligned}\tilde{u}_2(x,t,\alpha) &= \tilde{\phi}(v,\beta)p(p-2)(p-3)(p-5)x^{p-4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+2\tilde{\phi}^3(v,\beta)x^{3p}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &\quad -\tilde{\phi}^2(v,\beta)6p(p-2)x^{2p-2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.\end{aligned} \tag{45}$$

As a result, the following is the approximate series solution (up to three terms):

$$\begin{aligned}\tilde{u}(x,t,\alpha) &= \sum_{i=0}^{2}\tilde{u}_i(x,t,\alpha)\\ &= \tilde{\phi}(v,\beta)x^p+\tilde{\phi}(v,\beta)p(p-3)x^{p-2}\frac{t^\alpha}{\Gamma(\alpha+1)}\\ &\quad +\tilde{\phi}(v,\beta)p(p-2)(p-3)(p-5)x^{p-4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\tilde{\phi}^2(v,\beta)x^{2p}\frac{t^\alpha}{\Gamma(\alpha+1)}\\ &\quad -\tilde{\phi}^2(v,\beta)6p(p-2)x^{2p-2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+2\tilde{\phi}^3(v,\beta)x^{3p}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.\end{aligned} \tag{46}$$

7.3 Solving the model with CF derivative

Now consider the problem (42) equipped with the CF derivative:

$${}_0^{CF} D_t^\alpha \tilde{u}(\zeta,t,\alpha)=\frac{\partial^2\tilde{u}(x,t)}{\partial x^2}-\frac{2}{x}\frac{\partial\tilde{u}(x,t)}{\partial x}-\tilde{u}^2(x,t).$$

In a similar way as explained in Section 4, the first two terms are calculated below:

$$\begin{aligned}\tilde{u}_0(x,t,\alpha) &= \tilde{u}(x,0)=\tilde{\phi}(v,\beta)x^p,\\ \tilde{u}_1(x,t,\alpha) &= \left(\tilde{\phi}(v,\beta)p(p-1)x^{p-2}-\frac{2}{x}\tilde{\phi}(v,\beta)px^{p-1}-\tilde{\phi}^2(v,\beta)x^{2p}\right)\frac{1}{B(\alpha)}(1-\alpha(1-t)),\\ \tilde{u}_2(x,t,\alpha) &= \left(\frac{\partial^2 K_1}{\partial x^2}-\frac{2}{x}\frac{\partial K_1}{\partial x}-2\tilde{\phi}(v,\beta)x^pK_1\right)\frac{1}{B(\alpha)^2}\left((1-2\alpha+2\alpha t+\alpha^2-2\alpha^2 t+\frac{\alpha^2t^2}{2}\right),\end{aligned}$$

where

$$K_1 = \left(\tilde{\phi}(v,\beta)p(p-1)x^{p-2} - \frac{2}{x}\tilde{\phi}(v,\beta)px^{p-1} - \tilde{\phi}^2(v,\beta)x^{2p}\right).$$

We can find other terms in a similar way. Consequently, an approximate three-term solution is provided as:

$$\begin{aligned}\tilde{u}(x,t,\alpha) &= \tilde{\phi}(v,\beta)x^p + \left(\tilde{\phi}(v,\beta)p(p-1)x^{p-2} - \frac{2}{x}\tilde{\phi}(v,\beta)px^{p-1} - \tilde{\phi}^2(v,\beta)x^{2p}\right) \\ &\times \left(\frac{1}{B(\alpha)}(1-\alpha(1-t))\right) + \left(\frac{\partial^2 K_1}{\partial x^2} - \frac{2}{x}\frac{\partial K}{\partial x} - 2\tilde{\phi}(v,\beta)x^p K_1\right)\frac{1}{B(\alpha)^2} \\ &\times \left((1-2\alpha+2\alpha t+\alpha^2-2\alpha^2 t+\frac{\alpha^2 t^2}{2}\right) \end{aligned} \tag{47}$$

where

$$K_1 = \left(\tilde{\phi}(v,\beta)p(p-1)x^{p-2} - \frac{2}{x}\tilde{\phi}(v,\beta)px^{p-1} - \tilde{\phi}^2(v,\beta)x^{2p}\right).$$

7.4 Solving the model with ABC derivative

Now consider the problem (42) equipped with the ABC derivative

$${}^{ABC}_{0}D_t^{\alpha}\tilde{u}(\zeta,t,\alpha) = \frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} - \frac{2}{x}\frac{\partial \tilde{u}(x,t)}{\partial x} - \tilde{u}^2(x,t).$$

In a similar way as explained in Section 5, the first two terms are calculated below:

$$\begin{aligned}\tilde{u}_0(x,t,\alpha) &= \tilde{u}(x,0) = \tilde{\phi}(v,\beta)x^p, \\ \tilde{u}_1(x,t,\alpha) &= \left(\tilde{\phi}(v,\beta)p(p-1)x^{p-2} - \frac{2}{x}\tilde{\phi}(v,\beta)px^{p-1} - \tilde{\phi}^2(v,\beta)x^{2p}\right)\frac{1}{B(\alpha)}\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right), \\ \tilde{u}_2(x,t,\alpha) &= \left(\frac{\partial^2 K_1}{\partial x^2} - \frac{2}{x}\frac{\partial K}{\partial x} - 2\tilde{\phi}(v,\beta)x^p K_1\right)\frac{1}{B(\alpha)^2}\left((1-\alpha)^2 + \frac{2\alpha(1-\alpha)t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right),\end{aligned}$$

where

$$K_1 = \left(\tilde{\phi}(v,\beta)p(p-1)x^{p-2} - \frac{2}{x}\tilde{\phi}(v,\beta)px^{p-1} - \tilde{\phi}^2(v,\beta)x^{2p}\right).$$

Similarly, we can find other terms. Thus, a three-term approximate solution is given as:

$$\begin{aligned}\tilde{u}(x,t,\alpha) &= \tilde{\phi}(v,\beta)x^p + \left(\tilde{\phi}(v,\beta)p(p-1)x^{p-2} - \frac{2}{x}\tilde{\phi}(v,\beta)px^{p-1} - \tilde{\phi}^2(v,\beta)x^{2p}\right)\frac{1}{B(\alpha)} \\ &\times \left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right) + \left(\frac{\partial^2 K_1}{\partial x^2} - \frac{2}{x}\frac{\partial K}{\partial x} - 2\tilde{\phi}(v,\beta)x^p K_1\right)\frac{1}{B(\alpha)^2} \\ &\times \left((1-\alpha)^2 + \frac{2\alpha(1-\alpha)t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right),\end{aligned}$$

where

$$K_1 = \left(\tilde{\phi}(v,\beta)p(p-1)x^{p-2} - \frac{2}{x}\tilde{\phi}(v,\beta)px^{p-1} - \tilde{\phi}^2(v,\beta)x^{2p}\right).$$

8 Results and Discussion

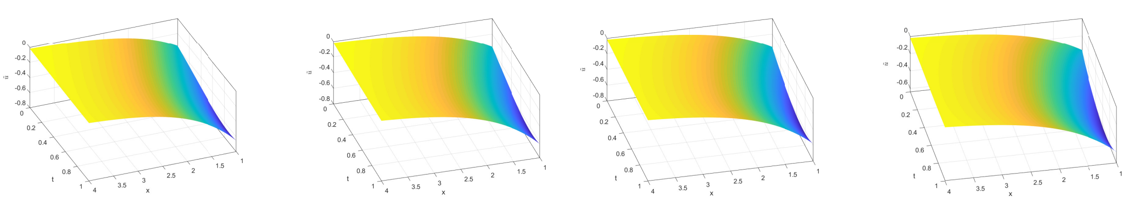
The fuzzy solution in numerical form for Example 1 is shown in Table 1 for six different values of $(0,0.2,0.4,0.6,0.8,1)$ of v and four different values $(0,0.4,0.6,1)$ of β, while the values of t, x, k, and α are fixed at 0.05, 4, -1, and 0.9, respectively. From Table 1, it can be observed that fuzzy solution $\tilde{u}(x,t,\alpha)$ depends significantly on the parameters v and β. As v increases from 0 to 1, $\tilde{u}(x,t,\alpha)$ approaches zero for all values of β, indicating a decrease in the solution's magnitude over this range. Furthermore, the absolute errors for both methods decrease as v approaches 1, with errors reaching zero at $v = 1$. When comparing the accuracy of the methods, the ADM consistently outperforms the EFDM by achieving lower absolute errors. Table 1 also presents both lower and upper approximations, showing symmetry around zero, which captures the fuzzy nature of the solution. In Table 2 the numerical solutions are achieved in all the cases i.e. Caputo, CF, ABC along with the exact solution at the respective values. This table provides a comparison of fuzzy solutions $\tilde{u}(x,t,\alpha)$ for Example 1 using the ADM across three different derivatives: Caputo, CF, and ABC. It is noted that in both lower and upper bounds, the solution reduces with an increase in v-cuts, with t, x, k, and α set at 0.05, 4, -1, and 0.9, respectively. The table also indicates that the absolute error is lower when comparing the ADM-generated solutions with those produced using the explicit finite difference method, providing compelling evidence that the ADM solution offers a better approximation. Additionally, the ABC derivative yields values slightly more from the exact solution as compared to Caputo and CF derivatives, particularly at higher v values. Overall, while all methods converge to zero at $v = 1$, the Caputo derivative generally provides values closest to the exact solution, highlighting its accuracy in modeling this fuzzy fractional system.

Table 3, shows the fuzzy solutions $\tilde{u}(x,t,\alpha)$ for Example 2 at fixed parameters $t = 0.05$, $x = 0.8$, $p = 1.2$, and $\alpha = 0.9$, with varying v-cuts and β values. The results, calculated using the ADM, display solutions for the Caputo, CF, and ABC derivatives across different values of β. It is noted that for both lower and upper approximations, the magnitude of the solution decreases with increasing v-cut values. The Caputo derivative produces values closest to zero as v approaches 1, particularly in lower approximation solutions, demonstrating its tendency towards stability. In comparison, the CF and ABC derivatives yield solutions that are further from zero, with ABC consistently showing the smallest values across most v-cuts. This suggests that, while the Caputo method remains closest to the initial conditions, the ABC method offers a more conservative approximation in this context.

For Example 1, the 3D graphical representations of the approximate solutions for both the lower bound ($\beta = 0$) and upper bound ($\beta = 1$) are displayed in Figures 1 and 2. These figures illustrate the behavior of the solutions at $v = 0$ while considering various fractional orders α set to 1, 0.9, 0.8, and 0.7, respectively. Each fractional order reflects a distinct solution behavior, showcasing the impact of varying α on the solution's progression. Similarly, for Example 2, the approximate solutions are presented in Figures 3 and 4 for the lower bound ($\beta = 0$) and upper bound ($\beta = 1$) at a fixed value of $p = 1.2$ and $v = 0$. As with Example 1, the graphical representations illustrate the solutions across different fractional orders $\alpha = 1$, 0.9, 0.8, and 0.7, demonstrating the solutions sensitivity to changes in the fractional order α and capturing how the solutions vary in response to these adjustments. The analysis of the 3D representations reveals an important insight into the behavior of the upper-bound and lower-bound solutions as the fractional order grows. In particular, it is seen that both the upper and lower bound solutions continuously shrink as the fractional parameter (α) grows. The variation of $\tilde{u}$ with respect to t for both the lower and upper bound approximation solutions for $x = 4$, $v = 0$, and for various Caputo fractional orders is shown in Example 1 in Figure 5, i.e., $\alpha = 1, 0.9, 0.8, 0.7$. These plots reveal that $\tilde{u}$ decreases as t increases in both the upper- and lower-bound scenarios. Additionally, the graphs demonstrate that over time, the cancer cell mortality rate rises, while the concentration of tumor cells gradually diminishes, mirroring findings observed in [33].

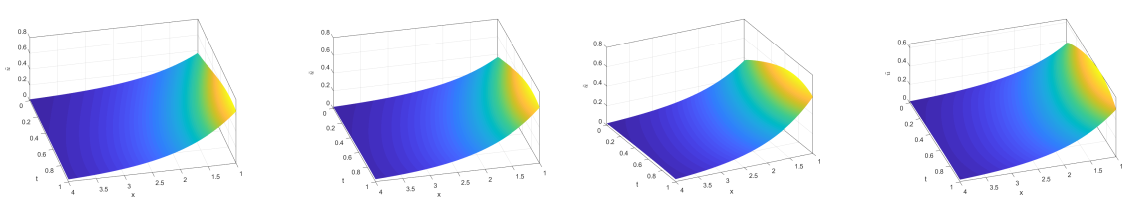
Figures 6 and Figures 7 show the behavior of $\tilde{u}$ versus t for various fractional derivatives Caputo, CF, and ABC of order one along with the exact solution for both the lower and upper bounds for Example 1 and Example 2, respectively. It is evident that the Caputo derivative provides the closest approximation to the exact solution compared to the CF and ABC derivatives. This provides compelling evidence for the Caputo derivative's dependability in the fuzzy fractional cancer model.

For Example 2, Figure 8 illustrates the variation of $\tilde{u}$ with t for the lower and upper limit approximations at $x = 0.8$, $p = 1.2$, $v = 0$, and different fractional orders ($\alpha = 1, 0.9, 0.8, 0.7$). Tumor growth dynamics require a thorough understanding of the fractional derivative. Although we have only approximated our series solution up to three series components, increasing the number of series components will allow for a more accurate approximation of the solution. It is observed that cancer cell concentration decreases for each α within the specified range, with numerical solution accuracy improving as α approaches 1. Given uncertainties in initial tumor cell production, a fuzzy approach is essential to manage the system's initial state effectively. As noted in [21], for the fractional cancer model solved via RPSE, $\alpha = 1.8$ is optimal when the net cell-killing rate depends on cell concentration. Conversely, for the fuzzy model under fuzzy initial conditions, $\alpha = 0.9$ is optimal. Understanding tumor development requires insight into the fractional derivative, as cancer cell concentration decreases across the range of α, with numerical accuracy improving as $\alpha \to 1$. Fuzziness remains critical due to initial-state ambiguity. Tumor growth dynamics depend on the fractional derivative, with cancer cell concentration decreasing across the specified α range and numerical accuracy improving as $\alpha \to 1$. Given uncertainties in initial tumor cell production, a fuzzy approach is essential.



(a) $\alpha = 1$ **(b)** $\alpha = 0.9$ **(c)** $\alpha = 0.8$ **(d)** $\alpha = 0.7$

Figure 1: For Example 1, 3D graphics of a lower approximate solution i.e., $\beta = 0$, obtained at $k = -1$ and $v = 0$ for different fractional orders.



(a) $\alpha = 1$ **(b)** $\alpha = 0.9$ **(c)** $\alpha = 0.8$ **(d)** $\alpha = 0.7$

Figure 2: For Example 1, 3D graphics of upper approximate solution i.e. $\beta = 1$, obtained at $k = -1$ and $v = 0$ for different fractional orders.

Table 1: Error analysis of a fuzzy solution to Example 1 by EFDM and ADM at $t = 0.05$, $x = 4$, $\alpha = 0.9$, $\forall \nu$, $\beta \in [0,1]$.

β	ν	$\tilde{u}(x,t,\alpha)$	Abs Error [33]	Abs Error (ADM)
$\beta = 0$, Lower Appr. Solution	0	-0.01964888	4.45721×10^{-5}	8.80×10^{-8}
	0.2	-0.015719104	3.56577×10^{-5}	7.10×10^{-8}
	0.4	-0.011789328	2.67433×10^{-5}	5.30×10^{-8}
	0.6	-0.0078595521	1.78288×10^{-5}	3.52×10^{-8}
	0.8	-0.0039297761	8.91442×10^{-6}	1.75×10^{-8}
	1	0	0	0
$\beta = 1$, Upper Appr. Solution	0	0.01964888	4.45721×10^{-5}	8.80×10^{-8}
	0.2	0.015719104	3.56577×10^{-5}	7.10×10^{-8}
	0.4	0.011789328	2.67433×10^{-5}	5.30×10^{-8}
	0.6	0.0078595521	1.78288×10^{-5}	3.52×10^{-8}
	0.8	0.0039297761	8.91442×10^{-6}	1.75×10^{-8}
	1	0	0	0
$\beta = 0.4$, Lower Appr. Solution	0	-0.0039297761	8.91442×10^{-6}	1.75×10^{-8}
	0.2	-0.0031438209	7.13154×10^{-6}	1.40×10^{-8}
	0.4	-0.0023578656	5.34865×10^{-6}	1.06×10^{-8}
	0.6	-0.0015719104	3.56577×10^{-6}	7.10×10^{-9}
	0.8	-0.00078595521	1.78288×10^{-6}	3.52×10^{-9}
	1	0	0	0
$\beta = 0.6$, Upper Appr. Solution	0	0.0039297761	8.91442×10^{-6}	1.75×10^{-8}
	0.2	0.0031438209	7.13154×10^{-6}	1.40×10^{-8}
	0.4	0.0023578656	5.34865×10^{-6}	1.06×10^{-8}
	0.6	0.0015719104	3.56577×10^{-6}	7.10×10^{-9}
	0.8	0.00078595521	1.78288×10^{-6}	3.52×10^{-9}
	1	0	0	0

Table 2: A fuzzy solution to Example 1 by ADM at $t = 0.05$, $x = 4$, $\alpha = 0.9$, $\forall \nu$, $\beta \in [0,1]$ for Caputo, CF, and ABC derivative.

β	ν	Exact	$\tilde{u}(x,t,\alpha)$ Caputo	$\tilde{u}(x,t,\alpha)$ CF	$\tilde{u}(x,t,\alpha)$ ABC
$\beta = 0$, Lower Appr. Solution	0	-0.019650134	-0.01964888	-0.02133121	-0.021750971
	0.2	-0.015720107	-0.015719104	-0.01706497	-0.017400777
	0.4	-0.01179008	-0.011789328	-0.01279872	-0.013050583
	0.6	-0.00786600536	-0.0078595521	-0.00853248	-0.0087003883
	0.8	-0.0039300268	-0.0039297761	-0.00426624	-0.0043501942
	1	0	0	0	0
$\beta = 1$, Upper Appr. Solution	0	0.019650134	0.01964888	0.02133121	0.021750971
	0.2	0.015720107	0.015719104	0.01706497	0.017400777
	0.4	0.01179008	0.011789328	0.01279872	0.013050583
	0.6	0.0078600536	0.0078595521	0.00853248	0.0087003883
	0.8	0.0039300268	0.0039297761	0.00426624	0.0043501942
	1	0	0	0	0
$\beta = 0.4$, Lower Appr. Solution	0	-0.0039300268	-0.0039297761	-0.00426624	-0.0043501942
	0.2	-0.0031440214	-0.0031438209	-0.00341299	-0.0034801553
	0.4	-0.0023580161	-0.0023578656	-0.00255974	-0.0026101165
	0.6	-0.0015720107	-0.0015719104	-0.00170650	-0.0017400777
	0.8	-0.00078600536	-0.00078595521	-0.00085325	-0.00087003883
	1	0	0	0	0
$\beta = 0.6$, Upper Appr. Solution	0	0.0039300268	0.0039297761	0.00426624	0.0043501942
	0.2	0.0031440214	0.0031438209	0.00341299	0.0034801553
	0.4	0.0023580161	0.0023578656	0.00255974	0.0026101165
	0.6	0.0015720107	0.0015719104	0.00170650	0.0017400777
	0.8	0.00078600536	0.00078595521	0.00085325	0.00087003883
	1	0	0	0	0

Table 3: A fuzzy solution to Example 2 by ADM at $t = 0.05$, $x = 0.8$, $p = 1.2$, $\alpha = 0.9$, $\forall\ v$, $\beta \in [0,1]$.

β	v	$\tilde{u}(x,t,\alpha)$ Caputo	$\tilde{u}(x,t,\alpha)$ CF	$\tilde{u}(x,t,\alpha)$ ABC
$\beta = 0$, Lower Appr. Solution	0	-0.5798454	-0.31829888	-0.24427739
	0.2	-0.45889475	-0.22465087	-0.15978955
	0.4	-0.34055161	-0.14685736	-0.094185198
	0.6	-0.22469925	-0.08405798	-0.0463997028
	0.8	-0.11122095	-0.03539232	-0.015357731
	1	0	0	0
$\beta = 1$, Upper Appr. Solution	0	0.5263297	0.00797533	-0.121079
	0.2	0.42664771	0.02604379	-0.074038544
	0.4	0.32128596	0.03514089	-0.037343105
	0.6	0.21613673	0.03440621	-0.012059996
	0.8	0.10908032	0.02297938	0.00074347558
	1	0	0	0
$\beta = 0.4$, Lower Appr. Solution	0	-0.11122095	-0.03539232	-0.015357731
	0.2	-0.088799909	-0.02727952	-0.011065814
	0.4	-0.066468224	-0.01969078	-0.0073926204
	0.6	-0.044224968	-0.01261919	-0.0043296126
	0.8	-0.022069205	-0.00605790	-0.0018682519
	1	0	0	0
$\beta = 0.6$, Upper Appr. Solution	0	0.10908031	0.2297938	0.00074347558
	0.2	0.087429905	0.01933524	0.0017126901
	0.4	0.065697598	0.01522212	0.0021314883
	0.6	0.043882468	0.01063312	0.0019913316
	0.8	0.02198358	0.00536138	0.0012836817
	1	0	0	0

9 Conclusion

The fuzzy fractional model of tumor cell growth is thoroughly examined in this work. The aforementioned study is significant concerning the mathematical model of tumor cells. Fuzzy logic is used to gain a better understanding of how tumor cells proliferate, since the early phases of cancer cell formation are inherently unpredictable. However, fractional derivatives such as ABC, CF, and Caputo help figure out how tumor cells respond over a brief period. The model's solution has been found using the ST and ADM. The suggested method's primary benefit is that it creates fewer errors; its drawback is the significant amount of time needed to identify the series components and then approximate them to obtain the solution. To more accurately approximate the solution, a large number of components are needed. Furthermore, the concentration of cancer cells has been found to steadily drop over time, eventually reaching zero in both the lower-bound and upper-bound solutions. To fully comprehend the mathematical model of tumor cells, this work will undoubtedly be crucial. This model facilitates the study of fractional derivatives and fuzziness, paving the way for more accurate cancer cell models and treatment strategies with the advancing technology. As more precise and alternative approaches become available soon, this work might be reconsidered to better understand tumor models and develop more effective techniques to combat these fatal diseases.

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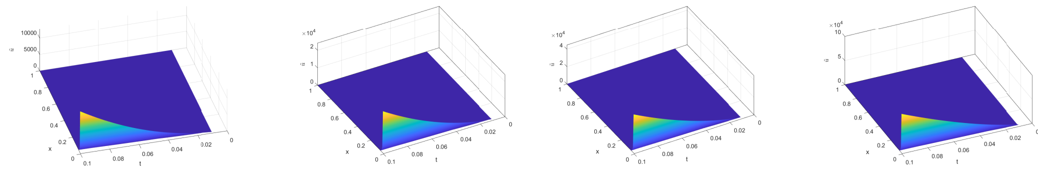
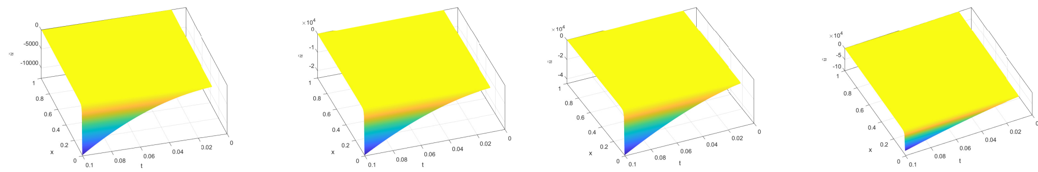


(a) $\alpha = 1$ **(b)** $\alpha = 0.9$ **(c)** $\alpha = 0.8$ **(d)** $\alpha = 0.7$

Figure 3: For Example 2, 3D graphics of lower approximate solution i.e. $\beta = 0$, obtained at $p = 1.2$ and $v = 0$ for different fractional orders.



(a) $\alpha = 1$ **(b)** $\alpha = 0.9$ **(c)** $\alpha = 0.8$ **(d)** $\alpha = 0.7$.

Figure 4: For Example 2, 3D graphics of upper approximate solution i.e. $\beta = 1$, obtained at $p = 1.2$ and $v = 0$ for different fractional orders.

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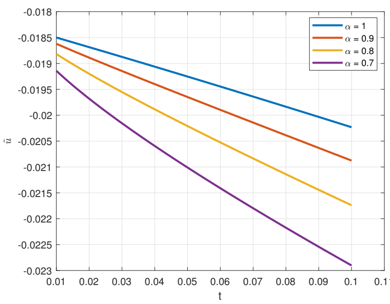
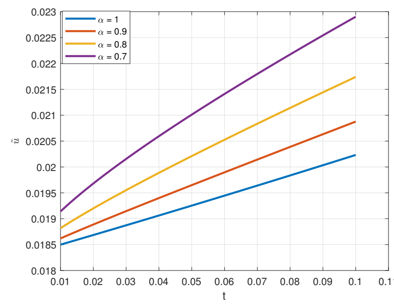
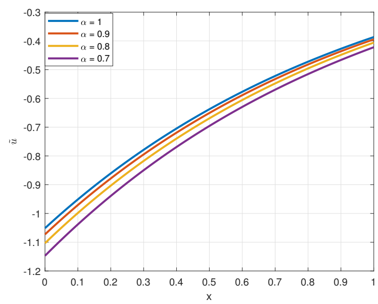


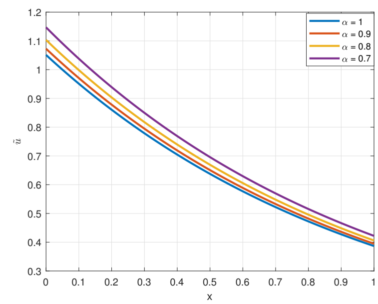
(a) Lower Bound



(b) Upper Bound

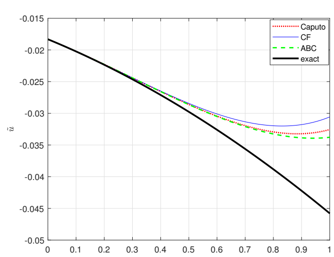


(c) Lower Bound

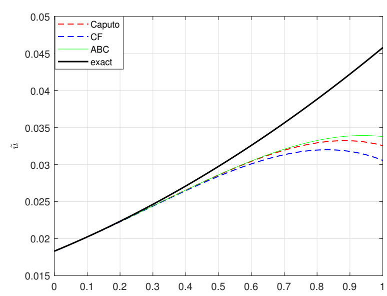


(d) Upper Bound

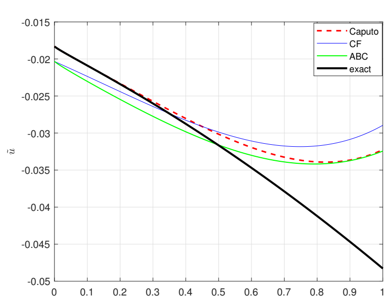
Figure 5: Plot for Example 1, $\tilde{u}$ versus t((a),(b)) and x((c),(d)) for lower and upper bounds at different Caputo fractional order.



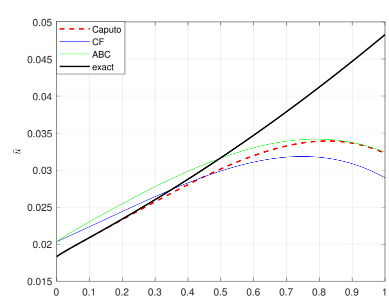
(a) Lower Bound at $\alpha = 1$



(b) Upper Bound at $\alpha = 1$

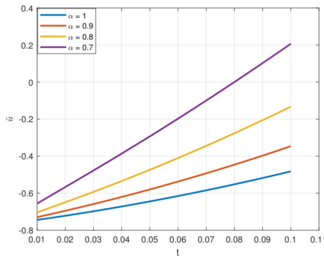


(c) Lower Bound at $\alpha = 0.9$

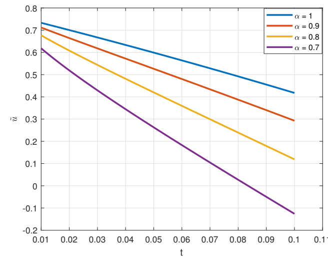


(d) Upper Bound at $\alpha = 0.9$

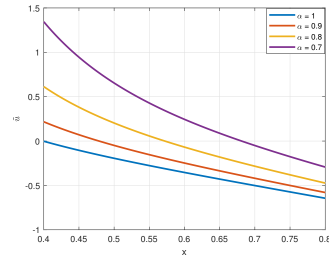
Figure 6: For Example 1, graphics of $\tilde{u}$ versus t at different fractional derivatives Caputo, CF, ABC.



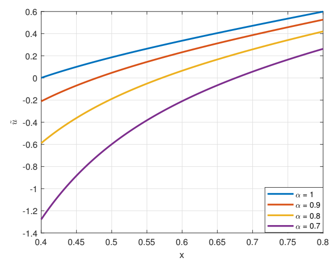
(a) Lower Bound



(b) Upper Bound

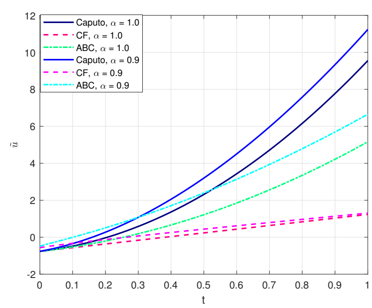


(c) Lower Bound

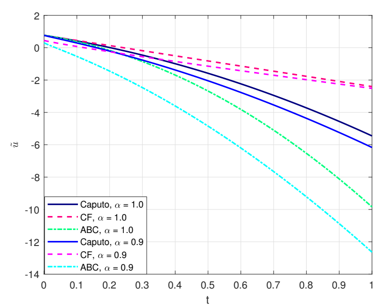


(d) Upper Bound

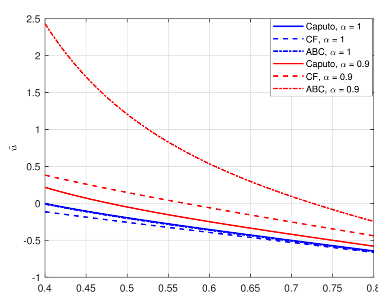
Figure 7: For Example 2, graphics of $\tilde{u}$ versus t((a),(b)) and x((c),(d)) for lower and upper bounds at different fractional order.



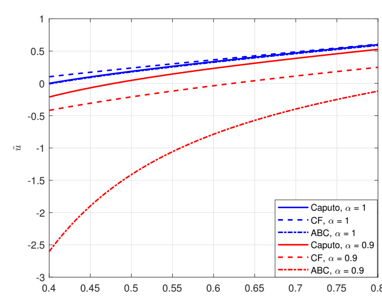
(a) Lower Bound



(b) Upper Bound



(c) Lower Bound



(d) Upper Bound

Figure 8: For Example 2, graphics of $\tilde{u}$ versus t ((a),(b)) and x ((c),(d)) for lower and upper bounds at different fractional orders 1 and 0.9 respectively.

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