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An analytical representation of bivariate isotropic stable density

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Abstract. Stable random vectors are characterized by their characteristic functions. The multivariate stable density and distribution functions generally do not have an analytic form. A few numerical methods have been developed to compute density functions of parametric stable random vectors. However, they have some limitations in terms of the range of the tail index. In this work, via the inversion formula, we present a new analytical representation of the density function of a bivariate isotropic stable random vector. We show that the analytical representation can be reduced to a closed form at the origin.

Keywords: Analytical representation, bivariate isotropic stable density, numerical computation, characteristic function.
AMS Subject Classification 2020: 60E07, 60E10.

1 Introduction

There are a few assumptions in interpreting risk management or portfolio optimization in financial data via mathematical modeling. The heavy-tail distribution is one of the most critical model-fitting assumptions [9]. Heavy tail distributions play important roles in the modeling extreme events in finance and risk management. They are crucial for capturing the empirical regularities of return distributions, particularly for effective assessments of the tail risk and modeling of extreme events in risk management. They help simulate portfolio values and assess risk, especially for assets with skewed and heavy-tailed returns [3].

Stable random vectors are heavy-tailed distributions and exhibit chaotic behavior [4, 5]. Their main limitation is the lack of a closed-form distribution and density functions.

To generalize the central limit theorem, stable random variables and vectors were characterized through their characteristic functions by Paul Lévy in 1920 and Feldheim in 1937 [8]. The characteristic function of an α-stable random vector is determined by a location vector and a spectral measure [8].

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Byczkowski et al. [2] introduced an approximation for the numerical computation of the densities of stable random vectors. Nolan and Balram [7] presented the necessary steps to calculate multivariate stable densities by numerically inverting the characteristic function. They proposed a program to calculate two-dimensional stable densities that uses a recent adaptive quadrature routine.

Zolotarev [10] introduces an integral form for the density function of isotropic stable vectors. The density function of a bivariate isotropic symmetric stable $\boldsymbol{X}$ is given by:

$$f_{\boldsymbol{X}}(x_1,x_2) = \begin{cases} \frac{1}{2\pi}\frac{1}{\sqrt{x_1^2+x_2^2}} f_R(\sqrt{x_1^2+x_2^2}) & \text{if } (x_1,x_2)\neq(0,0) \\ \frac{\Gamma(2/\alpha)}{\alpha 2\pi\gamma^2} & \text{if } (x_1,x_2)=(0,0) \end{cases} \tag{1}$$

where

$$f_R(r) = \int_0^\infty rtJ_0(rt)e^{-\gamma_0^\alpha t^\alpha}\,dt$$

, with stability index $\alpha \in (0,2]$, and scale parameter $\gamma > 0$, and

$$J_0(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{s}{2}\right)^{2m},$$

is the Bessel function of the first kind of order zero. A few integral form for isotropic stable vector density is introduced in [6].

The R, Mathematica, Python, and MATLAB packages compute stable random variables' density, distribution, and quantile functions. The only package that can compute a multivariate stable density function is available on `robustanalysis.com`. However, it is limited to a tail index between 0.8 and 2.

In this work, we present an analytical representation of the density functions of bivariate isotropic stable. One can calculate the exact convolution, reliability, or regression of isotropic stable distributions or check the robustness of bivariate models.

In the next section, we drive the analytical representation of the symmetric bivariate isotropic stable density function. Section 3 compares the calculation method of bivariate isotropic stable density. The paper concluded in Section 5. The Appendix contains sample codes of Section 3.

2 Bivariate isotropic stable density function

The characteristic function of a two dimensional isotropic symmetric stable random vector $\boldsymbol{X} = (X_1,X_2)^\intercal$ has the following form [8]:

$$\phi(\boldsymbol{u}) = E\exp(i<\boldsymbol{u},\boldsymbol{X}>) = \exp(-\gamma^\alpha|\boldsymbol{u}|^\alpha) = \exp\left(-\gamma^\alpha(u_1^2+u_2^2)^{\frac{\alpha}{2}}\right), \tag{2}$$

where $\mathbf{u} = (u_1,u_2)\in\mathbb{R}^2$.

Theorem 1. *Suppose $\boldsymbol{X} = (X_1,X_2)^\intercal$ has an isotropic characteristic function (2) with $\alpha\in(0,2]$, then the analytical representation and closed form of the density function are as follows:*

$$f_{\mathbf{X}}(x_1,x_2)=\begin{cases} \frac{2}{(2\pi)^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}x_1^{2i}x_2^{2j}}{(2i)!(2j)!}\frac{\Gamma\left(\frac{2i+2j+2}{\alpha}\right)}{\alpha(\gamma)^{2i+2j+2}}B\left(j+\frac{1}{2},i+\frac{1}{2}\right) & (x_1,x_2)\neq(0,0) \\ \frac{\Gamma\left(\frac{2}{\alpha}\right)}{2\pi\alpha\gamma^2} & (x_1,x_2)=(0,0) \end{cases}, \tag{3}$$

where B is a beta function and $B(r,s)=\frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$, for $r>0$ and $s>0$.

Proof. **Part 1:** $(x_1,x_2)\neq(0,0)$.
Using the inversion formula in [1], for the isotropic characteristic function (2), we have:

$$f_{\mathbf{X}}(x_1,x_2)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma^{\alpha}\left(u_1^2+u_2^2\right)^{\frac{\alpha}{2}}}\cos(u_1x_1)\cos(u_2x_2)\,du_1du_2. \tag{4}$$

Considering the Taylor expansion of $\cos(ux)$ as follows

$$\cos(u_1x_1)=\sum_{i=0}^{N}\frac{(-1)^i}{(2i)!}(u_1x_1)^{2i}+R_1, \tag{5}$$

$$\cos(u_2x_2)=\sum_{j=0}^{N}\frac{(-1)^j}{(2j)!}(u_2x_2)^{2j}+R_2, \tag{6}$$

where $R_1=\frac{(-1)^{N+1}}{(2N+2)!}(u_1x_1)^{2N+2}$ and $R_2=\frac{(-1)^{N+1}}{(2N+2)!}(u_2x_2)^{2N+2}$, and replacing (5) and (6) in (4) we have

$$f_{\mathbf{X}}(x_1,x_2)=\frac{1}{(2\pi)^2}\sum_{i=0}^{N}\sum_{j=0}^{N}\frac{(-1)^{i+j}x_1^{2i}x_2^{2j}}{(2i)!(2j)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma^{\alpha}\left(u_1^2+u_2^2\right)^{\frac{\alpha}{2}}}u_1^{2i}u_2^{2j}du_1du_2+\text{Error}, \tag{7}$$

where

$$\begin{aligned}\text{Error}=&\sum_{j=0}^{N}\frac{(-1)^j}{(2j)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma^{\alpha}\left(u_1^2+u_2^2\right)^{\frac{\alpha}{2}}}(u_2x_2)^{2j}R_1du_1du_2,\\ &+\sum_{i=0}^{N}\frac{(-1)^i}{(2i)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma^{\alpha}\left(u_1^2+u_2^2\right)^{\frac{\alpha}{2}}}(u_1x_1)^{2i}R_2du_1du_2,\\ &+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma^{\alpha}\left(u_1^2+u_2^2\right)^{\frac{\alpha}{2}}}R_1R_2du_1du_2.\end{aligned}$$

Since the factorial term $(2N+2)!$ dominates both the polynomial growth of $(ux)^{2N+2}$ and the exponential decay presented in the integrand, the factorial growth in the denominator primarily controls the error terms. This dominance leads to rapid convergence. Consequently, for N tends to infinity, we have

$$R_1\to 0 \quad \text{and} \quad R_2\to 0. \tag{8}$$

Using (8), the equation (7) reduces to:

$$f_{\mathbf{X}}(x_1,x_2)=\frac{1}{(2\pi)^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}x_1^{2i}x_2^{2j}}{(2i)!(2j)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma^{\alpha}\left(u_1^2+u_2^2\right)^{\frac{\alpha}{2}}}u_1^{2i}u_2^{2j}du_1du_2. \tag{9}$$

By considering $u_1 = r\cos(\theta)$, $u_2 = r\sin(\theta)$ and converting the integral part of (9) to polar coordinates we have

$$\int_0^{2\pi}\int_0^{\infty} e^{-\gamma^a r^\alpha}(r\cos(\theta))^{2i}(r\sin(\theta))^{2j} r dr d\theta,$$

which is equal to

$$\int_0^{2\pi}\cos^{2i}(\theta)\sin^{2j}(\theta)d\theta\cdot\int_0^{\infty} r^{2i+2j+1}e^{-\gamma^\alpha r^\alpha}dr. \tag{10}$$

By replacing (10) in (9) and rearranging, we get

$$f_{\mathbf{X}}(x_1,x_2) = \frac{1}{(2\pi)^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}x_1^{2i}x_2^{2j}}{(2i)!(2j)!}\int_0^{2\pi}\cos^{2i}(\theta)\sin^{2j}(\theta)d\theta\int_0^{\infty} r^{2i+2j+1}e^{-\gamma^\alpha r^\alpha}dr. \tag{11}$$

To evaluate the radial integral part of (10), using substitution $x = r^\alpha$, $dr = \frac{1}{\alpha}x^{\frac{1}{\alpha}-1}dx$, and simplifying, we obtain

$$\begin{aligned}\int_0^{\infty} r^{2i+2j+1}e^{-\gamma^\alpha r^\alpha}dr &= \int_0^{\infty}\left(x^{1/\alpha}\right)^{2i+2j+1}e^{-\gamma^\alpha x}\cdot\frac{1}{\alpha}x^{\frac{1}{\alpha}-1}dx\\ &= \frac{1}{\alpha}\int_0^{\infty} x^{\frac{2i+2j+2}{\alpha}-1}e^{-\gamma^\alpha x}dx\\ &= \frac{\Gamma\left(\frac{2i+2j+2}{\alpha}\right)}{\alpha(\gamma)^{2i+2j+2}}.\end{aligned} \tag{12}$$

Replacing (12) in (11) gives

$$f_{\mathbf{X}}(x_1,x_2) = \frac{1}{(2\pi)^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}x_1^{2i}x_2^{2j}}{(2i)!(2j)!}\frac{\Gamma\left(\frac{2i+2j+2}{\alpha}\right)}{\alpha(\gamma)^{2i+2j+2}}\int_0^{2\pi}\cos^{2i}(\theta)\sin^{2j}(\theta)d\theta. \tag{13}$$

To solve the integral part of (13), we use the symmetry properties of trigonometric functions as follows

$$\int_0^{2\pi}\cos^{2i}(\theta)\sin^{2j}(\theta)d\theta = 2\int_0^{\pi}\cos^{2i}(\theta)\sin^{2j}(\theta)d\theta. \tag{14}$$

Since, $2i$ and $2j$ are always even, the integral (14) over $[0,2\pi]$ is nonzero and symmetric. So we obtain

$$\int_0^{\pi}\sin^{2j}(\theta)\cos^{2i}(\theta)d\theta = 2\int_0^{\frac{\pi}{2}}\sin^{2j}(\theta)\cos^{2i}(\theta)d\theta. \tag{15}$$

Considering $t = \sin^2(\theta)$, we have $\sin^{2j}(\theta) = t^j$, $\cos^{2i}(\theta) = (1-t)^i$. Therefore, we calculate the integral on the right side of (15) as follows

$$\int_0^{\frac{\pi}{2}}\sin^{2j}(\theta)\cos^{2i}(\theta)d\theta = \int_0^1 t^j(1-t)^i\frac{dt}{2\sqrt{t(1-t)}} = \frac{1}{2}B\left(j+\frac{1}{2}, i+\frac{1}{2}\right), \tag{16}$$

where B is the Beta function. By substituting (16) in (15)

$$\int_0^{\pi}\sin^{2j}(\theta)\cos^{2i}(\theta)d\theta = \int_0^1 t^{j-\frac{1}{2}}(1-t)^{i-\frac{1}{2}}dt = B\left(j+\frac{1}{2}, i+\frac{1}{2}\right),$$

and (15) in (14) we have

$$\int_0^{2\pi} \sin^{2j}(\theta)\cos^{2i}(\theta)d\theta = 2\cdot B\left(j+\frac{1}{2}, i+\frac{1}{2}\right). \tag{17}$$

Finally, by replacing (17) in (13), we have (3), and the proof of part 1 is complete.

Part 2: $(x_1,x_2) = (0,0)$.
By replacing $(x_1,x_2) = (0,0)$ in (4), we have

$$f_{\mathbf{X}}(x_1,x_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\gamma^\alpha\left(u_1^2+u_2^2\right)^{\frac{\alpha}{2}}} du_1 du_2, \tag{18}$$

using $u_1 = r\cos(\theta)$, $u_2 = r\sin(\theta)$ and transforming the (18) to polar coordinates, we have

$$\frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{\infty} e^{-\gamma^\alpha r^\alpha} r dr = \frac{1}{2\pi}\int_0^{\infty} e^{-\gamma^\alpha r^\alpha} r dr. \tag{19}$$

Considering $t = r^\alpha$ and changing variables as

$$r = t^{1/\alpha} \quad \text{and} \quad dr = \frac{1}{\alpha} t^{\frac{1}{\alpha}-1} dt,$$

it follows from (19) that

$$\begin{aligned} f_{\mathbf{X}}(x_1,x_2) &= \frac{1}{2\pi}\int_0^{\infty} e^{-\gamma^\alpha r^\alpha} r dr = \frac{1}{2\pi\alpha}\int_0^{\infty} e^{-\gamma^\alpha t} t^{\frac{2}{\alpha}-1} dt \\ &= \frac{\Gamma\left(\frac{2+\alpha}{\alpha}\right)(\gamma^\alpha)^{-\frac{2}{\alpha}}}{4\pi} = \frac{\Gamma\left(\frac{2}{\alpha}\right)}{2\pi\alpha\gamma^2} \end{aligned}$$

and the proof of Part 2 is complete. □

Remark 1. The density function in (3) at the origin $\big((x_1,x_2) = (0,0)\big)$ reduces to the bivariate isotropic density function introduced in [6].

Figure 1 presents contour plots of the bivariate isotropic symmetric stable density function for four distinct parameter sets. The axes labeled x_1 and x_2, represent the respective variables of the bivariate system. Varying shades within these plots indicate density levels, with darker shades denoting higher densities. These plots effectively compare the behaviors of the density functions under stability indices $\alpha = 0.5, 0.7, 1$ and 1.7, alongside a scale parameter $\gamma = 1$, highlighting notable changes in density concentration around the origin. Table 3 in the next section reports the density value of a few points (x_1,x_2).

3 Computing density

The analytic density function for bivariate isotropic symmetric Cauchy is calculated, e.g., see [8]:

$$f_{\boldsymbol{X}}(x_1,x_2) = \frac{\gamma}{2\pi\left(x_1^2+x_2^2+\gamma^2\right)^{3/2}}, \quad -\infty < x_1,x_2 < \infty.$$

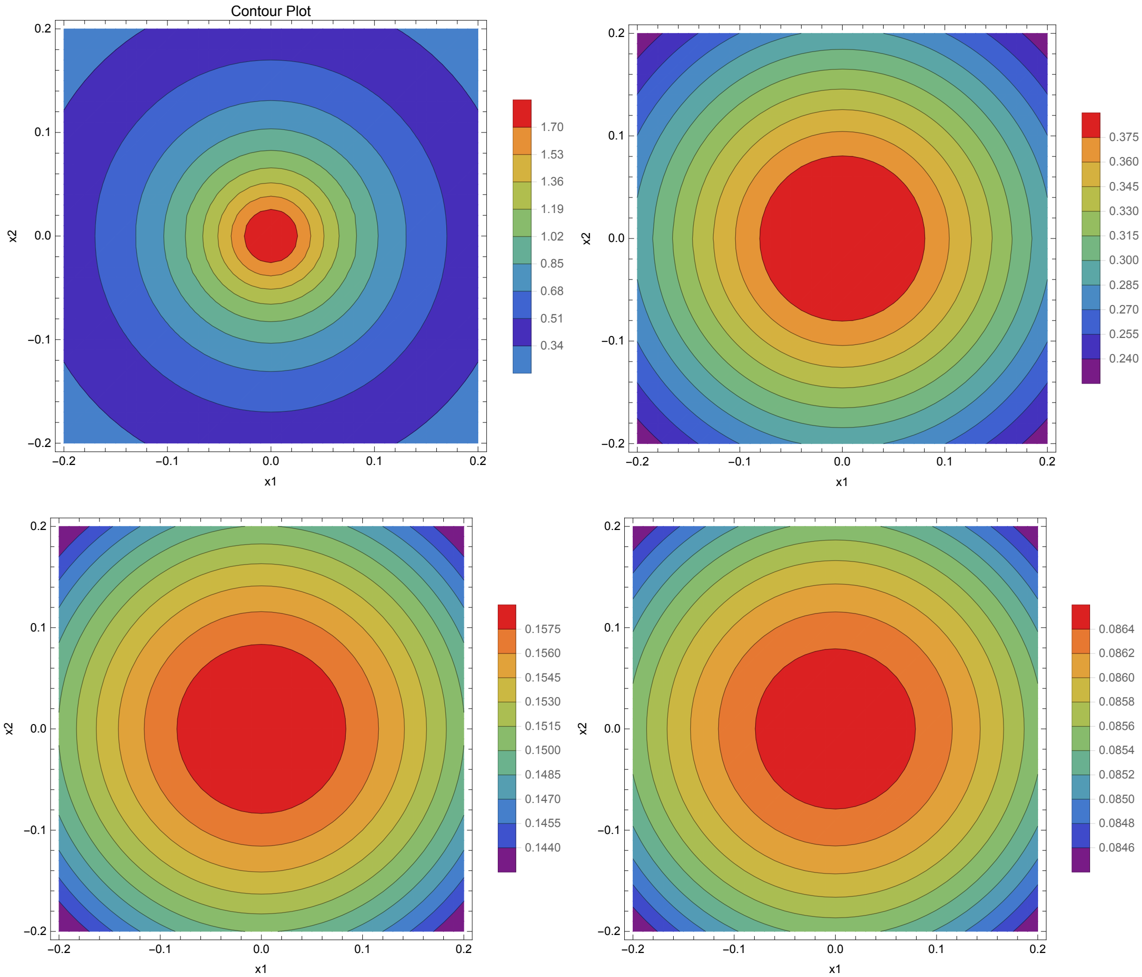

Figure 1: Contour graphs of bivariate isotropic symmetric stable density with $\gamma = 1$. Top left: $\alpha = 0.5$, top right: $\alpha = 0.7$, bottom left: $\alpha = 1$ and bottom right: $\alpha = 1.7$.

We use Robust Analysis's package version 5.3 to compute the density function in R. We use the inversion formula in (4) to see the accuracy. Table 1 presents the computation results. We see that all the methods have the same accuracy.

Table 1: Comparing bivariate isotropic symmetric stable density function approximations for $\alpha = 1$.

	$\alpha = 1$, γ=1					
$\boldsymbol{x} = (x_1, x_2)$	(-1,-1)	(0,1)	(1,0)	(1,1)	(1,2)	(2,1)
Exact	0.030629	0.056270	0.056270	0.030629	0.010829	0.010829
Nolan	0.030629	0.056270	0.056270	0.030629	0.010829	0.010829
Inversion formula	0.030629	0.056270	0.056270	0.030629	0.010829	0.010829

In Table 2, we increase the tail index to α=1.3 and compute the following densities. The exact value of the density function is not available. We use Theorem 1 to compute the density function via the proposed method. The computed values in each column are to six decimal points.

Table 2: Comparing bivariate symmetric isotropic stable density function approximations for $\alpha > 1$.

	$\alpha = 1.3$, γ=1					
(x_1, x_2)	(-1,-1)	(0,1)	(1,0)	(1,1)	(1,2)	(2,1)
Nolan	0.039453	0.062606	0.062606	0.039453	0.014231	0.014231
Proposed Method*	0.039450	0.062601	0.062601	0.039450	0.014230	0.014230

*The values at 0 are approximately computed at 0.0001.

Table 3 presents sample density values for different tail indices in Figure 1 using the inversion formula.

Table 3: Bivariate symmetric isotropic stable density function approximations via inversion formula.

		γ=1					
(x_1, x_2)	(-1,-1)	(0,0)	(0,1)	(1,0)	(1,1)	(1,2)	(2,1)
$\alpha = 0.5$	0.015	1.9	0.029	0.029	0.015	0.006	0.006
$\alpha = 0.7$	0.02	0.4	0.04	0.04	0.02	0.008	0.008
$\alpha = 1$	0.03	0.159	0.056	0.056	0.030	0.011	0.011
$\alpha = 1.7$	0.046	0.087	0.063	0.063	0.046	0.019	0.019

4 Running time comparison

This section compares the running time of computing bivariate isotropic stable density using Zolotarev formula (1), the inversion formula (4) and the proposed method (3). Table 4 demonstrates that the proposed method for computing bivariate isotropic stable densities has a lower average computation time than the inversion and Zolotarev formulas. This efficiency gain underscores the effectiveness of our approach in handling complex density computations within a reduced timeframe. Furthermore, the computational accuracy of the inversion formula and the proposed method was found to be identical and superior to that of the Zolotarev formula. All computations were conducted using Mathematica on a workstation equipped with an Intel® Pentium™ Processor G3250 operating at 3.20 GHz. For the proposed method and Zolotarev formula, the first 30 terms of the series were utilized.

Figure 2 illustrates the running time averages presented in Table 4. The blue and red bars represent the running time averages of the inversion and Zolotarev formulas, respectively. The green bar indicates the proposed method's running time average, which is shorter than the other methods.

5 Conclusion

This work introduced an analytical representation for bivariate isotropic stable density functions. We have demonstrated that using the inversion formula, bivariate isotropic stable density can be computed without any limitations on the range of the tail index. Analytical representation can be developed to calculate multivariate sub-Gaussian distribution and density functions as a more general

Table 4: Average computation times (in seconds) over 10 iterations and computed bivariate isotropic stable densities using the Inversion Formula (4), Zolotarev's Formula (1), and the Proposed Method (3).

	Average of computational time			Computed density		
(x_1, x_2)	Inversion	Zolatarev	Proposed	Inversion	Zolotarev	Proposed
(-2, -1.5)	2.77983	0.0379090	0.00912827	0.0103648	0.0103648	0.0103648
(-2, -1.4)	1.90631	0.0327080	0.00919678	0.0111092	0.0111092	0.0111092
(-2, 0.5)	1.48195	0.0319629	0.00919159	0.0176521	0.0176521	0.0176522
(-2, 1.5)	2.75197	0.0356501	0.00924685	0.0103648	0.0103648	0.0103648
(-2, -2)	2.75508	0.2474190	0.00916426	0.00712625	0.00712625	0.00714252
(-1.4, -1.5)	1.72137	0.0323467	0.00945639	0.0178672	0.0178672	0.0178672
(-1.4, -1.4)	2.52245	0.0268933	0.00925633	0.0195465	0.0195465	0.0195465
(-1.4, 0.5)	1.46872	0.0242274	0.00923911	0.0361464	0.0361464	0.0361464
(-1.4, 1.5)	1.72189	0.0291951	0.00931708	0.0178672	0.0178672	0.0178672
(-1.4, -2)	1.71350	0.0306215	0.00924075	0.0111092	0.0111092	0.0111092
(1.2, -1.5)	2.26589	0.0307974	0.00920460	0.0210454	0.0210454	0.0210454
(1.2, -1.4)	2.29585	0.0307990	0.00925497	0.0231821	0.0231821	0.0231821
(1.2, 0.5)	1.46706	0.0246160	0.00932812	0.0451417	0.0451417	0.0451417
(1.2, 1.5)	2.26798	0.0280298	0.00932533	0.0210454	0.0210454	0.0210454
(1.2, -2)	2.28792	0.0423362	0.00924429	0.0126575	0.0126575	0.0126575
(1.5, -1.5)	2.74446	0.0332080	0.00919001	0.0163893	0.0163893	0.0163893
(1.5, -1.4)	2.74142	0.0295952	0.00925244	0.0178672	0.0178672	0.0178672
(1.5, 0.5)	1.47970	0.0241755	0.00928791	0.0321886	0.0321886	0.0321886
(1.5, 1.5)	2.75232	0.0307820	0.00921067	0.0163893	0.0163893	0.0163893
(1.5, -2)	2.46838	0.0359593	0.00916300	0.0103648	0.0103648	0.0103648
(2, -1.5)	2.75803	0.0360555	0.00922833	0.0103648	0.0103648	0.0103648
(2, -1.4)	1.91134	0.0309745	0.00939851	0.0111092	0.0111092	0.0111092
(2, 0.5)	1.46690	0.0299433	0.00924763	0.0176521	0.0176521	0.0176522
(2, 1.5)	2.76243	0.0357955	0.00919519	0.0103648	0.0103648	0.0103648
(2, -2)	2.76798	0.2451580	0.00926998	0.00712625	0.00712625	0.00714252

parametric sub-class of stable distributions. However, complex calculations are required and may not be solvable. The analytical representation could be reduced to a closed form at the origin and used to verify the accuracy of stable distribution computation software. The Python and Mathematica codes for computing bivariate isotropic stable density using Theorem 1 and the inversion formula are provided in the Appendix.

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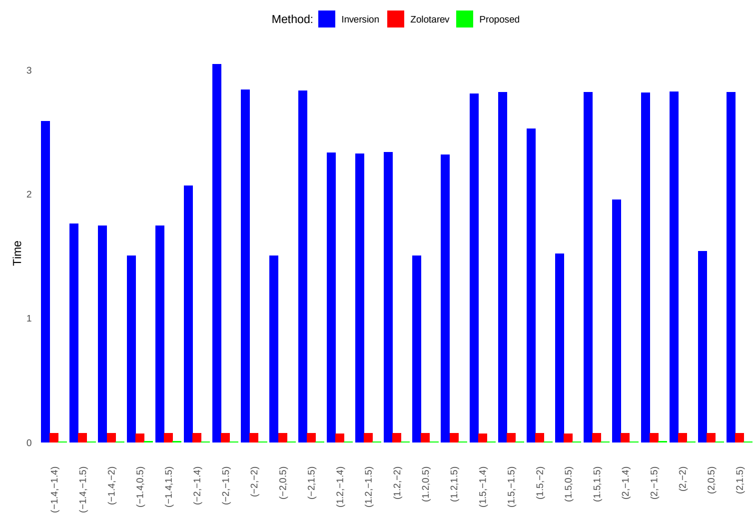

Figure 2: Comparison of computation time averages (in second) over ten iterations for computing bivariate isotropic stable densities using three different methods.

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Appendix

```
import numpy as np
from scipy.special import beta, factorial

# Parameters
alpha = 1.3     # tail index alpha in (1,2)
g=1             # scale parameter
max_terms = 20  # number of terms to sum
# Variables
x1=1
x2=1
# Compute the density function at (x1, x2)
sum_result = 0
for i in range(max_terms):
for j in range(max_terms):
gamma_term = gamma((2 * i + 2 * j + 2) / alpha)
term = ((-1)(i+j) * x1(2*i) * x2**(2*j) * gamma_term *\
beta(j + 0.5, i + 0.5)  )  / (factorial(2*i) *\
factorial(2*j) * alpha * g**(2*(i+j)+2))
sum_result += term

density = (2 / (2*np.pi)**2) * sum_result
print("Approximated density at (",x1,",",x2,") is:", density)

Approximated density at ( 1 , 1 ) is: 0.03944957569338398
```

Listing 1: Python code to calculate bivariate isotropic symmetric stable density function via analytic form in Theorem 1.

```
In[1]:= Remove["Global‘*"];
alpha = 1.3; x1 = 1; x2 = 1; g = 1; k = 20;
2/(2 Pi)^2 * Sum[Sum[(-1)^(i+j) x1^(2i) x2^(2j) Gamma[(2(i+j)
+2)/alpha] Beta[i + .5, j + .5]/(2i)!/(2j)!/ alpha^2 /g^(2(i+j)+2)
,   {i, 0, k}], {j, 0, k}] // N

Out[1]= 0.0394496

(* Inversion Formula Method *)
In[2]:=
alpha = 1.3; x1 = 1; x2 = 1; g = 1;
NIntegrate[Exp[-g^alpha (Sqrt[t1^2 + t2^2])^alpha] Cos[t1 x1]
 Cos[t2 x2], {t1, -Infinity, Infinity}, {t2, -Infinity, Infinity}]
 1/(2 Pi)^2 // N

Out[2]= 0.0394496
```

Listing 2: Mathematica codes to calculate bivariate isotropic symmetric stable density function via the analytic form in Theorem 1 and inversion formula in (3).