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Solving Uncertain Linear Equations Systems Using Monte Carlo Simulation

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ABSTRACT

This paper introduces a novel approach for solving uncertain linear equations systems through Monte Carlo simulation. The study delves into the uncertainty distributions of variables within a linear equation system, establishing a fresh concept for solving such systems. The proposed method utilizes both inverse uncertainty distribution techniques and Monte Carlo simulation. Through examples, the paper illustrates the efficacy of this approach in effectively solving linear equation systems.

1. Introduction

Linear equation systems are fundamental to many fields, including engineering, physics, finance, and social sciences. However, real-world problems often involve uncertainties in coefficients and parameters, making deterministic approaches insufficient. Traditional methods, such as fuzzy logic and probabilistic models, attempt to handle these uncertainties but face challenges in computational feasibility and scalability [1].

Uncertainty theory, first introduced by Liu [2], provides a mathematical foundation for dealing with imprecise data. One established approach is the inverse uncertainty distribution method [3], which transforms uncertain variables into deterministic equivalents. However, this method suffers from computational inefficiency when handling high-dimensional systems.

This paper proposes an alternative approach: Monte Carlo simulation. Unlike the inverse uncertainty distribution method, Monte Carlo does not require matrix inversion, making it more suitable for large-scale problems [4]. Our study contributes by developing a Monte Carlo-based framework for solving uncertain linear equation systems and validating the approach with numerical examples.

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The following sections explore the theoretical foundation, implementation details, and comparative analysis of the proposed method

2. Basic definitions and concepts

Several approaches have been proposed to solve uncertain linear systems. The most common include:

1. Fuzzy Logic Methods – These represent uncertainty using fuzzy sets and membership functions (Zadeh, 1965). However, they lack a rigorous probability interpretation and can yield inconsistent results [5].
2. Probabilistic Approaches – Bayesian inference and stochastic methods treat uncertainty as random variables [6]. While powerful, they require precise prior distributions, which may not always be available.
3. Inverse Uncertainty Distribution Method – Proposed by Li & Zhu [3], this approach transforms uncertain variables into deterministic equivalents. However, matrix inversion can become computationally infeasible in high-dimensional problems.

Monte Carlo simulation, widely used in risk analysis and statistical modeling, offers a scalable alternative. It approximates solutions by generating random samples from the uncertainty distributions and estimating the system behavior. This paper extends Monte Carlo methods to uncertain linear equation systems, demonstrating their efficiency and accuracy compared to the inverse uncertainty distribution method

Suppose we have a measurable space $(\Gamma, \mathcal{L})$. The initial step in uncertainty theory involves renaming the measurable set Λ as an event. It is important to note that every member Λ in $\mathcal{L}$ is a measurable set. The second task is to define an uncertain measure $\mathcal{M}$ on the σ-algebra. In this case, the value $\mathcal{M}\{\Lambda\}$ assigned to each event Λ represents the level of certainty that it will occur. It is evident that the assignment of certainty levels is not arbitrary, and the uncertain measure $\mathcal{M}$ must adhere to specific mathematical properties. To effectively handle degrees of belief, Liu proposed the following four axioms [7]:

Axiom 1 (Normality Axiom): For any universal set Γ, $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2 (Duality Axiom): For any event Λ, $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$.

Axiom 3 (Subadditivity Axiom): For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\{\cup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$$

Axiom 4 (Product Axiom): Let $(\Gamma_k, \mathcal{M}_k, \mathcal{L}_k)$ be uncertainty spaces for $k = 1,2,\ldots$. The uncertain measure $\mathcal{M}$ is an uncertain measure that satisfies

$$\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

Definition 1 An uncertain variable ξ is considered normal when it exhibits a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re$$

denoted by $\mathcal{N}(e,\sigma)$ where e and σ are real numbers with $\sigma > 0$.

Definition 2 The inverse uncertaint distribution for a normal uncertain variable $\mathcal{N}(\mathrm{e},\sigma)$ can be expressed as

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$$

Definition 3 The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are considered independent [1], if for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers

$$\mathcal{M}\{\cap_{i=1}^{n} (\xi_i \in B_i)\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

Theorem 1 Suppose $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distribution $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If f is a continuous and strictly increasing function, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)\right).$$

For proof, see [8].

Definition 4 Let $X = (X_1, X_2, \ldots, X_n)$ be a random vector generated from the cumulative distribution function $F_X(x)$. If $X = X_1, X_2, \ldots, X_n$ variables are independent, the combined density function will be as follows

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i)$$

where $f_i(x_i)$ is equal to the marginal probability density function of X_i.

The cumulative distribution function $F_X(x)$ can be used to generate the random vector $X = (X_1, X_2, \ldots, X_n)$ easily, the inverse transformation method can be used separately for each of the variables.

$$X_i = F_i^{-1}(U_i),\ X \in \mathcal{R}$$

Example 1 We want to simulate the uncertain variable $X \sim \mathcal{N}(\mathrm{e},\sigma)$. Where e and σ are real numbers and $\sigma > 0$. We know its uncertainty distribution as

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re$$

and its inverse uncertainty distribution is

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$$

herefore, we generate α from $u(0,1)$ andplace it in the inverse normal uncertainty distribution.In this way, this has simulated $X \sim \mathcal{N}(u,\sigma)$.

3. Uncertain linear equations system

*A*n uncertain linear equation system can be expressed as:

$$\boldsymbol{A}x = \xi,$$

where **A** is a known $n \times n$ coefficient matrix, x is the vector of unknown uncertain variables as $x = (x_1, x_2, \dots, x_n)^t$ and ξ is the known uncertain vector with a probability distribution. $\xi = (\xi_1, \xi_2, \dots, \xi_n)^t$ [3].

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = \xi_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = \xi_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = \xi_n \end{cases} \quad (1)$$

There are n equations and n unknowns in this equations system.

Give $\xi_1, \xi_2, \dots, \xi_n$ a set of uncertain variables with regular uncertainty distributions $\psi_1, \psi_2, \dots, \psi_n$, respectively. Lets assume $x_1, x_2, \dots, x_n$ are independent uncertain variables with regular uncertainty distributions $\phi_1, \phi_2, \dots, \phi_n$ respectively. Then we will define a concept of a solution to an uncertain linear system (1).

Definition 5 An uncertain vector $(x_1, x_2, \dots, x_n)^t$ is called a solution of the uncertain linear system (1) in distribution if for any $\alpha \in (0,1)$, we have

$$\begin{cases} a_{11}\Phi_1^{-1}(\varepsilon_{11}) + a_{12}\Phi_2^{-1}(\varepsilon_{12}) + \dots + a_{1n}\Phi_n^{-1}(\varepsilon_{1n}) = \Psi_1^{-1}(\alpha) \\ a_{21}\Phi_1^{-1}(\varepsilon_{21}) + a_{22}\Phi_2^{-1}(\varepsilon_{22}) + \dots + a_{2n}\Phi_n^{-1}(\varepsilon_{2n}) = \Psi_2^{-1}(\alpha) \\ \vdots \\ a_{n1}\Phi_1^{-1}(\varepsilon_{n1}) + a_{n2}\Phi_2^{-1}(\varepsilon_{n2}) + \dots + a_{nn}\Phi_n^{-1}(\varepsilon_{nn}) = \Psi_n^{-1}(\alpha) \end{cases} \quad (2)$$

where

$$\varepsilon_{ij} = \begin{cases} \alpha, & if \;\; a_{ij} \geq 0 \\ 1 - \alpha, & if \;\; a_{ij} < 0 \end{cases} \quad (3)$$

In general, the uncertainty distributions $\phi_1, \phi_2, \dots, \phi_n$ of the solutions $(x_1, x_2, \dots, x_n)$; in (1) with uncertain distributions of $\xi_1, \xi_2, \dots, \xi_n$ are related. In other words, $\xi_1, \xi_2, \dots, \xi_n$ are normal uncertain variables.

3.1. Solving the uncertain linear equations system by the inverse uncertain distribution method

Let $\xi \sim \mathcal{N}(e_i, \sigma_i), 1 \leq i \leq n$, where σ_i are larger than zero.

Theorem 2 Suppose that $x_i \sim \mathcal{N}(u_i, v_i), 1 \leq i \leq n$. Then (2) is equal to

$$\mathbf{Au} = \mathbf{b}_e \quad (4)$$

and

$$\mathbf{A}_1\mathbf{v} = \mathbf{b}_\sigma \tag{5}$$

where

$$\mathbf{A} = \left(a_{ij}\right)_{n\times n}, \mathbf{u} = (u_1, u_2, \ldots, u_n)^t, \mathbf{b}_e = (e_1, e_2, \ldots, e_n)^t$$

$$\mathbf{A}_1 = \left(|a_{ij}|\right)_{n\times n}, \mathbf{v} = (v_1, v_2, \ldots, v_n)^t, \mathbf{b}_\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)^t$$

Theorem 3 Suppose that $\mathrm{x_i} \sim \mathcal{N}(\mathrm{u_i}, \mathrm{v_i}), 1 \le \mathrm{i} \le \mathrm{n}$. If A and A_1 are non-singular matrices. So

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \mathbf{A}_1^{-1}\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix} \tag{6}$$

where in $\mathbf{A} = (a_{ij})_{n\times n}$ and $\mathbf{A}_1 = (|a_{ij}|)_{n\times n}$.

Example 2 Consider the uncertain linear equations system

$$\begin{cases} x_1 + 2x_2 = \xi_1 \\ x_1 - x_2 = \xi_2 \end{cases}$$

Where in $\xi_1 \sim \mathcal{N}(1{,}6)$, $\xi_2 \sim \mathcal{N}(2{,}4)$. Suppose $x_1 \sim \mathcal{N}(u_1, v_1)$ and $x_2 \sim \mathcal{N}(u_2, v_2)$.

The coefficient matrices are as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \mathbf{A}_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

From the above theorem, we can get

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$v_1 > 0$, $v_2 > 0$. Then

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{-1}{3} \end{pmatrix}$$

Therefore, $x_1 \sim \mathcal{N}(\frac{5}{3}, 2)$ and $x_2 \sim \mathcal{N}(\frac{-1}{3}, 2)$ are the solutions of the uncertain linear equations system.

Traditional methods solve for x using the inverse uncertainty distribution. However, as matrix dimensions increase, inversion becomes computationally expensive. This motivates the use of Monte Carlo simulation, which estimates x by repeatedly sampling from the probability distributions of ξ and solving the system numerically. The next section outlines our proposed simulation approach.

4. Solving uncertain linear equation system by Monte Carlo simulation method

Li and Zhu's approach demonstrates a decrease in efficiency as the dimension of the matrix increases. Additionally, during the process of solving, it is possible to encounter an invertible system of equations. However, the Monte Carlo simulation method does not suffer from these issues.

In this paper, we propose a novel technique for solving systems of linear uncertain equations by utilizing the Monte Carlo simulation. We consider the uncertain equations as $Ax = \xi$, where A represents a square matrix with known real dimensions, x denotes an unknown uncertain vector, and ξ represents a known uncertain vector. The objective is to determine the uncertain distributions for the variables in vector x. Our innovation lies in simulating the values of ξ using the Monte Carlo method. Once the values of ξ are determined, we solve the uncertain linear equations system $Ax = \xi$ using conventional numerical calculation methods. This simulation process is repeated N times, and the parameters of the uncertain variables in vector x are estimated using the values obtained from the simulation.

Unlike the inverse uncertainty distribution method, Monte Carlo generates multiple random samples from ξ, solving the system for each realization and estimating the distribution of x. The process follows these steps:

1. Assume $\xi \sim \mathcal{N}(e_i, \sigma_i)$. Input e_i, σ_i, where $i = 1,2, \ldots, n$.
2. Input the coefficient matrix A.
3. Simulate α_i, where $i = 1,2, \ldots, n$, from a uniform distribution on the interval $(0,1)$.
4. Calculate ξ_i, where $i = 1,2, \ldots, n$, using the inverse normal uncertainty distribution.
5. Compute $x_t = A \backslash \xi_i$, where $t = 1,2, \ldots, n$.
6. Repeat steps 3, 4, and 5 for $i = 1, \ldots, 100000$.
7. Return the mean of x_t, where $t = 1,2, \ldots, 100000$.

These steps outline the Monte Carlo algorithm for solving uncertain linear equations systems, which effectively addresses the limitations of Li and Zhu's method.

5. Numerical application of the solution of uncertain linear equations system by Monte Carlo method

To show the effectiveness of the presented method, examples have been examined and then we check in terms of mean and squared variance with the method of inverse uncertainty distribution.

Example 3 Consider the uncertain linear equations system

$$\begin{cases} x_1 + 2x_2 = \xi_1 \\ x_1 - x_2 = \xi_2 \end{cases}$$

where $\xi_1 \sim \mathcal{N}(1,6)$, $\xi_2 \sim \mathcal{N}(2,4)$. Suppose $x_1 \sim \mathcal{N}(u_1, v_1)$ and $x_2 \sim \mathcal{N}(u_2, v_2)$.

In solving by the inverse uncertainty distribution method, $x_1 = \mathcal{N}(\frac{5}{3}, 1)$ and $x_2 = \mathcal{N}(-\frac{1}{3}, 2)$ are obtained. By utilizing the Monte Carlo method, we have successfully resolved this system of uncertain equations. The simulation results have been presented in ***Table 1***. These findings provide evidence of the Monte Carlo method's ability to estimate the parameters of a normal uncertainty distribution. In the subsequent sections, we will apply the proposed Monte Carlo algorithm to solve a practical example introduced in [3].

Table 1. Comparison of the solutions of the uncertain linear equation system of the simulation method and the inverse uncertainty distribution method

inverse uncertainty distribution	simulation N=10000	simulation N=100000
$x_1 \sim \mathcal{N}(1.66667,2)$	$x_1^\star \sim \mathcal{N}(1.6716,2.15555)$	$x_1^\star \sim \mathcal{N}(1.6650,2.16054)$
$x_2 \sim \mathcal{N}(-0.33333,2)$	$x_2^\star \sim \mathcal{N}(-0.3529,1.9201)$	$x_2^\star \sim \mathcal{N}(-0.3319,1.9469)$

Example 4

A healthy diet is an important part of a healthy lifestyle in society. A proper diet helps to consume enough nutrients and avoid eating exces nutrients. Regultanously, the consumption of different nutrients for people is not very accurate. Experts suggest that people consume about 100 grams or approximately 100 ml [3]. These values are categorized as uncertain variables and can be represented by various uncertain distributions. These quantities are described using normal uncertain distributions. The example related to diet, based on the suggestions of nutrition experts, at a suitable time, the amount of daily consumption of calcium, vitamin C, and phosphor by a human being is approximately 1000, 850, and 700 mg, respectively. The diet of goat's milk, jujube, and beef may contain these elements. The relationships between elements (calcium, vitamin C and phosphor) and nutrients (goat's milk, jujube and beef) are shown in ***Table 2***.

Table 2. Nutrition diet relations

Nutrients	Nutrient content per gram of food(mg)			Demands of nutrients (mg)
2-4	Goat's milk	Jujube	Beef	
Calcium	1.5	0.4		ξ_1
Vitamin C	0.3	1.3		ξ_2
Phosphor		0.2	1.5	ξ_3

Let ξ_i be uncertainty orders for the ith nutrient elements, $i = 1,2,3$ respectively, where $\xi_1 \sim \mathcal{N}(1000,50)$, $\xi_2 \sim \mathcal{N}(850,40)$ and $\xi_3 \sim \mathcal{N}(700,30)$.

Let x_1, x_2 and x_3 be the diet amount of Goat's milk, Jujube and Beef, respectively. So according to the ***Table 2***, we have the following uncertain linear system

$$\begin{cases} 1.5x_1 + 0.4x_2 = \xi_1 \\ 0.3x_1 + 1.3x_2 = \xi_2 \\ 0.2x_2 + 1.5x_3 = \xi_3 \end{cases} \quad \textbf{(7)}$$

The solution obtained from the inverse uncertainty distribution method in [3] are $x_1 \sim \mathcal{N}(524.6,26.8)$, $x_2 \sim \mathcal{N}(532.8,24.6)$ and $x_3 \sim \mathcal{N}(395.6,16.7)$. Now we examine the problem by Monte Carlo simulation up to 100000 repeat.

The variables x_1, x_2 and x_3 from the Monte Carlo simulation method are denoted by $x_1^\star$, $x_2^\star$ and $x_3^\star$.

Table 3. Comparison of the solutions of the uncertain linear equation system from the Monte Carlo simulation method and the inverse uncertainty distribution method

inverse uncertainty distribution	Simulation N=10000	Simulation $N = 100000$
$x_1 \sim \mathcal{N}(524.6,26.8)$	$x_1^\star \sim \mathcal{N}(524.7769,29.781)$	$x_1^\star \sim \mathcal{N}(524.5562,29.5468)$
$x_2 \sim \mathcal{N}(532.8,24.6)$	$x_2^\star \sim \mathcal{N}(533.2285,28.0767)$	$x_2^\star \sim \mathcal{N}(532.80647,28.1952)$
$x_3 \sim \mathcal{N}(395.6,16.7)$	$x_1^\star \sim \mathcal{N}(395.8296,19.4585)$	$x_1^\star \sim \mathcal{N}(395.6851,19.7432)$

The provided example effectively showcases the capability of the proposed methodology in successfully determining the parameters of the normal uncertainty distribution. The inverse uncertain distribution method is based on linear algebra calculations. However, as the dimension of the coefficients matrix increases, there is a chance that the matrix of coefficients may not be invertible, thus impeding the resolution of the system of uncertain equations. In contrast, the Monte Carlo method is not reliant on the invertibility of the matrix of coefficients and continues to operate efficiently as the dimension of the matrix grows.

6. Conclusions

In this paper, we investigated the uncertain linear equations system model, where A is a clear matrix and ξ is an uncertain vector. Solutions were presented using the methods of inverse uncertainty distribution and Monte Carlo simulation in MATLAB. Simulating x values using inverse uncertainty distributions offers several features and advantages. The inverse uncertainty distribution method is well-suited for low-dimensional linear equation systems; however, as the system's dimension increases and the coefficient matrix A becomes invertible, this method may encounter limitations. Additionally, in high dimensions and with repeated computations, the inverse uncertainty distribution method can lead to computational challenges and time constraints. Therefore, the Monte Carlo simulation method emerges as a suitable alternative for solving uncertain linear equations systems. It provides a fast estimation of the solution, saving computational effort and time, particularly when dealing with high-dimensional problems. Monte Carlo simulation provides an efficient and scalable approach for solving uncertain linear equations, outperforming traditional methods in high-dimensional settings. Future research could explore its extension to nonlinear uncertain systems, improve computational performance, and apply the method to real-world engineering and financial problems.

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