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A study on the existence results for neutral functional random integro-differential equations with infinite delay

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Abstract. This study investigates the solutions of neutral functional integro-differential equations and second order neutral functional differential equations with delays and random effects. The Kakutani fixed-point theorem is used to prove the existence of mildly random solutions in the stochastic domain and to launch this investigation. The research heavily relies on core notions from functional analysis, and to make these concepts clearer, an explicit case is given.

Keywords: Random fixed point, neutral integro-differential equations, mild solution, infinite delay, semigroup theory.
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1 Introduction

In the vast landscape of mathematical research, the study of abstract nonlinear second order differential equations stands as a cornerstone, continually unveiling its profound significance. These equations, marked by their abstract and non-linear characteristics, have sparked extensive exploration into their properties, existence results, and diverse applications across mathematical domains. This article embarks on a comprehensive journey through this intriguing realm, guided by the seminal contributions of various researchers.

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The foundation of our exploration finds its roots in the pioneering work of Travis and Webb, who introduced the theory of cosine families and their profound relevance to abstract nonlinear second order differential equations [25]. Their insights have played a pivotal role in unraveling the intricate structure of such equations. Our journey takes us into the realm of stochastic differential equations, where Engl's general stochastic fixed-point theorem for continuous random operators on stochastic domains takes center stage [10]. This theorem provides valuable insight into understanding the solutions of differential equations influenced by stochastic processes, a crucial aspect of modeling real-world phenomena. Delving deeper into the field of functional differential equations, we encounter the innovative work of Hale and Kato on phase spaces for equations with infinite delay [12]. Their research has far-reaching implications, shedding light on the dynamics of systems with significant temporal dependencies. The study of functional differential equations with unbounded delay, as advanced by Hino, Murakami, and Naito, further enriches our understanding of these equations [13]. Their work finds application in diverse physical and biological systems, where the presence of unbounded delays is a common phenomenon.

The analysis of compactness, regularity, and uniform continuity properties in the context of strongly continuous cosine families, as presented by Travis and Webb, opens the doors to a wide range of mathematical investigations [24]. These properties are integral in the study of differential equations, enabling a deeper comprehension of solution behavior. Inequalities, both integral and finite differences, have proven to be indispensable tools in the analysis of solutions to differential equations under various constraints [17]. Pachpatte's work in this field provides a solid foundation for understanding the interplay of inequalities with differential equations. Moving forward, we explore the study of functional integro-differential equations, an area where researchers like Baghli [2], Dhage [7], Suresh [22], Gunasekar [11], Raghavendran [19], Dhakne and Kucche [8], and Pavlackova and Taddei [18] have contributed significantly to the existence results and their implications. Their collective research reflects the versatility and applicability of functional integro-differential equations across various mathematical landscapes. In the realm of neutral functional integro-differential equations, the works of Balachandran [3], Dong [9], Ye [28], and others emphasize the existence of solutions, particularly in nonlocal and perturbed scenarios. These investigations broaden our understanding of neutral differential equations, revealing their applicability in a wide range of mathematical contexts. Furthermore, the examination of second order neutral differential equations, as conducted by Benarab, Boussaada, Trabelsi, and Bonnet [4], offers a unique perspective on oscillation damping and the dynamic interplay between various components of the equation. The ever-expanding horizon of numerical methods for solving differential equations brings us to Lyu, Zhang, Chen, and Chen's deep mixed residual method (MIM) [14]. This innovative numerical approach shows promise in the treatment of high-order partial differential equations and offers efficient and accurate solutions to complex problems.

Dauer and Balachandran (2000) [6] established the existence of solutions for non-linear neutral integro-differential equations in Banach spaces. Bochenek (1991) [5] investigated second order nonlinear differential equations in an abstract framework. Recent works have extended these results to different settings. Tun and Tun (2023) [27] analyzed the Ulam stability of nonlinear iterative integro-differential equations, while Tun and Akyildiz (2024) [26] provided improved stability and instability results for neutral integro-differential equations with infinite delay. In the context of functional differential equations with infinite delay, Madhumitha et al. (2024) [15] studied the existence and controllability of second order functional differential equations involving random effects. Raghavendran et al. (2025) [20] examined the existence, uniqueness, and stability of impulsive neutral fractional Volterra-Fredholm equations, while another work by Raghavendran et al. (2025) [21] addressed similar equations with state-dependent delay. More studies can be found in [1, 16, 23].

To cover the broad and intricate field of abstract nonlinear second order differential equations, this study will explore their theoretical foundations, existence results, and applications across various mathematical settings, highlighting their significance in modeling real-world phenomena. In the forthcoming sections of this article, we shall dive deeper into each of these aspects, drawing upon the rich body of work established by the aforementioned researchers and exploring their implications in contemporary mathematics. In Section 3.1, we demonstrate that the second order functional differential equations with delay and random effects is of the form

$$\frac{d}{d\rho}[x'(\rho,\aleph)+\vartheta(\rho,x_\rho(.,\aleph),\aleph)] = Ax(\rho,\aleph)+\zeta(\rho,x_\rho(.,\aleph),\aleph), \quad \textbf{a.e. } \rho\in J=[0,T], \tag{1}$$

$$x(\rho,\aleph)=\phi_1(\rho,\aleph), \quad \theta\in(-\infty,0], \tag{2}$$

$$x'(0,\aleph)=\phi_2(\aleph), \tag{3}$$

where (Ω,F,p) is a complete probability space, $\aleph\in\Omega,(E,||.||)$ is a real Banach space, $\vartheta,\zeta : J*\mathfrak{B}*\Omega\to E$ and functions that depict the system's randomized nonlinearities $\phi\in J*\Omega$ and $A : D(A)\subset E\to E$ is the infinitesimal generator of a strongly continuous cosine family $T_1(\rho)$, $\rho\in J$ of bounded linear operator in E, and $\mathfrak{B}$ is the phase space. Let x be the function defined on $(-\infty,0]*\Omega$ and any $\rho\in J$ and $x_\rho(.,\aleph)\in\mathfrak{B}*\Omega$ be denoted by $x_\rho(\theta,\aleph)=x(\rho+\theta,\aleph)$ where $\theta\in(-\infty,0]$. We suppose that the histories $x_\rho(.,\aleph)$ are part of a general phase space $\mathfrak{B}$ where they represent the state's history from time $-\infty$ up to the present time ρ. Since there appears to be relatively little research on the local existence of random evolution equations with delay, the current study can be regarded as a new contribution to this field. T represents the end of the current time interval $J=[0,T]$, within which the equations is being analyzed or the solution is being determined. It signifies the point at which the behavior or state of the system is studied.

Consider that E is a Banach space with norm$||.||$. Assume that $\mathfrak{B}=C((-\infty,T],E)$ is a continuous banach space endowed with supremum norm $||x||_{\mathfrak{B}}=\sup\{||x(\rho)||:\rho\in J\}$. In Subsection 3.2 we demonstrate that the second order functional differential equations with delay and random effects is of the form

$$\frac{d}{d\rho}[x'(\rho,\aleph)+\vartheta(\rho,x_\rho(.,\aleph),\aleph)] = Ax(\rho,\aleph)+\zeta(\rho,x_\rho(.,\aleph),\int_0^\rho\gamma(\rho,s)\upsilon(s,x(s,\aleph))ds,\aleph),\rho\in J, \tag{4}$$

$$x(\rho,\aleph)=\phi_1(\rho,\aleph), \tag{5}$$

$$x'(0,\aleph)=\phi_2(\aleph), \tag{6}$$

where A is the infinetesimal generator of a strongly continuous cosine family $\{T_2(\rho):\rho\in\mathbb{R}\}$ of a bounded linear operator in a Banach space E with the norm $||.||$ and $\vartheta : J*\mathfrak{B}*\Omega\to E,\zeta : J*\mathfrak{B}*\Omega\to E,\upsilon : J*E\to E$ and $\gamma : J*J\to\mathbb{R}$ are continuous functions.

2 Preliminaries

Several notations, definitions, and theorems that are utilized throughout the rest of this work are presented in this section. We shall use the vocabulary from [12] and apply Hale and Kato's [13] axiomatic description of the phase space $\mathfrak{B}$. By satisfying the following axioms, $(\mathfrak{B}, ||.||_{\mathfrak{B}})$ will be a seminormed linear space of functions projecting $(-\infty, 0]$ into E:

(A) The following guidelines are applicable for all $\rho \in J$ if $x : (-\infty, T] \to E, T > 0$, is continuous on J and $x_0 \in \mathfrak{B}$:

1. $x_\rho \in \mathfrak{B}$,
2. there is a positive constant K such that $\|x(\rho)\| \leq K\|x_\rho\|_{\mathfrak{B}}$,
3. there are two functions $U, \sigma, \sigma' : \mathbb{R}_+ \to \mathbb{R}_+$ independent of x, with U continuous and bounded, σ, σ' locally bounded such that

$$||x_\rho||_{\mathfrak{B}} \leq U(\rho) \sup \{|X(m)| : 0 \leq m \leq \rho\} + \sigma(\rho)||x_0||_{\mathfrak{B}} + \sigma'(\rho)||x_0'||_{\mathfrak{B}}.$$

(B) x_ρ is a $\mathfrak{B}$-valued continuous function on J for the functions x in (A).

(C) The space $\mathfrak{B}$ is complete.

Definition 1. *A map $\zeta : J * \mathfrak{B} * \Omega \to E$ is said to be Caratheodory if*

1. *$\rho \to \zeta(\rho, x, \aleph)$ is measurable for each $x \in \mathfrak{B}$ and for all $\aleph \in \Omega$.*
2. *$x \to \zeta(\rho, x, \aleph)$ is continuous for almost each $\rho \in J$ and all $\aleph \in \Omega$.*
3. *$y \to \zeta(\rho, x, \aleph)$ is measurable for each $x \in \mathfrak{B}$ and for all $\rho \in J$.*

A random operator is a mapping $V : \Omega * X \to X$ such that V(.,x) is measurable for each $x \in X$ and is usually stated as $V(\aleph, x) = V(\aleph)x$. We will use both expressions likewise.

Theorem 1. *(Kakutani fixed point theorem) Let S be a non empty, compact and convex subset of some Euclidean space $\mathbb{R}^n$. Let $\phi : S \to 2^S$ be a set valued function on S with the following properties*

1. *ϕ has a closed graph .*
2. *$\phi(x)$ is non-empty and convex for all $x \in S$.*

Then ϕ has a fixed point.

Definition 2. *[10] Let Q be a mapping from Ω into 2^X. A mapping $V : (\aleph, x) : \aleph \in \Omega \wedge x \in Q(\aleph) \to X$ is called random operator with stochastic domain Q if Q is measurable (i.e., for all closed $B \subseteq X, \aleph \in \Omega : Q(\aleph) \cap B \neq \emptyset \in F$) and for every open set $C \subseteq X$ and all $x \in X$, $\{\aleph \in \Omega : x \in Q(\aleph) \wedge V(\aleph, x) \in C\} \in F$. We will say that V is continuous if every $V(\aleph)$ is continuous.*

Let $T > 0$ be a real number and let E be a Banach space with norm $||.||$. The Banach space of all continuous functions is denoted as $C, \phi : (-\infty, 0] \to E$ endowed with the sup-norm

$$||\phi|| = \sup\{\|\phi(\mu)\| : -\infty \leq \mu \leq 0\}.$$

Also for $x \in E$ we have $x_\rho \in \mathfrak{B} = C((-\infty, T], E)$ for $\rho \in J = [0, T]$ and $x_\rho(\mu) = x(\rho + \mu)$ for $\mu \in (-\infty, 0]$.

Definition 3. *A one parameter family of the strongly continuous cosine family is defined as $\{T_1(\rho) : \rho \in \mathbb{R}\}$ of bounded linear operators mapping the Banach space E into itself if*

1. $T_1(s+\rho)+T_1(s-\rho)=2T_1(s)T_1(\rho)$ *for each* $s,\rho\in\mathbb{R}$.
2. $T_1(0)=I$.
3. $T_1(\rho)x$ *is continuous in* ρ *on* $\mathbb{R}$ *for all fixed* $x\in E$.

The linked sine family $T_2(\rho),\rho\in\mathbb{R}$, *should be defined by*

$$T_2(\rho)x=\int_0^{\rho}T_1(s)xds,\ \ x\in E,\rho\in\mathbb{R}.$$

Consider A under the following circumstances:
(G_1) The adjoint operator A^* *is densely defined, and A is the infinitesimal generator of the strongly continuous cosine family* $T_1(\rho),\rho\in\mathbb{R}$, *of bounded linear operators from E into itself (i.e.,*$\overline{D(A^*)}=E^*$*)* . *The operator A is defined by acts as the infinitesimal generator of the strongly continuous cosine family* $T_1(\rho),\rho\in\mathbb{R}$,

$$Ax=\Big[\frac{d^2}{d\rho^2}T_1(\rho)x\Big]_{\rho=0},\ \ x\in D(A),$$

where $D(A)=\{x\in E:T_1(\rho)x$ *is twice continuously differentiable in* $\rho\}$. *Define* $F=\{x\in E:T_1(\rho)x$ *is once continuously differentiable in* $\rho\}$.

The following lemmas are required in order to prove our main theorem.

Lemma 1. *[25] Let (G_1) hold. Then*

1. *there exist a constant* $N\geq 1$ *and* $\eta\geq 0\ni|T_1(\rho)|\leq Ne^{\eta|\rho|}$ *and* $|T_2(\rho)-T_2(\rho^*)|\leq N|\int_{\rho}^{\rho^*}e^{\eta|s|}ds|$ *for* $\rho,\rho^*\in\mathbb{R}$.
2. $T_2(\rho)E\subset F$ *and* $T_2(\rho)F\subset D(A)$ *for* $\rho\in\mathbb{R}$.
3. $\frac{d}{d\rho}T_1(\rho)x=AT_2(\rho)x$ *for* $x\in F$ *and* $\rho\in\mathbb{R}$.
4. $\frac{d^2}{d\rho^2}T_1(\rho)x=AT_1(\rho)x$ *for* $x\in D(A)$ *and* $\rho\in\mathbb{R}$.

Lemma 2. *(see [24, 25]) Assume that (G_1) is true, let* $v:\mathbb{R}\to E$ *be continuously differentiable, and suppose* $p(\rho)=\int_0^{\rho}T_2(\rho-s)v(s)ds$.*Then p is twice countinuously differentiable and for* $\rho\in\mathbb{R},q(\rho)\in D(A)$,

$$p'(\rho)=\int_0^{\rho}T_1(\rho-s)v(s)ds,$$

and $p''(\rho)=Aq(\rho)+v(\rho)$.

Lemma 3. *[17, p-47] Let* $z(\rho),u(\rho),v(\rho),\aleph(\rho)\in C([\alpha,\beta],\mathbb{R}_+),\gamma\geq 0$ *be a real constant and*

$$z(\rho)\leq\gamma+\int_{\alpha}^{\rho}u(s)[z(s)+\int_{\alpha}^{s}v(\sigma)z(\sigma)d\sigma+\int_{\alpha}^{\beta}\aleph(\sigma)z(\sigma)d\sigma]ds,\quad for\quad \rho\in[\alpha,\beta].$$

If

$$r^*=\int_{\alpha}^{\beta}\aleph(\sigma)e^{(\int_{\alpha}^{\sigma}[u(\tau)+v(\tau)]d\tau)d\sigma}<1,$$

then

$$z(\rho)\leq\frac{\gamma}{1-r^*}e^{(\int_{\alpha}^{\rho}[u(s)+v(s)]ds)}\quad for\quad \rho\in[\alpha,\beta].$$

3 Existence results

3.1 Functional differential equation

We now present our primary existence result regarding problem (1)-(3). The definition of a mild random solution comes first.

Definition 4. *Let $X = C[J,E]$ and $T > 0$. Define a function $x(\rho,\aleph) = \phi_1(\rho,\aleph)$, where $x : J\times\Omega \to E$ is continuous. If x satisfies the integral equation, it is called a random mild solution of problem (1)-(3) and we get:*

$$x(\rho,\aleph) = T_1(\rho)\phi_1(0,\aleph) + T_2(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)] - \int_0^\rho T_1(\rho-s)\vartheta(s,x_s(.,\aleph),\aleph)ds + \int_0^\rho T_2(\rho-s)\zeta(s,x_s(.,\aleph),\aleph)ds, \quad \rho\in J.$$

Assume $\sigma = \{||T_1(\rho)||_{\mathfrak{B}(E)} : \rho\geq 0\}$ and $\sigma' = \{||T_2(\rho)||_{\mathfrak{B}(E)} : \rho\geq 0\}$. Our primary findings will be attained under the given situations:
(H_1) A strongly continuous semigroup $T(\rho), \rho\in J$, that is compact for $\rho > 0$ in the Banach space E, has an infinitesimal generator called $A : D(A)\subset E\to E$,
(H_2) the function $\zeta : J*\mathfrak{B}*\Omega\to E$ is Caratheodory,
(H_3) there exist functions $\delta_1,\delta_2 : \mathfrak{B}*\Omega\to\mathbb{R}^+$and $q_1,q_2 : J*\Omega\to\mathbb{R}^+$ such that for all $\aleph\in\mathfrak{B}$, $\delta(.,\aleph)$ is a continuous non-decreasing function, $q(.,\aleph)$ is integrable and

$$\|\vartheta(\rho,u,\aleph)\| \leq q_1(\rho,\aleph)\delta_1(||u||_{\mathfrak{B}},\aleph),$$

$$\|\zeta(\rho,u,\aleph)\| \leq q_2(\rho,\aleph)\delta_2(||u||_{\mathfrak{B}},\aleph),$$

for every $\rho\in J$ and each $u\in\mathfrak{B}$,
(H_4) there exists a random function $P : \Omega\to\mathbb{R}^+\backslash\{0\}$ such that

$$\sigma||\phi_1||_{\mathfrak{B}} + \sigma'||\phi_2 + q_1\delta_1||_{\mathfrak{B}} + \sigma\delta_1(C_T,\aleph)||q_1||_{L^1} + \sigma'\delta_2(C_T,\aleph)||q_2||_{L^1} \leq P(\aleph),$$

where $C_T = U_T P(\aleph) + \sigma_T||\phi_1||_{\mathfrak{B}} + \sigma'_T||\phi_2||_{\mathfrak{B}}$. Assume $U_T = \sup\{U(\rho) : \rho\in J\}$, $\sigma_T = \sup\{\sigma(\rho) : \rho\in J\}$ and $\sigma'_T(\rho) = \sup\{\sigma'(\rho) : \rho\in J\}$,
(H_5) for all $\aleph\in\Omega, \phi(.,\aleph)$ is continuous and $\phi(\rho,.)$ is measurable for every ρ.

Theorem 2. *Assume that the conditions $(H_1)-(H_5)$ are true. Then on $(-\infty,T]$, there is at least one mild random solution to the problems (1) - (3).*

Proof. Let $X = \{u\in C(J,E) : u(0,\aleph) = \phi(0,\aleph)\}$ be endowed with the uniform convergence topology, and let $N : \Omega*X\to 2^X$ be the random operator specified by convergence topology defined by

$$(N(\aleph)x)(\rho) = T_1(\rho)\phi_1(0,\aleph) + T_2(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)] - \int_0^\rho T_1(\rho-s)\vartheta(s,\bar{x}_s(.,\aleph),\aleph)ds + \int_0^\rho T_2(\rho-s)\zeta(s,\bar{x}_s(.,\aleph),\aleph)ds, \quad \rho\in J,$$

where $\bar{x} : (-\infty, T] * \Omega \to E$ is formed so that $\bar{x}_0(., \aleph) = \phi(., \aleph)$ and $\bar{x}_0(\aleph) = \phi(\aleph)$ on J, and $\bar{x}(., \aleph) = y(., \aleph)$ on J. Let $\overline{\phi} : (-\infty, T] * \Omega \to E$ be the extension of ϕ to $(-\infty, T]$ such that $\overline{\phi}(0, \aleph) = \overline{\phi}(0, \aleph) = 0$ on J.

We will demonstrate that the above mapping definition is a random operator. To accomplish this, we must demonstrate that $N(.)(x) : \Omega \to X$ is a random variable for any $x \in X$. Since the mapping $\zeta(\rho, x, .)$ for $\rho \in J$ and $x \in X$ is measurable under conditions (H_2), it is first important to note that $N(.)(x) : \Omega \to Y$ is measurable (H_5). Let $C : \Omega \to 2^X$ be defined by $C(\aleph) = \{x \in X : ||x|| \leq P(\aleph)\}$.

Step 1: To prove that $N(.)(\aleph)$ is a closed graph, it is enough to prove that $x_n \to x, y_n \to y$ and if $y_n \in \phi(x_n)$ then prove $y \in N(.)(x)$. Let us have $x^n \to x$ in X, then we have to prove that N is convergent for all $x \in X$. We have

$$\begin{aligned}
\|(N(\aleph)x^n)(\rho) - (N(\aleph)x)(\rho)\| &= \| \int_0^\rho T_1(\rho - s)[\vartheta(s, \bar{x}_s, \aleph) - \vartheta(s, \bar{x}_s^n, \aleph)]ds \\
&\quad + \int_0^\rho T_2(\rho - s)[\zeta(s, \bar{x}_s^n, \aleph) - \zeta(s, \bar{x}_s, \aleph)]ds\| \\
&\leq \sigma \int_0^\rho \|\vartheta(s, \bar{x}_s^n, \aleph) - \vartheta(s, \bar{x}_s, \aleph)\|ds \\
&\quad + \sigma' \int_0^\rho \|\zeta(s, \bar{x}_s^n, \aleph) - \zeta(s, \bar{x}_s, \aleph)\|ds.
\end{aligned}$$

According to the Lebesgue dominated convergence theorem, $\zeta(s, ., \aleph)$ is continuous. Therefore

$$\begin{aligned}
&||\vartheta(., \bar{x}_s^n, \aleph) - \vartheta(., \bar{x}_s, \aleph)||_{L^1} \quad \to 0 \quad \text{as} \quad n \to \infty, \\
&||\zeta(., \bar{x}_s^n, \aleph) - \zeta(., \bar{x}_s, \aleph)||_{L^1} \quad \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Similarly, we can prove that $N(\aleph)(y^n) \to N(\aleph)(y)$ as $n \to \infty$ and if $y_n \in N(.)(x_n)$, we know that $x_n \to x$ and $y_n \to y$ then we get $y \in N(.)(x)$ i.e., $y_n \in N(.)(x_n) \implies y \in N(.)(x)$. Hence it has a closed graph.

Step 2: To show that $N(.)(\aleph)$ is uniformly bounded, from (H_3) and (H_4) we have

$$\begin{aligned}
\|(N(\aleph)x)(\rho)\| &\leq \|T_1(\rho)\|\|\phi_1(0, \aleph)\| + \|T_2(\rho)\|\|[\phi_2(\aleph) + \vartheta(0, \phi_1(0, \aleph), \aleph)]\| \\
&\quad + \sigma \int_0^\rho \|\vartheta(s, \bar{x}_s, \aleph)\|ds + \sigma' \int_0^\rho \|\zeta(s, \bar{x}_s, \aleph)\|ds \\
&\leq \sigma\|\phi_1\|_{\mathfrak{B}} + \sigma'\|\phi_2 + q_1\delta_1\|_{\mathfrak{B}} + \sigma \int_0^\rho q_1(s, \aleph)\delta_1(\|\bar{x}_s\|_{\mathfrak{B}}, \aleph)ds \\
&\quad + \sigma' \int_0^\rho q_2(s, \aleph)\delta_2(\|\bar{x}_s\|_{\mathfrak{B}}, \aleph)ds \\
&\leq \sigma\|\phi_1\|_{\mathfrak{B}} + \sigma'\|\phi_2 + q_1\delta_1\|_{\mathfrak{B}} \\
&\quad + \sigma\delta_1(C_T, \aleph) \int_0^\rho q_1(s, \aleph)ds + \sigma'\delta_2(C_T, \aleph) \int_0^\rho q_2(s, \aleph)ds \\
&\leq \sigma\|\phi_1\|_{\mathfrak{B}} + \sigma'\|\phi_2 + q_1\delta_1\|_{\mathfrak{B}} + \sigma\delta_1(C_T, \aleph)\|q_1\|_{L^1} + \sigma'\delta_2(C_T, \aleph)\|q_2\|_{L^1} \\
&\leq P(\aleph).
\end{aligned}$$

Hence $N(.)(\aleph)$ is uniformly bounded.

Step 3: To show that N has convex values, let $x_1, x_2 \in X$, then there exists $g_1, g_2 \in E$ such that

$$N(\aleph)y_i(\rho) = T_1(\rho)\phi_1(0,\aleph) + T_2(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)] - \int_0^\rho T_1(\rho - s) f_i(s,\bar{x}_s,\aleph)ds + \int_0^\rho T_2(\rho - s) g_i(s,\bar{x}_s,\aleph)ds, \quad \rho \in J.$$

Let $\alpha \in [0,1]$. Then for each $\rho \in J$ we obtain that

$$\begin{aligned} N(\aleph)(\alpha x_1 + (1-\alpha)x_2)(\rho) &= T_1(\rho)\phi_1(0,\aleph) + T_2(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)] \\ &\quad - \int_0^\rho T_1(\rho - s)(\alpha f_1(s,\bar{x}_s,\aleph) + (1-\alpha) f_2(s,\bar{x}_s,\aleph))ds \\ &\quad + \int_0^\rho T_2(\rho - s)(\alpha g_1(s,\bar{x}_s,\aleph) + (1-\alpha) g_2(s,\bar{x}_s,\aleph))ds. \end{aligned}$$

We further have

$$\begin{aligned} ||N(\aleph)(\alpha x_1 + (1-\alpha)x_2)(\rho)|| &\leq ||T_1(\rho)\phi_1(0,\aleph)|| + ||T_2(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)]|| \\ &\quad + \int_0^\rho ||T_1(\rho - s)(\alpha f_1(s,\bar{x}_s,\aleph) + (1-\alpha) f_2(s,\bar{x}_s,\aleph))||ds \\ &\quad + \int_0^\rho ||T_2(\rho - s)(\alpha g_1(s,\bar{x}_s,\aleph) + (1-\alpha) g_2(s,\bar{x}_s,\aleph))ds|| \\ &\leq \sigma||\phi_1||_{\mathfrak{B}} + \sigma'||\phi_2 + q_1\delta_1||_{\mathfrak{B}} \\ &\quad + \sigma\int_0^\rho ||\alpha f_1(s,\bar{x}_s,\aleph) + (1-\alpha) f_2(s,\bar{x}_s,\aleph)||ds \\ &\quad + \sigma'\int_0^\rho ||\alpha g_1(s,\bar{x}_s,\aleph) + (1-\alpha) g_2(s,\bar{x}_s,\aleph)||ds. \end{aligned}$$

Since E has convex values and $f_i, g_i, i = 1,2$ are linear, it holds that $\alpha x_1 + (1-\alpha x_2) \in X$.

Step 4: Showing that $N(.)(\aleph)$ maps a bounded set into a relatively compact set. Next, we establish that $\{x \in C(\aleph) : N(\aleph)x = x\} \neq \emptyset$ for every $\aleph \in \Omega$. To accomplish this, we shall use Arzela-Ascoli theorem. First, we demonstrate how N maps bounded sets into equicontinuous sets in $C(\aleph)$. Let $\rho_0, \rho \in [0,T]$ with $\rho > \rho_0$, $C(\aleph)$ be a bounded set and let $x \in C(\aleph)$, then

$$\begin{aligned} &\|(N(\aleph)x)(\rho - a) - (N(\aleph)x)(\rho_0 - a)\| \\ &\quad \leq \|T_1(\rho - a) - T_1(\rho_0 - a)\|\|\phi_1\|_{\mathfrak{B}} + \|T_2(\rho - a) - T_2(\rho_0 - a)\|\|\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)\|_{\mathfrak{B}} \\ &\qquad + \|\int_0^{\rho_0 - a}[T_1(\rho - a - s) - T_1(\rho_0 - a - s)]\vartheta(s,\bar{x}_s,\aleph)ds)\| + \|\int_{\rho_0 - a}^{\rho - a} T_1(\rho - a - s)\vartheta(s,\bar{x}_s,\aleph)ds\| \\ &\qquad + \|\int_0^{\rho_0 - a}[T_2(\rho - a - s) - T_2(\rho_0 - a - s)]\zeta(s,\bar{x}_s,\aleph)ds)\| + \|\int_{\rho_0 - a}^{\rho - a} T_2(\rho - a - s)\zeta(s,\bar{x}_s,\aleph)ds\| \\ &\quad \leq \|T_1(\rho - a) - T_1(\rho_0 - a)\|\|\phi_1\|_{\mathfrak{B}} + \|T_2(\rho - a) - T_2(\rho_0 - a)\|\|\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)\|_{\mathfrak{B}} \\ &\qquad + \int_0^{\rho_0 - a}\|T_1(\rho - a - s) - T_1(\rho_0 - a - s)\|\|\vartheta(s,\bar{x}_s,\aleph)\|ds + \int_{\rho_0 - a}^{\rho - a}\|T_1(\rho - a - s)\|\|\vartheta(s,\bar{x}_s,\aleph)\|ds \\ &\qquad + \int_0^{\rho_0 - a}\|T_2(\rho - a - s) - T_2(\rho_0 - a - s)\|\|\zeta(s,\bar{x}_s,\aleph)\|ds + \int_{\rho_0 - a}^{\rho - a}\|T_2(\rho - a - s)\|\|\zeta(s,\bar{x}_s,\aleph)\|ds \end{aligned}$$

$$\leq \|T_1(\rho-a)-T_1(\rho_0-a)\|\|\phi_1\|_{\mathfrak{B}}+\|T_2(\rho-a)-T_2(\rho_0-a)\|\|\phi_2+q_1\delta_1\|_{\mathfrak{B}}$$
$$+\delta_1(C_T,\aleph)\int_0^{\rho_0-a}\|T_1(\rho-a-s)-T_1(\rho_0-a-s)\|q_1(s,\aleph)ds+\sigma\delta_1(C_T,\aleph)\int_{\rho_0-a}^{\rho-a}q_1(s,\aleph)ds$$
$$+\delta_2(C_T,\aleph)\int_0^{\rho_0-a}\|T_2(\rho-a-s)-T_2(\rho_0-a-s)\|q_2(s,\aleph)ds+\sigma'\delta_2(C_T,\aleph)\int_{\rho_0-a}^{\rho-a}q_2(s,\aleph)ds.$$

Given that $T_1(\rho),T_2(\rho)$is uniformly continuous, the right-hand side of the inequality above tends to zero as $\rho_0-\rho\to 0$. The Arzela-Ascoli theorem can be applied since N is bounded and equicontinuous if we can demonstrate that the operator N maps $C(\aleph)$ into a precompact set in E. Therefore, let be a real number satisfying $0<\varepsilon<\rho$ and let $\rho\in[0,T]$ be fixed. We define $x\in C(\aleph)$ as

$$\begin{aligned}(N(\aleph)x)(\rho) &= T_1(\rho)\phi_1(0,\aleph)+T_2(\rho)[\phi_2(\aleph)+\vartheta(0,\phi_1(0,\aleph),\aleph)]\\ &\quad+\int_0^{\rho}T_1(\rho-s)\vartheta(s,\bar{x}_s(.,\aleph),\aleph)ds\\ &\quad+\int_0^{\rho}T_2(\rho-s)\zeta(s,\bar{x}_s(.,\aleph),\aleph)ds.\end{aligned}$$

The set $Z_\varepsilon(\rho,\aleph)=\{(N_\varepsilon(\aleph)x)(\rho):x\in C(\aleph)\}$ is precompact in E for every ε with $0<\varepsilon<\rho$ because $T_1(\rho)$ is a compact operator. Moreover,

$$\begin{aligned}\|(N(\aleph)x)(\rho)-(N_\varepsilon(\aleph)x)(\rho)\| &\leq \int_{\rho-\varepsilon}^{\rho}\|T_1(\rho-s)\|\cdot\|\vartheta(s,\bar{x}_s(.,\aleph),\aleph)\|ds\\ &\quad+\int_{\rho-\varepsilon}^{\rho}\|T_2(\rho-s)\|\cdot\|\zeta(s,\bar{x}_s(.,\ \aleph),\aleph)\|ds\\ &\leq \sigma\delta_1(C_T,\aleph)\int_{\rho-\varepsilon}^{\rho}q_1(s,\aleph)ds+\sigma'\delta_2(C_T,\aleph)\int_{\rho-\varepsilon}^{\rho}q_2(s,\aleph)ds.\end{aligned}$$

Therefore the set $Z(\rho,\aleph)=\{(N(\aleph)x)(\rho):x\in C(\aleph)\}$ is precompact in E. Hence $N(\aleph):C(\aleph)\to C(\aleph)$ is a closed graph, compact and has convex values. Thus by Kakutani fixed point theorem the random operator N has a unique fixed point which in turn is a mild solution of the problem (1)-(3). □

3.2 Functional integro-differential equation

We now present our primary existence result regarding problem (4) - (6). The definition of a mild random solution comes first.

Definition 5. *Let $B=C[J,E]$ and $T>0$. Define a function $x(\rho,\aleph)=\phi_1(\rho,\aleph)$, where $x:J\times\Omega\to E$ is continuous. If x satisfies the integral equation, it is called a random mild solution of problem* (4)-(6) *and we get:*

$$\begin{aligned}x(\rho,\aleph)= \quad & T_1(\rho)\phi_1(0,\aleph)+T_2(\rho)[\phi_2(\aleph)+\vartheta(0,\phi_1(0,\aleph),\aleph)]-\int_0^{\rho}T_1(\rho-\tau)\vartheta(\tau,x_\tau(.,\aleph),\aleph)\,d\tau\\ &+\int_0^{\rho}T_2(\rho-\tau)\zeta(\tau,x_\tau(.,\aleph),\int_0^{s}\gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta)\,d\tau.\end{aligned}$$

We have listed the hypotheses that will be discussed in the following section:
(G_2) there exist a constant $K>0$ such that $||\upsilon(\rho,x(\rho))||\leq KH(||x(\rho)||)$, for every $\rho\in[0,T]=J$ and

$x \in E$ where $H : \mathbb{R}_+ \to (0,\infty)$ is continuous non-decreasing function,
(G_3) there exist a continuous function $v : J * \Omega \to \mathbb{R}_+$ such that $||\vartheta(\rho, x, \aleph)|| \leq v(\rho, \aleph)||x, \aleph||$,
(G_4) there exist a continuous function $q : J * \Omega \to \mathbb{R}_+$ such that

$$||\zeta(\rho, x, y, \aleph)|| \leq q(\rho, \aleph)((||x|| + ||y||), \aleph),$$

for all $\rho \in J$ and $x, y \in E$,
(G_5) for all $\rho \in J$, the function $\upsilon(\rho, .) : J * E \to E$ is continuous and for all $x \in E$ the function $\zeta(., x) : J * E \to E$ is strongly measurable,
(G_6) for all $\rho \in J$ the function $\zeta(\rho, ., ., .) : J * \mathfrak{B} * E * \Omega \to E$ is continuous and for all $x \in E, \aleph \in \Omega$ the function $\vartheta(., x, \aleph) : J * \mathfrak{B} * \Omega \to E$ is strongly measurable,
(G_7) for all positive integer m, there exist $\alpha_m \in L^1(0,T)$ and $\beta_m \in L^1(0,T) \ni ||\zeta(\rho, x, y, \aleph)|| \leq \alpha_m(\rho, \aleph)$ and $||\vartheta(\rho, x, \aleph)|| \leq \beta_m(\rho, \aleph)$ for x, y satisfying $||x|| \leq m, ||y|| \leq m$ and for almost everywhere $\rho \in J$.
(G_8) $T_1(\rho), \rho > 0$ is compact.

Theorem 3. *If the hypotheses $(G_1) - (G_7)$ are satisfactory, system (4) - (6) has at least one mild solution on $(-\infty, T]$, assuming that*

$$r = \int_0^\rho \exp\left(\int_0^\tau [(\sigma + \sigma')p(\sigma) + (L+N)q(\sigma)]d\sigma\right)d\tau < 1,$$

where $\sigma = \sup\{||T_1(\rho)|| : \rho \in J\}$.

Proof. Let the operator $F : B \to 2^B$ be defined by

$$(F(\aleph)x)(\rho) = \begin{cases} \phi(\rho, \aleph), \quad \rho \in (-\infty, 0], \\ T_1(\rho)\phi_1(0, \aleph) + T_2(\rho)[\phi_2(\aleph) + \vartheta(0, \phi_1(0, \aleph), \aleph)] - \int_o^\rho T_1(\rho - \tau), \\ \vartheta(\tau, x_\tau(., \aleph), \aleph)d\tau + \int_0^\rho T_2(\rho - \tau)\zeta(\tau, x_\tau(., \aleph), \int_0^s \gamma(\tau, \eta)\upsilon(\eta, x(\eta, \aleph))d\eta, \aleph)d\tau, \end{cases} \tag{7}$$

where $\rho \in J$. Using the hypothesis G_2, G_3 and G_3 for each $\rho \in J$, we have

$$\begin{aligned} ||(F(\aleph)x)(\rho)|| \leq \; & ||T_1(\rho)\phi_1(0, \aleph)|| + ||T_2(\rho)[\phi_2(\aleph) + \vartheta(0, \phi_1(0, \aleph), \aleph)]|| \\ & + ||\int_0^\rho T_1(\rho - \tau)\vartheta(\tau, x_\tau(., \aleph), \aleph)d\tau|| \\ & + \int_0^\rho ||T_2(\rho - \tau)\zeta(\tau, x_\tau(., \aleph), \int_0^s \gamma(\tau, \eta)\upsilon(\eta, x(\eta, \aleph))d\eta, \aleph)d\tau|| \\ \leq \; & \sigma||\phi_1|| + \sigma' b||\phi_2 + c_1\phi_1 + c_2|| + \int_0^\rho \sigma v(s, \aleph)||x_s(., \aleph), \aleph||ds \\ & + \int_0^\rho \sigma q(s, \aleph)[(||x_s(., \aleph)|| + L_1 LH||x(s)), \aleph]ds, \end{aligned}$$

where $\rho \in J$. Since γ is continuous on compact set $J * J$, there exist $L_1 > 0$ such that

$$||\gamma(\rho, s)|| \leq L_1 \quad \text{for} \quad 0 \leq s \leq \rho \leq T.$$

Consider the function μ_1 given by $\mu(\rho,\aleph) = \sup\{||(F(\aleph)x)(\rho)|| : -\infty, s \le \rho\}, \rho \in J$. Let $\rho^* \in (-\infty, T]$ be such that $\mu(\rho,\aleph) = ||(F(\aleph)x)(\rho)||$. If $\rho^* \in [0,T] = J$ by the previous inequality, we have

$$\begin{aligned}\mu(\rho,\aleph) \le \; & \sigma||\phi_1||_C + \sigma'||\phi_2 + c_1\phi_1 + c_2||_C + \int_0^\rho \sigma v(s,\aleph)||\mu_\rho(.,\aleph),\aleph||ds \\ & + \int_0^\rho \sigma' q(s,\aleph)[(||\mu_\rho(.,\aleph)|| + L_1 LH||\mu(\rho,\aleph)),\aleph]ds.\end{aligned} \tag{8}$$

Let $\mu'(\rho,\aleph) = \sup||(F'(\aleph)x)(\rho)|| : 0 \le s \le \rho, \rho \in J$. If $\rho^* \in (-\infty, 0]$ then $\mu(\rho,\aleph)b \le ||\phi_1||_C$ and the previous inequality obviously holds. But for $\rho \in J$ using equation (7), we get

$$\begin{aligned}(F'(\aleph)x)(\rho) = \; & AT_2(\rho)\phi_1(0,\aleph) + T_1(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)] - \int_0^\rho AT_2(\rho-\tau)\zeta(\tau,x_\tau(.,\aleph),\aleph)d\tau \\ & + \int_0^\rho T_1(\rho-\tau)\vartheta(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\zeta(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau,\end{aligned}$$

and using hypothesis (G_2), (G_3) and (G_4), we obtain

$$\begin{aligned}||(F'(\aleph)x)(\rho)|| \le \; & ||AT_2(\rho)\phi_1(0,\aleph)|| + ||T_1(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)]|| \\ & + ||\int_0^\rho AT_2(\rho-\tau)\vartheta(\tau,x_\tau(.,\aleph),\aleph)d\tau|| \\ & + \int_0^\rho ||T_1(\rho-\tau)\zeta(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau|| \\ \le \; & ||AT_2(\rho)||||\phi_1|| + N||\phi_2 + c_1\phi_1 + c_2|| + \int_0^\rho Lv(s,\aleph)||x_s(.,\aleph),\aleph||ds \\ & + \int_0^\rho Nq(s,\aleph)[(||x_s(.,\aleph)||, L^1 LH||x(s)||),\aleph]ds.\end{aligned}$$

Let $\rho^{**} \in J$ be such that $\mu'(\rho,\aleph) = ||x(\rho^{**})||$ and let $L = \sup\{||AS(\rho)|| : \rho \in J\}$, then we have

$$\begin{aligned}\mu'(\rho,\aleph) \quad & \le L||\phi_1|| + \sigma||\phi_2 + c_1\phi_1 + c_2|| + \int_0^\rho Lv(s,\aleph)||\mu_\rho(.,\aleph),\aleph||ds \\ & + \int_0^\rho \sigma q(s,\aleph)[||\mu_\rho(.,\aleph)||, L^1 LH||\mu(\rho,\aleph)||]ds.\end{aligned} \tag{9}$$

Adding (8) and (9), we get

$$\begin{aligned}\mu(\rho,\aleph) + \mu'(\rho,\aleph) \le \; & ||\phi_1||[L + \sigma + c_1(\sigma+\sigma')] + (\sigma+\sigma')||\phi_2|| + (L+\sigma)\int_0^\rho v(s,\aleph)||\mu_\rho(.,\aleph)|| \\ & + (\sigma+\sigma')\int_0^\rho q(s,\aleph)[\mu_\rho(.,\aleph) + L_1 LH\mu(\rho,\aleph)]ds \\ \mu(\rho,\aleph) + \mu'(\rho,\aleph) \le \; & \frac{||\phi_1||[L + \sigma + c_1(\sigma+\sigma')] + (\sigma+\sigma')||\phi_2|| + (c_2(\sigma+\sigma'))}{1-r^*} \\ & \times \exp\left(\int_0^b [(\sigma+\sigma')q(s,\aleph) + (L+\sigma)v(s,\aleph)]ds\right).\end{aligned}$$

Therefore $||(F(\aleph)x)(\rho)|| \le \mu(\rho,\aleph) \le \gamma$ and $||(F'(\aleph)x)(\rho)|| \le \mu'(\rho,\aleph) \le \gamma, \rho \in J$ and hence $||x||_{\mathfrak{B}} \le \gamma$.
Step 1: Now we have to prove that the operator defined in (7) is uniformly bounded .

Let $B_k = \{x \in B : ||x||_{\mathfrak{B}} \leq \gamma, \rho \in J\}$ for $\gamma \geq 1$. Using the hypothesis (G_2),(G_3),(G_4) and (G_6), we have

$$\begin{aligned}
||(F(\aleph)x)(\rho)|| \leq\ & ||T_1(\rho)\phi_1(0,\aleph)|| + ||T_2(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)]|| \\
& + ||\int_0^\rho T_1(\rho-\tau)\vartheta(\tau,x_\tau(.,\aleph),\aleph)d\tau|| \\
& + \int_0^\rho ||T_2(\rho-\tau)\zeta(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau|| \\
\leq\ & ||T_1(\rho)||||\phi_1|| + ||T_2(\rho)||||\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)|| + \int_0^\rho ||T_1(\rho-s)||\beta_m(\rho,\aleph)d\rho \\
& + \int_0^\rho ||T_2(\rho-s)||\alpha_m(\rho,\aleph)d\rho, \\
||(F(\aleph)x)'(\rho)|| \leq\ & ||AT_2(\rho)\phi_1(0,\aleph)|| + ||T_1(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)]|| \\
& + ||\int_0^\rho AT_2(\rho-\tau)\vartheta(\tau,x_\tau(.,\aleph),\aleph)d\tau|| \\
& + \int_0^\rho ||T_1(\rho-\tau)\zeta(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau|| \\
\leq\ & ||AT_2(\rho)\phi_1(0,\aleph)|| + ||T_1(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)]|| \\
& + ||\int_0^\rho ||AT_2(\rho-s)||\beta_m(\rho,\aleph)d\rho + \int_0^\rho ||T_1(\rho-s)||\alpha_m(\rho,\aleph)d\rho.
\end{aligned}$$

We are aware that $T_1(\rho), T_2(\rho)$ are uniformly bounded, and the compactness of $T_1(\rho), T_2(\rho)$ implies that the uniform operator topology is uniformly bounded for $\rho > 0$. Hence F maps bounded set into a bounded set.

Step 2: To show that the operator in (7) has a closed graph.

It is enough to show that $(F(\aleph)x_n)(\rho) \to (F(\aleph)x)(\rho), (F(\aleph)y_n)(\rho) \to (Fy)(\rho)$ and if $y_n \in \phi(x_n)$ then $y \in \phi(x_n)$. We have

$$\begin{aligned}
||(F(\aleph)x_n)(\rho)|| \leq & ||T_1(\rho)\phi_1(0,\aleph)|| + ||T_2(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)]|| \\
& + ||\int_0^\rho T_1(\rho-\tau)\vartheta(\tau,x_{n\tau}(.,\aleph),\aleph)d\tau|| \\
& + \int_0^\rho ||T_2(\rho-\tau)\zeta(\tau,x_{n\tau}(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x_n(\eta,\aleph))d\eta,\aleph)d\tau||, \\
||(F(\aleph)x_n)(\rho) - (F(\aleph)x)(\rho)|| \leq & ||\int_0^\rho T_1(\rho-\tau)(\vartheta(\tau,x_{n\tau}(.,\aleph),\aleph)d\tau - \vartheta(\tau,x_\tau(.,\aleph),\aleph)d\tau)|| \\
& + ||\int_0^\rho T_2(\rho-\tau)[\zeta(\tau,x_{n\tau}(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x_n(\eta,\aleph))d\eta,\aleph)d\tau \\
& - \zeta(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau]|| \\
\leq & \int_0^\rho ||T_1(\rho-\tau)||||(\vartheta(\tau,x_{n\tau}(.,\aleph),\aleph)d\tau - \vartheta(\tau,x_\tau(.,\aleph),\aleph)||d\tau) \\
& \int_0^\rho ||T_2(\rho-\tau)||||\zeta(\tau,x_{n\tau}(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x_n(\eta,\aleph))d\eta,\aleph)d\tau \\
& - \zeta(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau||,
\end{aligned}$$

$$\begin{aligned}||(F(\aleph)x_n)'(\rho)|| \leq & ||AT_2(\rho)\phi_1(0,\aleph)|| + ||T_1(\rho)[\phi_2(\aleph) + \vartheta(0,\phi_1(0,\aleph),\aleph)]|| \\ & + ||\int_0^\rho AT_2(\rho-\tau)\vartheta(\tau,x_{n\tau}(.,\aleph),\aleph)d\tau|| \\ & + \int_0^\rho ||T_1(\rho-\tau)\zeta(\tau,x_{n\tau}(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x_n(\eta,\aleph))d\eta,\aleph)d\tau||, \\ ||(F(\aleph)x_n)'(\rho)-(F(\aleph)x)(\rho)|| \leq & ||\int_0^\rho AT_2(\rho-\tau)[\vartheta(\tau,x_{n\tau}(.,\aleph),\aleph)-\vartheta(\tau,x_\tau(.,\aleph),\aleph)]d\tau|| \\ & ||\int_0^\rho T_1(\rho-\tau)[\zeta(\tau,x_{n\tau}(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x_n(\eta,\aleph))d\eta,\aleph)d\tau \\ & -\zeta(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau]|| \\ \leq & \int_0^\rho ||AT_2(\rho-\tau)||||\vartheta(\tau,x_{n\tau}(.,\aleph),\aleph)-\vartheta(\tau,x_n(.,\aleph),\aleph)||d\tau \\ & + \int_0^\rho ||T_1(\rho-\tau)||||\zeta(\tau,x_{n\tau}(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x_n(\eta,\aleph))d\eta,\aleph)d\tau \\ & -\zeta(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau||.\end{aligned}$$

We prove $(F(\aleph)x_n)(\rho) \to (F(\aleph)x)(\rho)$. Since $T_1(\rho)$ is compact, let $\{x_n\} \subseteq B$ with $x_n \to x$ in B. This implies that $\vartheta(\tau,x_{n\tau}(.,\aleph),\aleph) \to \vartheta(\tau,x_{(n)}(.,\aleph),\aleph)$ and

$$\zeta\left(\tau,x_{n\tau}(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x_n(\eta,\aleph))d\eta,\aleph\right) \to \zeta\left(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph\right).$$

By using hypothesis (G_5) and (G_6), we have $\left\|\vartheta(\tau,x_{n\tau}(.,\aleph),\aleph)-\vartheta(\tau,x_{(n)}(.,\aleph),\aleph)\right\| \leq 2\beta_m(\tau,\aleph)$ and

$$\begin{aligned}&\Bigg\|\zeta\left(\tau,x_{n\tau}(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x_n(\eta,\aleph))d\eta,\aleph\right) \\ &\quad -\zeta\left(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph\right)\Bigg\| \leq 2\alpha_m(\tau,\aleph),\end{aligned}$$

where $m = 2k\ max\{1+l(\rho),\rho \in J\}$. We have by dominated convergence theorem $||(F(\aleph)x_n)(\rho) - (F(\aleph)x)(\rho)|| \to 0$, and $||(F(\aleph)x_n)'(\rho)-(F(\aleph)x)(\rho)|| \to 0$, when $n \to \infty$ that is $Fx_n \to Fx$. Similarly, $Fy_n \to Fy$ in B_k as $y_n \to y$ in B_k and if $y_n \in Fx_n$, we know that $x_n \to x$ and $y_n \to y$, then we get $y \in Fx$ i.e., $y_n \in Fx_n \implies y \in Fx$. Hence it has a closed graph.

Step 3: To show that the operator in (7) has a convex values. Let us we take $y_1, y_2 \in \mathfrak{B}_k$ and $f_1, f_2 \in E$, then consider

$$\begin{aligned}(F(\aleph)(\alpha y_1+(1-\alpha)y_2))(\rho) = \; & T_1(\rho)\phi_1(0,\aleph)+T_2(\rho)[\phi_2(\aleph)+\vartheta(0,\phi_1(0,\aleph),\aleph)] \\ & -\int_0^\rho T_1(\rho-\tau)(\alpha f_1(\tau,x_\tau(.,\aleph),\aleph)+(1-\alpha)f_2(\tau,x_\tau(.,\aleph),\aleph)d\tau \\ & +\int_0^\rho T_2(\rho-\tau)(\alpha g_1(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau \\ & +(1-\alpha)g_2(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau).\end{aligned}$$

Now taking norm on both sides, we get

$$\begin{aligned}
||(F(\aleph)(\alpha y_1+(1-\alpha)y_2))(\rho)|| \leq{}& ||T_1(\rho)\phi_1(0,\aleph)||+||T_2(\rho)[\phi_2(\aleph)+\vartheta(0,\phi_1(0,\aleph),\aleph)]||\\
&+||\int_0^\rho T_1(\rho-\tau)(\alpha f_1(\tau,x_\tau(.,\aleph),\aleph)+(1-\alpha)f_2(\tau,x_\tau(.,\aleph),\aleph)d\tau||\\
&+\int_0^\rho ||T_2(\rho-\tau)[\alpha g_1(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau\\
&+(1-\alpha)g_2(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta,\aleph)d\tau]||\\
&\leq ||T_1(\rho)\phi_1(0,\aleph)||+||T_2(\rho)[\phi_2(\aleph)+\vartheta(0,\phi_1(0,\aleph),\aleph)]||\\
&+||\int_0^\rho T_1(\rho-s)||||\alpha\beta_{m_1}(\rho)+(1-\alpha)\beta_{m_2}(\rho,\aleph)+\\
&\int_0^\rho ||T_2(\rho-\tau)||[\alpha\alpha_{m_1}(\rho,\aleph)+(1-\alpha)\alpha_{m_2}(\rho,\aleph))]||d\rho,\\
(F(\aleph)(\alpha y_1+(1-\alpha)y_2)')(\rho) ={}& AT_2(\rho)\phi_1(0,\aleph)+T_1(\rho)[\phi_2(\aleph)+\vartheta(0,\phi_1(0,\aleph),\aleph)]\\
&-\int_0^\rho AT_2(\rho-\tau)[\alpha f_1(\tau,x_\tau(.,\aleph),\aleph)+(1-\alpha)f_2(\tau,x_\tau(.,\aleph),\aleph)]d\tau\\
&+\int_0^\rho T_1(\rho-\tau)[\alpha g_1(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta)\\
&-(1-\alpha)g_1(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta)]d\tau,\\
||(F(\aleph)(\alpha y_1+(1-\alpha)y_2)')(\rho)|| \leq{}& ||AT_2(\rho)\phi_1(0,\aleph)||+||T_1(\rho)[\phi_2(\aleph)+\vartheta(0,\phi_1(0,\aleph),\aleph)]||\\
&+||\int_0^\rho AT_2(\rho-\tau)[\alpha f_1(\tau,x_\tau(.,\aleph),\aleph)+(1-\alpha)f_2(\tau,x_\tau(.,\aleph),\aleph)]d\tau||\\
&+\int_0^\rho ||T_1(\rho-\tau)||[||\alpha g_1(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta)\\
&-(1-\alpha)g_1(\tau,x_\tau(.,\aleph),\int_0^s \gamma(\tau,\eta)\upsilon(\eta,x(\eta,\aleph))d\eta)||]d\tau\\
&\leq ||AT_2(\rho)\phi_1(0,\aleph)||+||T_1(\rho)[\phi_2(\aleph)+\vartheta(0,\phi_1(0,\aleph),\aleph)]||\\
&+\int_0^\rho AT_2(\rho-\tau)||\alpha\beta_{m_1}(\rho,\aleph)+(1-\alpha)\beta_{m_2}(\rho,\aleph)d\rho\\
&+\int_0^\rho ||T_1(\rho-\tau)||[||\alpha\alpha_{m_1}(\rho,\aleph)+(1-\alpha_m)\alpha_{m_2}(\rho,\aleph))||]d\rho,
\end{aligned}$$

where $\beta_m=\sup\{||f_i(s,x,y)||, 0\leq s\leq\rho\}, \alpha_m=\sup\{||g_i(s,x,y)||, 0\leq s\leq\rho\}\in E$ which has convex values and $T_1(\rho),T_2(\rho)$ is bounded and linear hence it holds that $\alpha y_1+(1-\alpha)y_2\in\mathfrak{B}_k$. Thus F has convex values.

Step 4: To show that $(F(\aleph)x)(\rho)$ maps a bounded set into a relatively compact set. We already proved that the operator in (7) maps a bounded set into a closed graph and bounded set. We all know that compact set is nothing but closed and bounded set. Hence it maps bounded set into a relatively compact set.

Therefore by Kakutani fixed point theorem F has a fixed point in B. This implies that any fixed point of F satisfying $(F(\aleph)x)(\rho)=x(\rho,\aleph)$ is a mild solution of (4) - (6) on $(-\infty,T]$. Thus initial value problem (4) - (6) has at least one mild solution on $(-\infty,T]$. □

Remark 1. *To ensure that the assumptions of Theorem 1 and Theorem 2 hold, we consider a special case of equations (1) where the functional terms are simplified. Specifically, we assume that the integral term in equations (1) is structured to satisfy the necessary boundedness and continuity conditions. Additionally, the random effects functions ϑ, ζ, and υ are restricted to well-defined forms that align with the theorem's assumptions. Under these conditions, the equations is reduced to a form that allows for direct application of the results established in this work.*

4 Example

Consider the following non-linear mixed partial integro-differential equations

$$\begin{aligned}
&\tfrac{\partial}{\partial\rho}\big(\tfrac{\partial}{\partial\rho}z(y,\rho,\aleph)-R(\rho,z(y,\rho+s),\aleph)\big)\\
&=\tfrac{\partial^2}{\partial y^2}z(y,\rho,\aleph)+F(\rho,z(y,\rho+s),\int_0^\rho\gamma(\rho,s)G(\rho,\rho,z(y,\rho+s)),\aleph)ds
\end{aligned} \tag{10}$$

$$z(0,\rho,\aleph)=z(\pi,\rho,\aleph)=0,\rho\in J, \tag{11}$$

$$z(y,\rho,\aleph)=z_0(y,\aleph),y\in(0,\pi),\rho\in(-\infty,0], \tag{12}$$

$$\frac{\partial}{\partial\rho}z(y,0,\aleph)=\phi(y,\aleph),y\in(0,\pi), \tag{13}$$

and

$$z(y,s,\aleph)=z_0(y,s,\aleph),s\in(-\infty,0],$$

$$\frac{\partial}{\partial\rho}z(y,s,\aleph)=\phi(y,s,\aleph),y\in(0,\pi),$$

where $\phi:[0,T]*(0,\pi)*\Omega\to(0,\pi)$ is continuous and the formulas of the functions F and G are given below. Consider the space $X=L^2[0,\pi]=E$ with usual norm $||.||_{L^2}$ and the operator $A:X\to X$ defined by $Aw=w''$, where $w\in D(A)=\{w\in X: w,w'$ are absolutely coninuous, $w''\in X$ and $w(0)=w(\pi)=0\}$. It is well known that

$$Aw=\sum_{n=1}^{\infty}-n^2(\aleph,w_n)w_n,\ \aleph\in D(A),$$

where $w_n(s)=(\sqrt{\frac{2}{\pi}})\sin ns, n=1,2,3,...$ is the orthogonal set of eigen vectors of A. Then A is the infinitesimal generator of a strongly continuous cosine family $T_2(\rho)$, $\rho\in\mathbb{R}$ in X .

$$T_2(\rho)w=\sum_{n=1}^{\infty}\cos nt(w,w_n)w_n,\ \aleph\in X,$$

and the sine family connected with it is

$$T_2(\rho)w=\sum_{n=1}^{\infty}\frac{1}{n}\sin nt(w,w_n)w_n,w\in X.$$

Let $r:J*\mathfrak{B}\to X$ be defined by $r(\rho,\mu)(v)=R(\rho,\mu(v))$, where $v\in(0,\pi),\mu\in\mathfrak{B}$. We define the function to represent the preceding partial differential equations (10) through (13) as an abstract form

(4) to (6), that is $\vartheta : J * \mathfrak{B} * X * \Omega \to X, \vartheta(\rho,\mu,x,\aleph)(v) = F(\rho,\mu(s)(v),x(v),\aleph), v \in (0,\pi),$ where $F : [0,b] * (0,\pi) * (0,\pi) * \Omega \to (0,\pi)$ is continuous and strongly measurable.
Even more we define $\zeta : J * \mathfrak{B} * E * \Omega \to X$ by,

$$\zeta(\rho,\mu,x,\aleph)(v) = G(\rho,\mu(s)(v),x(v),\aleph),$$

where $G : [0,b] * (0,\pi) \to (0,\pi)$ are continuous and strongly measurable.
Let us consider $F : [0,b] * (0,\pi) * (0,\pi) * \Omega \to (0,\pi)$ and $G : [0,b] * (0,\pi) \to (0,\pi)$ meeting the fundamental assumption:
(F_1) there exist a function $p_1 : [0,b] * \Omega \to (0,\pi), q_1 : [0,b] \to (0,\pi), s_1 : [0,b] * \Omega \to (0,\pi), L_1 : [0,b] \to (0,\pi)$, such that
$(I)||F(\rho,\mu,x)|| \leq p_1(\rho,\aleph)((\|\mu\| + \|x\|),\aleph),$
$(II)||G(\rho,x)|| \leq q_1(\rho)(\|x\|),$
$(III)||R(\rho,\mu)|| \leq s_1(\rho,\aleph)(\|\mu\|,\aleph)$, for every $\rho \in [0,b]$ and $x \in (0,\pi)$.
Since γ is continuous on a compact set $J * J\ \exists\ L_1 > 0$, such that
$(III)\|\gamma(\rho,s)\| \leq L_1,$
(F_2) for each positive integer $m_1 > 0$ there exist $\alpha_m \in L^1(0,b)$, such that

$$\|F(\rho,\mu,x)\| \leq \alpha_m(\rho,\aleph), \quad \text{for all} \quad \|\mu\| \leq m_1, \|x\| \leq m_1,$$

and almost all $\rho \in [0,T]$. The problem (4) - (6) is an abstract formulation of a (10) - (13) using the functions ϑ and ζ and the operator A chosen above. Since all assumptions of Theorem 3.2 are satisfied, (10) - (13) has a solution on $(-\infty,T]$.

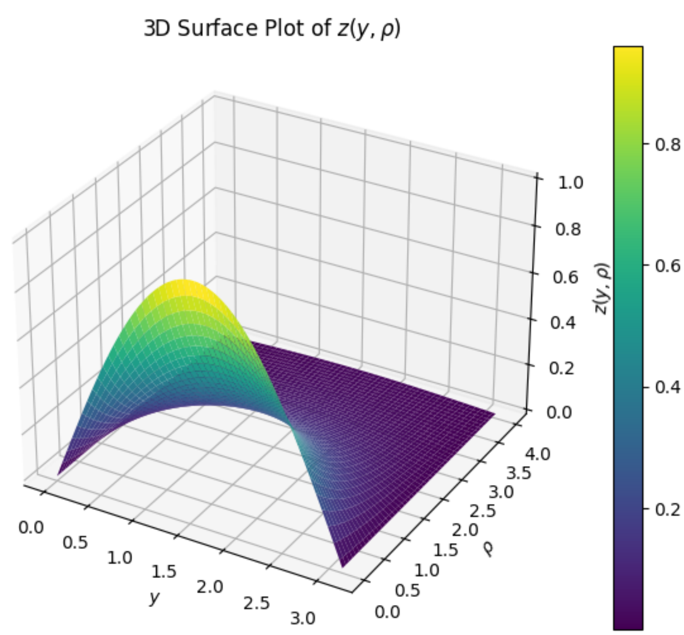

Figure 1: Graphical representation of the solution $z(y,\rho)$ to the non-linear mixed partial integrodifferential equation.

The solution $z(y,\rho)$ to the given non-linear mixed partial integrodifferential equation is graphically represented in Figure 1. The figure shows how the solution varies with respect to the two variables: y and ρ. The solution exhibits a sine wave pattern along the y-axis, indicating periodic oscillations due to the second order derivative in y. This oscillatory behavior is typical for differential equation involving spatial variables. In contrast, the solution decays exponentially in the ρ-direction, reflecting the diminishing influence of ρ on the system as it increases. This exponential decay is modeled by the $\exp(-\rho)$ term in the equation, which captures the weakening effect of ρ over time or distance. The plot demonstrates the interaction between y and ρ, with more pronounced oscillations for smaller ρ and a gradual approach to zero as ρ increases. This interpretation highlights the combined effect of both variables on the behavior of the solution.

5 Conclusion

In this paper, the study delves into the existence of solutions for second order functional differential equation and functional integro-differential equation featuring delays and random effects. The Kakutani fixed point theorem is employed for this purpose. These findings can be extended to explore the controllability of both types of equation. Additionally, instead of using integer powers, the paper considers replacing them with fractional powers denoted as α for further discussions on controllability.

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