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Solutions of nonlinear fractional partial differential-difference equations using the generalized-exponential-rational-function

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ABSTRACT

The central topic of the present article is the investigation of solutions of nonlinear fractional partial differential models (NFPDDEs) using the generalized-exponential-rational-function (GEERAF) Method. In this regard, the jumarie's modified-riemann-liouville (JMRL) derivative has been used to convert the proposed model into ordinary differential-difference model (ODDEM). This efficient proposed method can be used as a replacement for generating novel types of solutions to NFPDDEs in Scientific issues. According to the scientific literature, our findings have not been published before in any other sources.

1. Introduction

Fractional differential calculus is of great importance in problems in engineering, mathematics, and physics. In recent decades, researches have shown that these types of equations can be very useful and efficient by modeling various phenomena in terms of fractional derivatives. They have many applications in many fields such as nonlinear earthquake oscillations, quantum mechanics [1,2], physics and plasma physics [3,4], and light propagation. Therefore, solving fractional differential equations or NFPDDEs is one of the problems of the day. Several methods used to find exact soliton wave solutions including: Extended tanh technique [5-7], Generalized and Modified Kudryashov approach [8,9], Extended hyperbolic technique [10], $\frac{G'}{G^2}$-expansion approach [11], Double $\left(\frac{G'}{G},\frac{1}{G}\right)$ -expansion technique [12], Generalized projective Riccati equation approach [13], and etc. [14-17].

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The theory of the GEERAF method was first introduced by Ghanbari et al. for nonlinear partial differential equations [18]. In recent years, this method has been used by scientists to solve various partial differential equations [19-25].

Sometimes, the modeling of real-world phenomena using differential-difference equations (DDEs) will be described and introduced. Many of the fundamental concepts of partial differential equations were also extended to DDEs after the introduction of Fermi's theory in the early 1960 [26].

Scientists apply nonlinear lattice equations for physical model development including applications in electric current research along with electric and biological circuit pulse behavior [27-28]. After Fermi and over the years, various lattices of differential equations have been introduced, such as the Hybrid (Wadati) lattice, the Diederik Korteweg lattice, the Toda lattice, the Ablowitz Ladik lattice, and the Generalized Volterra lattice [29,30].

After the emergence of soliton theory, in addition to solutions to partial differential equations, discussions began about solutions to DDEs and FDDEs, including the Hirota's approach [31], the $\left(\frac{G'}{G}\right)$-expansion method [32,33], the Generalized differential transform approach [34,35], the Extended Riccati Sub-ODE technique [36] which are the most common methods. This paper seeks to explore FDDEs of the form

$$D_{\beta}^{t} w_n = Q(w_{n-1}, w_n, w_{n+1}), \quad 0 < \beta \leq 1$$

where $\boldsymbol{D_{\beta}^{t}}$ represents the JMRL of order $\boldsymbol{\beta}$, $\boldsymbol{Q}$ is a rational function and \$ $\boldsymbol{w_n = w(n, t)}$ represents how much the nth particle has shifted from its equilibrium position.

In the section ‘Description of the GEERAF technique ‘, we describe the basic steps of the GEERAF method for FDDEs. In the section ‘Procedure of Solution ‘, the linearization process of FDDEs and the results obtained by the method for solving an NFPDDEs are presented. The paper concludes with a 3D graphical representation of the results in these sections and some concluding remarks.

2. 2. Preliminaries of the JMRL derivative

In this section, we introduce The JMRL derivative. Let $\boldsymbol{h}$: $\mathbb{R} \to \mathbb{R}$ represent a continuous function (but it may not be differentiable). The JMRL derivative of order $\boldsymbol{\beta}$ is expressed as follows [37]:

$$ {}^{J}D_{z}^{\beta}\, h(z) = \begin{cases} \dfrac{1}{\Gamma(-\beta)} \displaystyle\int_{0}^{t} (z-\zeta)^{-\beta-1} \left(h(\zeta) - h(0)\right) d\zeta, & \beta < 0, \\ \dfrac{1}{\Gamma(1-\beta)} \dfrac{d}{dt} \displaystyle\int_{0}^{t} (z-\zeta)^{-\beta} \left(h(\zeta) - h(0)\right) d\zeta, & 0 < \beta < 1, \\ {}^{J}D_{z}^{\beta-k}\, D^{k} h(z), & k \leq \beta \leq k+1, \quad k > 1, \end{cases} $$

Some important properties of the JMRL derivative are as follows:

$$
{}^{J}D_z^{\beta-k}\, z^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta+1-\beta)}\, z^{-\beta+\eta}; \quad \eta > 0,
$$
$$
{}^{J}D_z^{\beta}\,\big(h(z)g(z)\big) = g(z)\,{}^{J}D_z^{\beta-k} h(z) + h(z)\,{}^{J}D_z^{\beta} g(z),
$$
$$
{}^{J}D_z^{\beta}\,\Big(h\big(g(z)\big)\Big) = h'_g\big(g(z)\big)\,{}^{J}D_z^{\beta} g(z) = {}^{J}D_g^{\beta} h\big(g(z)\big)(g'_z)^{\beta},
$$

3. Description of the GEERAF Technique

As mentioned, in recent years, many models have been solved using the GEERAF method. In this section, we briefly present an application of this technique to determine solutions to NFPDDEs. The general form of the NFPDDEs is as follows:

$$
\mathbb{P}(w_{n+s_1}, \dots, w_{n+s_c}, \dots, D_t^{\beta} w_{n+s_1}, \dots, D_t^{\beta} w_{n+s_c}, \dots, D_t^{\beta}\, w_{n+s_1}, \dots, D_t^{\beta} w_{n+s_c}) = 0, \quad 0 < \beta \leq 1. \tag{1}
$$

Where $w_n$is the unknown function, and ${}^{J}D_t^{\beta} w_n$ is the JMRL derivative of order $0 < \beta \leq 1$.

Step 1. According to the following fractional complex transformation

$$
w_{n+s_p}(t) = \chi_{n+s_p}(\xi_n), \;\; \xi_n = \sum_{\mathbb{i}=1}^{\mathbb{Q}} d_{\mathbb{i}}\, n_{\mathbb{i}} + \sum_{\mathbb{j}=1}^{\mathbb{N}} \frac{k_{\mathbb{j}} t_{\mathbb{j}}^{\alpha}}{\Gamma(1+\gamma)} + \zeta, \;\; p = 1,2, \dots, c. \tag{2}
$$

Then, ***Eq. (1)*** becomes a system of ODDEs of integer order, as shown below:

$$
\mathbb{P}(\chi_{n+s_1}, \dots, \chi_{n+s_c}, \dots, \chi'_{n+s_1}, \dots, \chi'_{n+s_c}, \dots, \chi_{n+s_1}^{(r)}, \dots, \chi_{n+s_c}^{(rn)}) = 0, \quad 0 < \beta \leq 1. \tag{3}
$$

Step 2. We seek solutions of ***Eq. (3)*** that can be expressed as a finite series in $\Omega(\xi_n)$ as follows:

$$
\chi(\xi_n) = \Theta_0 + \sum_{\mathbb{i}=1}^{\mathbb{M}} \Theta_{\mathbb{i}} \left(\Omega(\xi_n)\right)^{\mathbb{i}} + \sum_{\mathbb{j}=1}^{\mathbb{M}} \Lambda_{\mathbb{j}} \left(\Omega(\xi_n)\right)^{-\mathbb{j}}, \tag{4}
$$

Where

$$
\Omega(\xi_n) = \frac{\theta_1\, exp(\alpha_1\, \xi_n) + \theta_2\, exp(\alpha_2\, \xi_n)}{\theta_3\, exp(\alpha_3\, \xi_n) + \theta_4\, exp(\alpha_4\, \xi_n)}, \tag{5}
$$

In the assumed structures ***Eq. (4)*** and ***Eq. (5)***, the values of the constants $\theta_{\mathbb{J}}$, $\alpha_{\mathbb{j}}$'s (where $1 \leq \mathbb{j} \leq 4$), Θ_0 , $\Theta_{\mathbb{j}}$', and $\Lambda_{\mathbb{J}}$'s} (where $1 \leq \mathbb{j} \leq 4$) are ascertained by substituting the solution ***Eq. (4)*** into ***Eq. (3)***. In addition, the positive number of $\mathbb{M}$ can be ascertained by using balance rules.

Step 3. Substituting ***Eq. (4)*** into ***Eq. (3)***, collecting all powers of $exp\big(\alpha_{\mathbb{j}}\, \xi_n\big)$ for $\mathbb{j} = 1, \dots, 4$ and equating the coefficients to zero results in a system of nonlinear equations.

Step 4. Eventually, solutions to ***Eq. (1)*** are acquired after solving the obtained system and inserting the obtained values into ***Eq. (4)***.

4. Procedure of solution

We will obtain analytical solutions of the following NFPDDEs equation, given by [38]

$$ {}^{J}D_t^{\beta}\, w_n = \frac{w_{n-1} - w_{n+1} + 2w_n - w_{n-1}w_{n+1} + 2w_n w_{n+1} - 2w_n^2}{1 + w_{n-1} - w_{n+1}}, \quad 0 < \beta \leq 1, \tag{6} $$

To find new analytical solutions of ***Eq. (6)***, the following transformations in ***Eq. (2)*** are utilized, where $\mathbb{Q} = \mathbb{N} = 1$, and we assume $d_1 = d, k_1 = k$ and $t_1 = t$. we get

$$ k\chi_n'(1 + \chi_{n-1} - \chi_{n+1}) - \chi_{n-1} - \chi_{n+1} + 2\chi_{n-1}\chi_n + 2\chi_n\chi_{n+1} - 2\chi_{n-1}\chi_{n+1} - 2\chi_n^2 = 0, \tag{7} $$

Then, by balancing χ_n^2 and χ_n' in ***Eq. (7)***, $\mathbb{M} = 1$. Substituting $\mathbb{M} = 1$ into ***Eq. (4)*** yields the form:

$$ \chi(\xi_n) = \Theta_0 + \Theta_1\big(\Omega(\xi_n)\big) + \Lambda_1\big(\Omega(\xi_n)\big)^{-1}, \tag{8} $$

After implementing the GEERAF method, we obtain the following various solutions of ***Eq. (6)***:

Family I. By assuming $\theta = [-1,1,1,1]$, $\alpha = [1,-1,1,-1]$ in ***Eq. (5)***,

$$ \Omega(\xi_n) = \frac{-e^{\xi_n} + e^{-\xi_n}}{e^{\xi_n} + e^{-\xi_n}} = \frac{\frac{-e^{\xi_n} + e^{-\xi_n}}{2}}{\frac{e^{\xi_n} + e^{-\xi_n}}{2}} = -\,tanh(\xi_n), \tag{9} $$

By substituting ***Eq. (9)*** into ***Eq. (8)***, the following values can be obtained

$$ \Theta_0 = \Theta_0, \qquad \Theta_1 = \frac{1 - e^{8d}}{4 + 4e^{8d}}, \qquad \Lambda_1 = \frac{1 - e^{8d}}{4 + 4e^{8d}}, \qquad k = -\frac{1}{4}e^{-4d}(-1 + e^{8d}), $$

Taking the above values and substituting them into ***Eq. (4)*** gives the following result

$$ \chi(\xi_n) = \Theta_0 + \frac{1}{2}\, coth(2\xi_n)\, tanh(4d), \tag{10} $$

Thus, an analytical solution to ***Eq. (6)*** is obtained as follows:

$$ w(n,t) = \Theta_0 + \frac{1}{2}\, coth\left(2(dn + \zeta) - \frac{t\, sinh(4d)}{\Gamma(1+\gamma)}\right) tanh(4d), \tag{11} $$

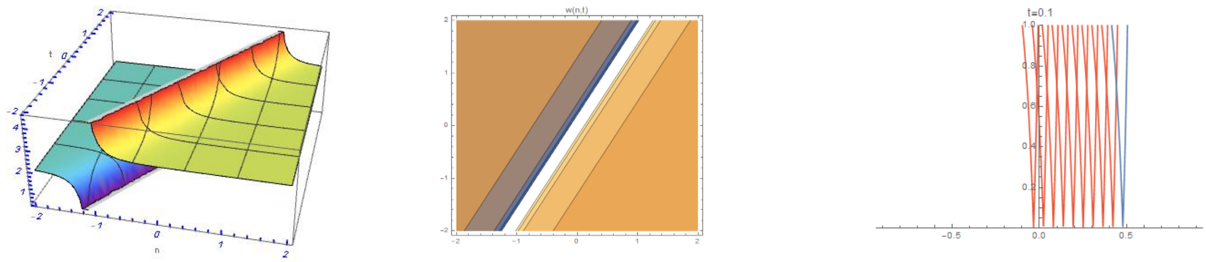

Figure 1. 3D, contour and 2D plots of the solution $w_1(n,t)$ for $\Theta_0 = 2.5, d = 1, \gamma = 4$ and $\zeta = 0$.

Family II. By assuming $\theta = [1,-1,2,0]$, $\alpha = [1,-1,-1,0]$ in ***Eq. (5)***,

$$\Omega(\xi_n) = cosh(\xi_n)\, sinh(\xi_n) + sinh(\xi_n)^2, \tag{12}$$

By substituting ***Eq. (12)*** into ***Eq. (8)***, the following values can be obtained

$$\Theta_0 = \Theta_0, \qquad \Theta_1 = 0, \qquad \Lambda_1 = \frac{1}{2}\, tanh(2d), \qquad k = -sinh(2d),$$

Taking the above values and substituting them into ***Eq. (4)*** gives the following result

$$\chi(\xi_n) = \Theta_0 + \frac{tanh(2d)}{-1 + e^{2\xi_n}}, \tag{13}$$

Thus, an analytical solution to ***Eq. (6)*** is obtained as follows:

$$w(n,t) = \Theta_0 + \frac{tanh(2d)}{-1 + e^{2+2dn-\frac{t\, sinh(4d)}{\Gamma(1+\gamma)}}}, \tag{14}$$

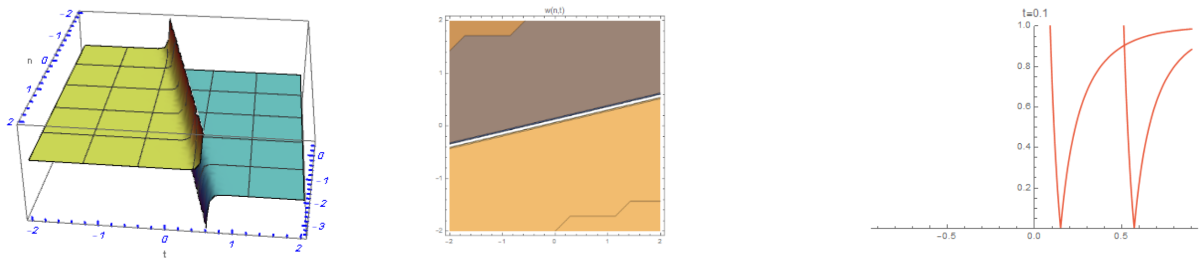

Figure 2. 3D, contour and 2D plots of the solution $\boldsymbol{w_2(n,t)}$ for $\boldsymbol{\Theta_0 = -1, d = 2.4, \gamma = 3}$ and $\zeta = 1$.

Family III. By assuming $\theta = [1,-1,2,0]$, $\alpha = [1,-1,-1,0]$ in ***Eq. (5)***,

$$\Omega(\xi_n) = -\frac{2}{5} cosh(\xi_n)\, sinh(\xi_n) + \frac{2\, sinh(\xi_n)^2}{5}, \tag{15}$$

By substituting ***Eq. (12)*** into ***Eq. (8)***, the following values can be obtained

$$\Theta_0 = \Theta_0, \qquad \Theta_1 = 0, \qquad \Lambda_1 = \frac{1 - 4e^{4d}}{5 + 5e^{4d}}\,, \qquad k = -\frac{1}{2}\, e^{-2d}(-1 + e^{4d}),$$

Taking the above values and substituting them into ***Eq. (4)*** gives the following result

$$\chi(\xi_n) = \Theta_0 + \frac{e^{2\xi_n}(-1 + e^{4d})}{(1 - e^{4d})(-1 + e^{2\xi_n})}, \tag{16}$$

Thus, an analytical solution to ***Eq. (6)*** is obtained as follows:

$$w(n,t) = \Theta_0 + coth\left(dn + \zeta - \frac{t\, sinh(2d)}{\Gamma(1+\gamma)}\right) tanh(2d), \tag{17}$$

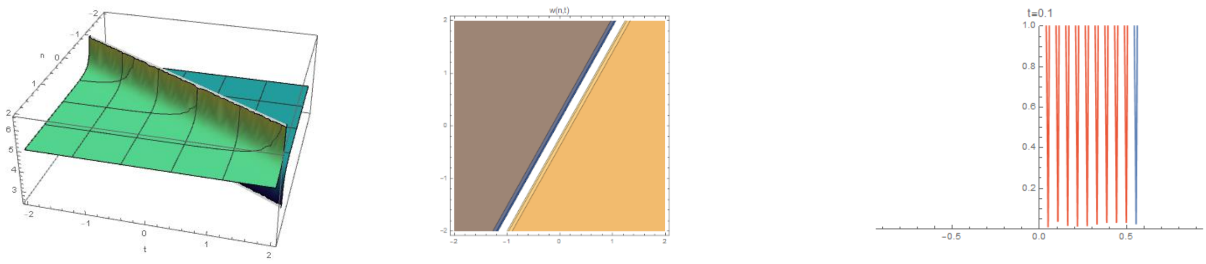

Figure 3. 3D, contour and 2D plots of the solution $\boldsymbol{w_3(n,t)}$ for $\boldsymbol{\Theta_0 = -1, d = 2.4, \gamma = 3}$ and $\boldsymbol{\zeta = 1}$.

Family IV. By assuming $\theta = [-3,-1,1,1]$, $\alpha = [1,-1,1,-1]$ in ***Eq. (5)***,

$$\Omega(\xi_n) = -2 - tanh(\xi_n), \tag{18}$$

By substituting ***Eq. (18)*** into ***Eq. (8)***, the following values can be obtained

$$\Theta_0 = \Theta_0, \qquad \Theta_1 = 0, \qquad \Lambda_1 = \frac{3}{2}\, tanh(2d), \qquad k = -2\, cosh(d) sinh(2d),$$

Taking the above values and substituting them into ***Eq. (4)*** gives the following result

$$\chi(\xi_n) = \Theta_0 + \frac{3\left(1 + e^{2\xi_n}\right) tanh(2d)}{2 + 6e^{2\xi_n}}, \tag{19}$$

Thus, an analytical solution to ***Eq. (6)*** is obtained as follows:

$$w(n,t) = \Theta_0 + \frac{e^{2(dn+\zeta)}(-1 + e^{4d})}{(1 + e^{4d})e^{2(dn+\zeta) - \frac{t\, sinh(22d)}{\Gamma(1+\gamma)}}}, \tag{20}$$

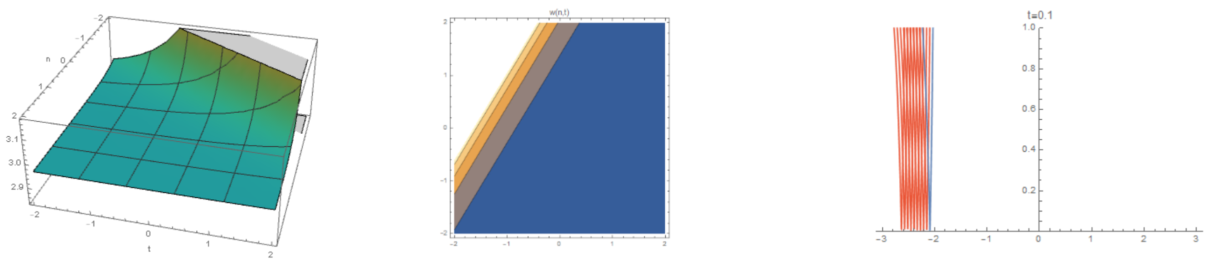

Figure 4. 3D, contour and 2D plots of the solution $\boldsymbol{w_4(n,t)}$ for $\boldsymbol{\Theta_0 = 2, d = 1, \gamma = 3}$ and $\boldsymbol{\zeta = 2.5}$.

Family V. By assuming $\theta = [2 + \mathrm{i}, 2 - \mathrm{i}, 1,1]$, $\alpha = [-\mathrm{i}, \mathrm{i}, -\mathrm{i}, \mathrm{i}]$ in ***Eq. (5)***,

$$\Omega(\xi_n) = 2 + tan(\xi_n), \tag{21}$$

By substituting ***Eq. (21)*** into ***Eq. (8)***, the following values can be obtained

$$\Theta_0 = \Theta_0, \qquad \Theta_1 = 0, \qquad \Lambda_1 = -\frac{5i\,(-1+e^{4id})}{2(1+e^{4id})}, \qquad k = \frac{1}{2}\, i\, e^{-2id}\,(-1+e^{4id}),$$

Taking the above values and substituting them into ***Eq. (4)*** gives the following result

$$\chi(\xi_n) = \Theta_0 + \frac{5\,cos(\xi_n)tan(2d)}{2\left(2cos(\xi_n) + sin(\xi_n)\right)}, \tag{22}$$

Thus, an analytical solution to ***Eq. (6)*** is obtained as follows:

$$w(n,t) = \Theta_0 + \frac{e^{2(dn+\zeta)+\frac{i\,e^{-2id}(-1+e^{4id})\,t}{\Gamma(1+\gamma)}}\,(-1+e^{4d})}{(1+e^{4d})\left(-1+e^{2dn+2\zeta+\frac{i\,e^{-2id}(-1+e^{4id})\,t}{\Gamma(1+\gamma)}}\right)}, \tag{23}$$

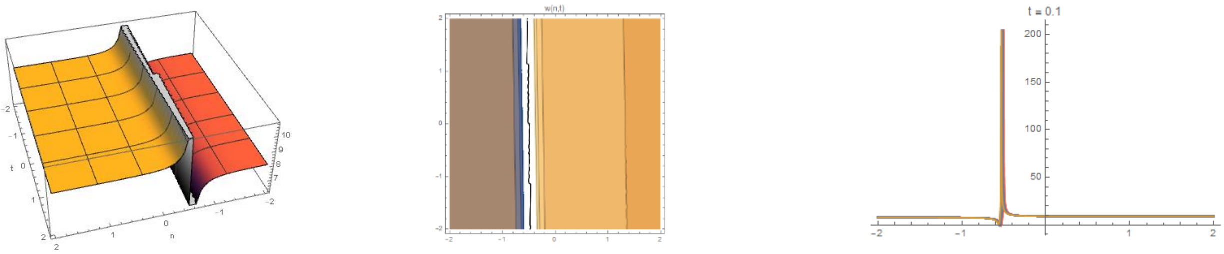

Figure 5. 3D, contour and 2D plots of the solution $\boldsymbol{w_5(n,t)}$ for $\boldsymbol{\Theta_0 = 8, d = 2, \gamma = 4}$ and $\boldsymbol{\zeta = 1}$.

Family VI. By assuming $\theta = [\mathrm{i}, -\mathrm{i}, 1,1]$, $\alpha = [-\mathrm{i}, \mathrm{i}, -\mathrm{i}, \mathrm{i}]$ in ***Eq. (5)***,

$$\Omega(\xi_n) = tan(\xi_n), \tag{24}$$

By substituting ***Eq. (24)*** into ***Eq. (8)***, the following values can be obtained

$$\Theta_0 = \Theta_0, \qquad \Theta_1 = \frac{i\,(-1+e^{2id})(1+e^{2id}+e^{4id})^2}{4(1+3e^{2id}+3e^{8id}+e^{10id})}, \qquad \Lambda_1 = -\frac{i\,(-1+e^{2id})(1+e^{2id}+e^{4id})^2}{4(1+3e^{2id}+3e^{8id}+e^{10id})},$$
$$k = \frac{1}{4}\, i\, e^{-4\,id}\,(-1+e^{8id}),$$

Taking the above values and substituting them into ***Eq. (4)*** gives the following result

$$\chi(\xi_n) = \Theta_0 + \frac{\left(1+2\,cos(2d)\right)^2 cot(\xi_n)\,sin(2d)}{2\left(3\ cos(3d) +\ cos(5d)\right)}, \tag{25}$$

Thus, an analytical solution to ***Eq. (6)*** is obtained as follows:

$$w(n,t) = \Theta_0 + \frac{1}{2}\left(1 + cot\left(dn + \zeta - \frac{t\,sin(4d)}{2\Gamma(1+\gamma)}\right)\right) tan(2d), \tag{26}$$

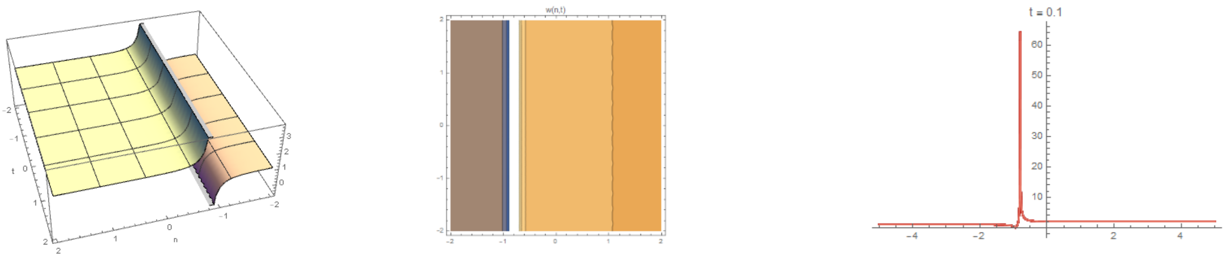

Figure 6. 3D, contour and 2D plots of the solution $\boldsymbol{w_6(n,t)}$ for $\boldsymbol{\Theta_0 = 1, d = 2.5, \gamma = 6}$ and $\boldsymbol{\zeta = 2}$.

5. Conclusion

The advances that have been made in various analytical methods are due to years of effort by scientists. In this paper, we analyze and investigate NFPDDEs using the GEERAF method. It is worth noting that the basic idea of this technique is to convert FDDEs into ODEs. The notable point in this method is the search for soliton-type solutions of ODEs derived from the given FDDEs. This method is more general than other methods because it produces a variety of exact soliton solutions, such as rotations, gaps, compactons, and so on. In addition, graphical diagrams related to the solutions presented in the article are provided, and these diagrams provide significant assistance in predicting the dynamic behaviors of the phenomenon under study. The results obtained can be very useful in advancements related to quantum mechanics, physics, and beyond.

References

[1] Iomin, A.: Fractional evolution in quantum mechanics. Chaos, Solitons & Fractals: X 1, 100001 (2019). doi:10.1016/j.csfx.2018.100001

[2] Nasrolahpour, H.: Time fractional formalism: classical and quantum phenomena. arXiv preprint arXiv:1203.4515 (2012). doi:10.48550/arXiv.1203.4515

[3] Kumar, D., Baleanu, D.: Fractional calculus and its applications in physics. Frontiers Media SA (2019). doi:10.3389/fphy.2019.00081

[4] Faridi, W.A., Asjad, M.I., Jhangeer, A.: The fractional analysis of fusion and fssion process in plasma physics. Physica Scripta 96(10), 104008 (2021). doi:10.1088/1402-4896/ac0dfd

[5] Pandir, Y., Yildirim, A.: Analytical approach for the fractional differential equations by using the extended tanh method. Waves in Random and Complex Media 28(3), 399–410 (2018). doi:10.1080/17455030.2017.1356490

[6] Dubey, S., Chakraverty, S.: Application of modifed extended tanh method in solving fractional order coupled wave equations. Mathematics and Computers in Simulation 198, 509–520 (2022). doi:10.1016/j.matcom.2022.03.007

[7] Raslan, K., Ali, K.K., Shallal, M.A.: The modifed extended tanh method with the riccati equation for solving the space-time fractional ew and mew equations. Chaos, Solitons & Fractals 103, 404–409 (2017). doi:10.1016/j.chaos.2017.06.029

[8] Tuluce Demiray, S., Pandir, Y., Bulut, H.: Generalized kudryashov method for time-fractional differential equations. In: Abstract and Applied Analysis, vol. 2014, p. 901540 (2014). doi:10.1155/2014/901540

[9] Kumar, D., Seadawy, A.R., Joardar, A.K.: Modifed kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology. Chinese journal of physics 56(1), 75–85 (2018). doi:10.1016/j.cjph.2017.11.020

[10] Eslami, M., Sharif, A.: Extended hyperbolic method to the perturbed nonlinear chen–lee–liu equation with conformable derivative. Partial Differential Equations in Applied Mathematics 11, 100838 (2024). doi:10.1016/j.padiff.2024.100838

[11] Mohyud-Din, S.T., Bibi, S.: Exact solutions for nonlinear fractional differential equations using GG2′ -expansion method. Alexandria Engineering Journal 57(2), 1003–1008 (2018). doi:10.1016/j.aej.2017.01.035

[12] Al-Shawba, A.A., Abdullah, F.A., Azmi, A., Akbar, M.A.: An extension of the double (GG′ , G1)-expansion method for conformable fractional differential equations. Complexity 2020(1), 7967328 (2020). doi:10.1155/2020/7967328

[13] Rezazadeh, H., Korkmaz, A., Eslami, M., Vahidi, J., Asghari, R.: Traveling wave solution of conformable fractional generalized reaction duffng model by generalized projective riccati equation method. Optical and Quantum Electronics 50, 1–13 (2018). doi:10.1007/s11082-018-1416-1

[14] Zheng, B., Wen, C.: Exact solutions for fractional partial differential equations by a new fractional subequation method. Advances in Difference Equations 2013, 1–12 (2013). doi:10.1186/1687-1847-2013-199

[15] Bekir, A., Güner, Ö., Cevikel, A.C.: Fractional complex transform and exp-function methods for fractional differential equations. In: Abstract and Applied Analysis, vol. 2013, p. 426462 (2013). doi:10.1155/2013/426462. Wiley Online Library

[16] Bekir, A., Güner, Ö., Ünsal, Ö.: The frst integral method for exact solutions of nonlinear fractional differential equations. Journal of Computational and Nonlinear Dynamics 10(2), 021020 (2015). doi:10.1115/1.4028065

[17] Baleanu, D., Jassim, H.K.: Exact solution of two-dimensional fractional partial differential equations. Fractal and Fractional 4(2), 21 (2020). doi:10.3390/fractalfract4020021

[18] Ghanbari, B., Inc, M.: A new generalized exponential rational function method to fnd exact special solutions for the resonance nonlinear schrödinger equation. The European Physical Journal Plus 133(4), 142 (2018). doi:10.1140/epjp/i2018-11984-1

[19] Ali, K.K., Yokus, A., Seadawy, A.R., Yilmazer, R.: The ion sound and langmuir waves dynamical system via computational modifed generalized exponential rational function. Chaos, Solitons & Fractals 161, 112381 (2022). doi:10.1016/j.chaos.2022.112381

[20] Aldhabani, M.S., Nonlaopon, K., Rezaei, S., Bayones, F.S., Elagan, S., El-Marouf, S.A.: Abundant solitary wave solutions to a perturbed schrödinger equation with kerr law nonlinearity via a novel approach. Results in Physics 35, 105385 (2022). doi:10.1016/j.rinp.2022.105385

[21] Nonlaopon, K., Günay, B., Mohamed, M.S., Elagan, S., Najati, S., Rezapour, S.: RETRACTED: On extracting new wave solutions to a modifed nonlinear Schrödinger's equation using two integration methods. Elsevier (2022). doi:10.1016/j.rinp.2022.105589

[22] Peng, S.: Optical solutions of the generalized (2+ 1)-dimensional dynamical schrödinger equation using the generalized exponential rational function method. Journal of Applied Science and Engineering 27(1) (2023). doi:10.6180/jase.202401_27(1).0009

[23] Abbas, N., Hussain, A., Riaz, M.B., Ibrahim, T.F., Birkea, F.O., Tahir, R.A.: A discussion on the lie symmetry analysis, travelling wave solutions and conservation laws of new generalized stochastic potential-kdv equation. Results in Physics 56, 107302 (2024). doi:10.1016/j.rinp.2023.107302

[24] Dhiman, S.K., Kumar, S.: Analyzing specifc waves and various dynamics of multi-peakons in (3+ 1)-dimensional p-type equation using a newly created methodology. Nonlinear Dynamics 112(12), 10277–10290 (2024). doi:10.1007/s11071-024-09588-7

[25] Sağlam Özkan, Y.: New exact solutions of the conformable space-time two-mode foam drainage equation by two effective methods. Nonlinear Dynamics 112(21), 19353–19369 (2024). doi:10.1007/s11071-024-10010-5

[26] Ulam, S.: Collected Papers of Enrico Fermi. University of Chicago Press, US (1965)

[27] Toda, M.: Theory of Nonlinear Lattices vol. 20. Springer, (2012)

[28] Ablowitz, M.J., Ladik, J.F.: Nonlinear differential- difference equations. Journal of Mathematical Physics 16(3), 598–603 (1975). doi:10.1063/1.522558

[29] Wadati, M.: Transformation theories for nonlinear discrete systems. Progress of Theoretical Physics Supplement 59, 36–63 (1976). doi:10.1143/PTPS.59.36

[30] Kevrekidis, P., Kevrekidis, I., Malomed, B.: Stability of solitary waves in fnite ablowitz–ladik lattices. Journal of Physics A: Mathematical and General 35(2), 267 (2002). doi:10.1088/0305-4470/35/2/307

[31] Hu, X.-B., Tam, H.-W.: Application of bilinear method to integrable differential-difference equations. Glasgow Mathematical Journal 43(A), 43–51 (2001). doi:10.1017/S0017089501000052

[32] Zhang, S., Dong, L., Ba, J.-M., Sun, Y.-N.: The (GG′)-expansion method for nonlinear differentialdifference equations. Physics Letters A 373(10), 905–910 (2009)

[33] Bekir, A., Güner, Ö., Ayhan, B.: Exact solutions of some systems of fractional differential-difference equations. Mathematical Methods in the Applied Sciences 38(17), 3807–3817 (2015). doi:10.1002/mma.3318
[34] Arikoglu, A., Ozkol, I.: Solution of differential–difference equations by using differential transform method. Applied Mathematics and Computation 181(1), 153–162 (2006). doi:10.1016/j.amc.2006.01.022
[35] Zou, L., Wang, Z., Zong, Z.: Generalized differential transform method to differential-difference equation. Physics Letters A 373(45), 4142–4151 (2009). doi:10.1016/j.physleta.2009.09.036
[36] Feng, Q.-H.: Exact solutions for fractional differential-difference equations by an extended riccati sub-ode method. Communications in Theoretical Physics 59(5), 521 (2013). doi:10.1088/0253-6102/59/5/01
[37] Wu, G.-c., Lee, E.: Fractional variational iteration method and its application. Physics Letters A 374(25), 2506–2509 (2010). doi:10.1016/j.physleta.2010.04.034
[38] Narita, K.: New discrete modifed kdv equation. Progress of theoretical physics 86(4), 817–824 (1991). doi:10.1143/ptp/86.4.817