Journal of Mathematical Modeling
Vol. 13, No. 3, 2025, pp. 519-531. Research Article

JMM

An approximation technique for a system of time-fractional differential equations arising in population dynamics

Jugal Mohapatra[†] [*], **Siba Prasad Mohapatra**[‡], **Anasuya Nath**[§]

[†] *Department of Mathematics, National Institute of Technology Rourkela, India*
[‡] *Department of Mathematics, Konark Institute of Science and Technology, India*
[§] *Department of Mathematics, Utkal University, Bhubaneswar, India*
Email(s): jugal@nitrkl.ac.in, spmohapatra71@gmail.com, anasuya_nath@yahoo.com

Abstract. In this work, we develop and analyze an approximation technique for the system of time-fractional nonlinear differential equations arising in population dynamics. The fractional of order $\sigma \in (0,1)$ is taken in the Caputo sense. The proposed technique uses L1 discretization on the uniform mesh to approximate the differential operator. The fractional model is transformed into a system of nonlinear algebraic equations. The generalized Newton-Raphson method is employed to solve the corresponding nonlinear system. A rigorous error estimation is presented. It is shown that the proposed scheme achieved $(2-\sigma)$ order of accuracy. Lastly, numerical experiment is conducted to demonstrate the validity of the proposed technique.

Keywords: System of fractional model, Caputo derivative, L1 scheme, error analysis.
AMS Subject Classification 2010: 26A33, 37M05, 92D25, 65L05.

1 Introduction

Fractional calculus is a generalization of differentiation and integration to non-integer orders. The concept of fractional calculus dates back to the origins of calculus itself. The idea was first mentioned in a letter from L'Hopital to Leibniz in 1695, where L'Hopital inquired about the meaning of a half-order derivative. Over the centuries, mathematicians made significant contributions to formalize and expand the theory of fractional calculus. In this regard, fractional order differential equations (FDEs) provide a powerful framework for modeling systems with memory and hereditary properties. Their applications span a wide range of fields such as physics, mechanics, chemistry, and engineering, offering more accurate and flexible models compared to traditional integer-order differential equations (see for

*Corresponding author
Received: 16 October 2024 / Revised: 29 January 2025 / Accepted: 8 February 2025
DOI: 10.22124/jmm.2025.28718.2548

applications [2,4,6,8,9]). Despite the challenges in solving and interpreting these equations, their benefits in capturing the complexity of real-world phenomena make them an important area of research and application.

The present work consider the following system of time-fractional differential equations:

$$\begin{cases} \mathscr{D}_t^{\sigma} W(t) = G(t, W(t)) \quad \text{for } t \in (0, T], \\ W(0) = \chi, \end{cases} \tag{1}$$

where $\mathscr{D}_t^{\sigma}$ denotes the Caputo fractional operator,

$$W(t) = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}, \quad G(t, W(t)) = \begin{bmatrix} G_1(t, X(t), Y(t)) \\ G_2(t, X(t), Y(t)) \end{bmatrix},$$

and $\chi = [\eta, \zeta]$ is a Lipschitz continuous function. η, ζ are prescribed real constants. If

$$G = \begin{bmatrix} a(t)X(t) - b(t)X(t)Y(t) \\ c(t)X(t)Y(t) - d(t)Y(t) \end{bmatrix},$$

then (1) is known as time-fractional Lotka-Volterra model, which is the generalization of the classical Lotka-Volterra model arising in population dynamics. In this case, the unknown variable $X(t)$, called prey and $Y(t)$, called predator, represents the number of individuals or population density in species X and Y respectively in a habitat at time t. The competition coefficients a, b, c, d are positive and smooth functions describe the interaction rates. The Lotka-Volterra model, also known as the predator-prey model, is a fundamental concept in ecological and mathematical biology. Developed independently by Alfred J. Lotka [16] in 1925 and Vito Volterra [18] in 1926, this model describes the dynamics between two interacting species: a predator and its prey. The Lotka-Volterra model's principles apply broadly beyond ecology, including in economics to model competition between companies and in epidemiology to describe the spread of diseases, chemical reactions, dynamic systems, laser and plasma physics, and control system theory; see [10] and the references therein. The fractional model can describe the memory properties of biological systems better than the usual classical model.

Finding the exact solutions of the fractional order system is rarely accessible and usually complicated because of its nonlinear and nonlocal nature. Consequently, suitable work has been done on the development of the numerical and semi-analytical techniques to find the desired approximate solution. In [13], Lin and Liu presented higher-order multiple step methods for nonlinear fractional-order ordinary differential equations. Das and Gupta employed the homotopy perturbation method to obtain an approximate analytic solution of the fractional order LotkaVolterra equations in [3]. Li and Zeng studied stability and convergence analysis based on the generalized discrete Gronwall inequality for the fractional order Euler, Adams high order methods in [12] for solving nonlinear equations involving nonlocal derivatives. In [19], Yan et al. constructed higher-order numerical methods to solve nonlinear fractional differential equations. They used two different approaches: direct discretization and discretization of the integral form. The stability analysis was presented using sufficiently smooth initial data. Li et al. [11] analyzed predictorcorrector based finite difference schemes on non-uniform meshes for solving nonlinear fractional differential equations. In [1], Alqudah et al. presented existence theory and an approximate analytical solution using the Adomian decomposition method for a coupled nonlinear system involving a nonsingular kernel. Jafari et al., in [7], proposed a numerical scheme based on the three-step Adams-Bashforth method to obtain a numerical solution of the fractional order population dynamics models. Recently, an efficient finite difference scheme on a non-uniform mesh for the time-fractional LotkaVolterra competitive model involving singularity is presented in [5].

From the said literature, one can observe that an efficient numerical technique for the fractional order nonlinear system is not so enriched. In this article, we propose an efficient finite difference scheme to approximate the nonlinear system (1) involving arbitrary order derivative. The L1 formula is used to discretize the fractional operator on uniform mesh. The generalized Newton-Raphson method is employed to solve the corresponding nonlinear system. The global error analysis is established and measured in terms of the discrete maximum norm. It is shown that the proposed technique achieves its optimal $(2-\sigma)$ order of convergence. The theoretical estimation and illustrative test examples demonstrate the deficiency of the proposed method. The solution methodology and its supporting analysis are the main novelty of this study. The rest of the paper is listed as follows: In Section 2, we elaborate on some basic definitions of fractional calculus and solution properties of the model problem. Section 3 constructs the numerical scheme to solve (1). Section 4 deals with the error analysis of the proposed approach and illustrates the test example. Finally, concluding remarks are given in Section 5.

Notation: $\mathcal{C}^k(D)$ denotes the space of k'th order continuously differentiable real-valued functions on a domain D, use $\mathcal{C}(D)$ for $\mathcal{C}^0(D)$. Set $\phi_i = \phi(t_i)$ for any function ϕ defined on a domain D. Define the discrete maximum norm by $||\phi||_\infty = \max_i |\phi(t_i)|$. In several inequalities, C denotes a generic positive constant which has different values at different occurrences.

2 Definitions and notations

The following definitions and properties will be useful for this work. For more details one may refer [4, 17].

Definition 1. *Let $\phi(x) \in \mathcal{C}[a,b]$. The Riemann-Liouville fractional integral of order $\nu \in \mathbb{R}^+$ of $\phi(x)$ is defined by:*

$$J_t^{\nu}\phi(x) := \frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}\phi(t)\,dt.$$

$\Gamma(\cdot)$ *being Euler's gamma function.*

Definition 2. *Let $n-1<\nu\le n, n\in\mathbb{N}$. The Caputo fractional derivative of order $\nu\in\mathbb{R}^+$ of the function $\phi(x)\in\mathcal{C}^n[a,b]$, is defined by:*

$$\mathscr{D}_t^{\nu}\phi(x) := J_t^{n-\nu}\phi^{(n)}(x) = \begin{cases} \dfrac{1}{\Gamma(n-\nu)}\displaystyle\int_a^x (x-t)^{n-\nu-1}\phi^{(n)}(t)\,dt, & \text{if} \quad n-1<\nu<n, \\ \phi^{(n)}(x), & \text{if} \quad \nu = n. \end{cases}$$

Definition 3. *The two parameters $\beta,\gamma>0$ Mittag-Leffler function is defined by:*

$$E_{\beta,\gamma}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+\gamma)}, \qquad z\in\mathbb{C}.$$

The following are some basic properties of the fractional calculus:

1. For $x \in [a,b]$, $m-1 < \nu \le m$, we have

$$\mathscr{D}_x^{\nu} J_x^{\nu} \varphi(x) = \varphi(x) \quad \text{and} \quad J_x^{\nu} \mathscr{D}_x^{\nu} \varphi(x) = \varphi(x) - \sum_{k=0}^{n-1} \varphi^{(k)}(a^+) \frac{(x-a)^k}{k!}, \quad x > a.$$

2. For $m-1 < \widetilde{\sigma} \le m$, $m \in \mathbb{N}$, we have

$$\mathscr{D}_x^{\widetilde{\sigma}} (x-a)^{\nu} = \begin{cases} \dfrac{\Gamma(\nu+1)}{\Gamma(\nu-\widetilde{\sigma}+1)} x^{\nu-\widetilde{\sigma}}, & \text{if} \quad \nu > m-1, \\ 0, & \text{if} \quad \nu \in \{0,1,\ldots,m-1\}. \end{cases}$$

3. $\mathscr{D}_x^{\widetilde{\sigma}} e^{\lambda(x-a)} = \lambda^m (x-a)^{m-\widetilde{\sigma}} E_{1,m-\widetilde{\sigma}+1}(\lambda(x-a))$ for any $\lambda \in \mathbb{C}$ and $m-1 < \widetilde{\sigma} \le m$, $m \in \mathbb{N}$.

2.1 Regularity of the solution

Definition 4. *The column vector $W(t) \in \mathcal{C}[0,T]$ means each components $X(t), Y(t) \in \mathcal{C}[0,T]$. The norm of $W(t)$ is define by*

$$\|W(t)\| = \sup_{t\in[0,T]} |X(t)| + \sup_{t\in[0,T]} |Y(t)|.$$

Lemma 1. *If $G(t,W(t))$ bounded continuous vector valued function and satisfies Lipschitz condition with respect to $W(t)$ i.e., there is a positive constant L independent of $t, Z_1(t), Z_2(t)$ such that*

$$\|G(t,Z_1(t)) - G(t,Z_2(t))\| \le L \|Z_1(t) - Z_2(t)\|,$$

then the initial value problem (1) has a unique solution.

Proof. One can find the detailed proof in Theorem 6.1, 6.5 and 6.28 of [4]. □

3 The discretized problem

In this section, we derived an approximation scheme on a uniform mesh for (1) by combining the L1 technique and Newton-Raphson method. Consider N be a positive integer and construct a uniform grid $\{t_j = j\tau \ \text{ for } \ j = 0,1,\ldots,N\}$ with step length $\tau = T/N$. The Caputo fractional derivative $\mathscr{D}_t^{\sigma} X(t)$ at t_j for $j = 1,2,\ldots,N$ can be written as

$$\mathscr{D}_t^{\sigma} X(t_j) = \frac{1}{\Gamma(1-\sigma)} \sum_{k=0}^{j-1} \int_{s=t_k}^{t_{k+1}} (t_j - s)^{-\sigma} X'(s)\, ds,$$

which can be approximated using L1 scheme [14] as

$$\begin{aligned}
\mathscr{D}_N^{\sigma} X(t_j) &= \frac{1}{\Gamma(1-\sigma)} \sum_{k=0}^{j-1} \frac{X(t_{k+1}) - X(t_k)}{\tau} \int_{s=t_k}^{t_{k+1}} (t_j - s)^{-\sigma} ds \\
&= \frac{\tau^{-\sigma}}{\Gamma(2-\sigma)} \sum_{k=0}^{j-1} \left[X(t_{k+1}) - X(t_k)\right] d_{j-k} \\
&= \frac{\tau^{-\sigma}}{\Gamma(2-\sigma)} \left[X(t_j) d_1 + \sum_{k=1}^{j-1} \left(d_{j-k+1} - d_{j-k}\right) X(t_k) - X(t_0) d_j\right],
\end{aligned} \tag{2}$$

where $d_p = p^{1-\sigma} - (l-p)^{1-\sigma}$ for $p = 1,2,\ldots,N$. The local truncation error $\varepsilon_j^{(1)} = \left(\mathscr{D}_t^{\sigma} - \mathscr{D}_N^{\sigma}\right) X(t_j)$. Similarly, the L1 discretization of $\mathscr{D}_t^{\sigma} Y(t)$ at t_j for $j = 1,2,\ldots,N$ can be expressed as

$$\mathscr{D}_N^{\sigma} Y(t_j) = \frac{\tau^{-\sigma}}{\Gamma(2-\sigma)} \sum_{k=0}^{j-1} \left[Y(t_{k+1}) - Y(t_k)\right] d_{j-k}, \tag{3}$$

with the local truncation error $\varepsilon_j^{(2)} = \left(\mathscr{D}_t^{\sigma} - \mathscr{D}_N^{\sigma}\right) Y(t_j)$. Using (2) and (3), the model (1) transformed to

$$\begin{cases} \mathscr{D}_N^{\sigma} W(t_j) = G(t_j, W(t_j)) + \mathscr{E}_j, \quad \text{for } j = 1,2,\ldots,N, \\ W(t_0) = \chi, \end{cases} \tag{4}$$

where $\mathscr{E}_j = [\varepsilon_j^{(1)}, \varepsilon_j^{(2)}]^T$ is the truncation error. Now, the difference equation (4) can be written as

$$\begin{cases} \mathscr{D}_N^{\sigma} X(t_j) = G_1(t_j, X(t_j), Y(t_j)) + \varepsilon_j^{(1)}, \\ \mathscr{D}_N^{\sigma} Y(t_j) = G_2(t_j, X(t_j), Y(t_j)) + \varepsilon_j^{(2)}, \\ X(t_0) = \eta,\ Y(t_0) = \zeta, \end{cases} \tag{5}$$

for $j = 1,2,\ldots,N$. Neglecting the error term, the discrete problem (4) corresponding to (1) reduces to

$$\begin{cases} \mathscr{D}_N^{\sigma} W_j = G(t_j, W_j), \quad \text{for } j = 1,2,\ldots,N, \\ W_0 = \chi, \end{cases} \tag{6}$$

and the corresponding discrete system (5) reduces to

$$\begin{cases} F_1(X_j, Y_j) \equiv \dfrac{\tau^{-\sigma}}{\Gamma(2-\sigma)} t_j - G_1(t_j, X_j, Y_j) - \dfrac{\tau^{-\sigma}}{\Gamma(2-\sigma)} \left[t_{j-1} + \sum_{k=0}^{j-2} \left(X_k - X_{k+1}\right) d_{j-k}\right] = 0, \\ F_2(X_j, Y_j) \equiv \dfrac{\tau^{-\sigma}}{\Gamma(2-\sigma)} Y_j - G_2(t_j, X_j, Y_j) - \dfrac{\tau^{-\sigma}}{\Gamma(2-\sigma)} \left[Y_{j-1} + \sum_{k=0}^{j-2} \left(Y_k - Y_{k+1}\right) d_{j-k}\right] = 0, \\ X_0 = \eta,\ Y_0 = \zeta, \end{cases} \tag{7}$$

for $j = 1,2,\ldots,N$. W_j denotes the approximation value of $W(t)$ at $t = t_j$. Similar notation is used for X_j and Y_j. Clearly, (7) is a nonlinear system of explicit algebraic equation. We can solve the above system of equations using standard Newton-Raphson method. Consider, $[X_j^{(0)}, Y_j^{(0)}]^T = [0,0]^T$ be the initial guess. Then after n^{th} iteration, the approximate root of the system (7) is

$$\begin{bmatrix} X_j^{(n)} \\ Y_j^{(n)} \end{bmatrix} = \begin{bmatrix} X_j^{(n-1)} \\ Y_j^{(n-1)} \end{bmatrix} - J_{(n-1)}^{-1} \begin{bmatrix} F_1(X_j^{(n-1)}, Y_j^{(n-1)}) \\ F_2(X_j^{(n-1)}, Y_j^{(n-1)}) \end{bmatrix},$$

where

$$J_{(p)} = \begin{bmatrix} \dfrac{\tau^{-\sigma}}{\Gamma(2-\sigma)} - \dfrac{\partial G_1}{\partial s}(t_j, s, Y_j^{(p)})\Big|_{s=X_j^{(p)}} & \dfrac{\partial G_1}{\partial s}(t_j, X_j^{(p)}, s)\Big|_{s=Y_j^{(p)}} \\ -\dfrac{\partial G_2}{\partial s}(t_j, s, Y_j^{(p)})\Big|_{s=X_j^{(p)}} & \dfrac{\tau^{-\sigma}}{\Gamma(2-\sigma)} - \dfrac{\partial G_2}{\partial s}(t_j, X_j^{(p)}, s)\Big|_{s=Y_j^{(p)}} \end{bmatrix}.$$

Then the approximate solution of (1) for each $j = 1, 2, \ldots, N$ becomes

$$\begin{bmatrix} X_j \\ Y_j \end{bmatrix} \approx \begin{bmatrix} X_j^{(n)} \\ Y_j^{(n)} \end{bmatrix}.$$

The stopping criterion for the successive iteration is $\|W_j^{(n)} - W_j^{(n-1)}\| \leq TOL$.

4 Error analysis and numerical discussions

In this section, first, we establish the truncation error associated with approximating the Caputo fractional derivative. Then, the accuracy of the proposed difference scheme is computed. Finally, numerical test examples validate the theoretical result. All computations are implemented with MATLAB R2016a.

Lemma 2. *Let $X(t), Y(t) \in \mathcal{C}^2[0,T]$, then for $j = 1, 2, \ldots, N$,*

$$\left|\varepsilon_j^{(i)}\right| \leq CN^{-(2-\sigma)}, \text{ for } i = 1, 2.$$

Proof. Using definition

$$\begin{aligned} \left|\varepsilon_j^{(1)}\right| &= \left|(\mathscr{D}_t^\sigma - \mathscr{D}_N^\sigma) X(t_j)\right| \\ &= \left| \frac{1}{\Gamma(1-\sigma)} \sum_{k=0}^{j-1} \int_{s=t_k}^{t_{k+1}} (t_j - s)^{-\sigma} \left[X'(s) - \frac{X(t_{k+1}) - X(t_k)}{\tau} \right] ds \right| \\ &\leq C \left| \frac{1}{\Gamma(1-\sigma)} \sum_{k=0}^{j-1} \int_{s=t_k}^{t_{k+1}} \frac{t_j + t_{j-1} - 2s}{(t_j - s)^\sigma} ds + \mathscr{O}(N^{-2}) \right|. \end{aligned}$$

From [14], we get

$$\left| \frac{1}{\Gamma(1-\sigma)} \sum_{k=0}^{j-1} \int_{s=t_k}^{t_{k+1}} \frac{t_j + t_{j-1} - 2s}{(t_j - s)^\sigma} ds \right| \leq CN^{-(2-\sigma)}.$$

This however means $|\varepsilon_j^{(1)}| \leq CN^{-(2-\sigma)}$. In a similar way, we can show that $|\varepsilon_j^{(2)}| \leq CN^{-(2-\sigma)}$. Hence completes the proof. □

Denote $e_j^{(1)} = X(t_j) - X_j$, $e_j^{(2)} = Y(t_j) - Y_j$, and $E_j = W(t_j) - W_j = [e_j^{(1)}, e_j^{(2)}]^T$ for $j = 1, 2, \ldots, N$ be the error at t_j. By using (1) and (4) the following error equation is obtained

$$\begin{cases} \mathscr{D}_N^{\sigma} E_j = G(t_j, W(t_j)) - G(t_j, W_j) + \mathscr{E}_j & \text{for } j = 1, 2, \ldots, N, \\ E_0 = 0. \end{cases} \tag{8}$$

Using Lemma (1), the error equation can be written as

$$\begin{cases} \mathscr{D}_N^{\sigma} \|E_j\| \le L\|E_j\| + \|\mathscr{E}_j\|, & \text{for } j = 1, 2, \ldots, N, \\ E_0 = 0. \end{cases} \tag{9}$$

Theorem 1. *Solution of the error equation (9) satisfies the following stability result*

$$\|E_j\| \le C \sup_{1 \le k \le j-1} \left\{ \|\mathcal{E}_k\| \right\}.$$

Proof. Using (2), the inequality (9) can be written as

$$\frac{\tau^{-\sigma}}{\Gamma(2-\sigma)} \left[d_1 \|E_j\| - \sum_{k=1}^{j-1} \left(d_{j-k} - d_{j-k+1} \right) \|E_k\| \right] \le L\|E_j\| + \|\mathcal{E}_j\|,$$

or

$$\left(\frac{\tau^{-\sigma}}{\Gamma(2-\sigma)} - L \right) \|E_j\| \le \frac{\tau^{-\sigma}}{\Gamma(2-\sigma)} \sum_{k=1}^{j-1} \left(d_{j-k} - d_{j-k+1} \right) \|E_k\| + \|\mathcal{E}_j\|,$$

or

$$\|E_j\| \le \frac{M}{1 - L\tau^{\sigma}\Gamma(2-\sigma)} \sum_{k=1}^{j-1} \|E_k\| + \frac{1}{\dfrac{\tau^{-\sigma}}{\Gamma(2-\sigma)} - L} \|\mathcal{E}_j\|,$$

where $M = \max_{1 \le k \le j-1} \left\{ d_{j-k} - d_{j-k+1} \right\}$. From [15], for $L\Gamma(2-\sigma) < \tau^{-\sigma}$ and $\lim_{h \to 0} \sum_{k=1}^{j-1} \|E_k\| = 0$, we have

$$\|E_j\| \le C \sup_{1 \le k \le j-1} \left\{ \|\mathcal{E}_k\| \right\}.$$

Hence the desired result. □

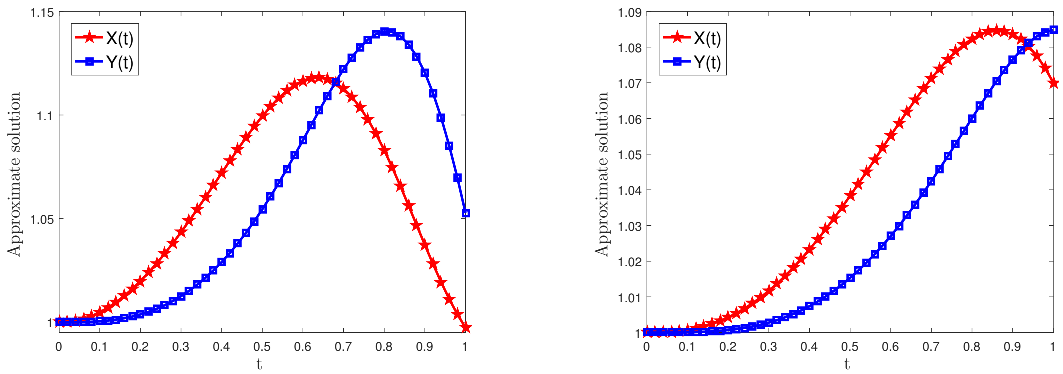
Theorem 2. *Suppose* $\{W(t_j)\}_{j=1}^N$ *be the exact solution and* $\{W_j\}_{j=1}^N$ *be the approximate solution of (1), then we have*

$$\max_{1 \le j \le N} \left\| W(t_j) - W_j \right\| \le CN^{-(2-\sigma)}.$$

Proof. The Lemma 2 and Theorem 1 yields

$$\|E_j\| \le C \sup_{1 \le k \le j-1} \left\{ \|\mathcal{E}_k\| \right\} \le C \sup_{1 \le k \le j-1} \left\{ N^{-(2-\sigma)} \right\} \le CN^{-(2-\sigma)}.$$

This completes the proof. □



(a) $\sigma = 0.2$ **(b)** $\sigma = 0.8$

Figure 1: Numerical solutions with $N = 50$ for Example 1.

4.1 Numerical experiments

Example 1. Consider the test problem with $T = 1$:

$$\begin{cases} D_t^\sigma X(t) = t^2 X(t) - t^3 X(t) Y(t), \\ D_t^\sigma Y(t) = t^3 X(t) Y(t) - t^4 Y(t), \\ X(0) = 1,\ Y(0) = 1. \end{cases}$$

Due to the unavailability of the analytical solutions in the test example, the maximum error (Σ_N) is estimated using the double mesh principle. Now, if z_j is the computed solution at t_j with N number of partitions, then

$$\Sigma_N = \max_{0 \le j \le N} \left| z_j - \tilde{z}_j \right|,$$

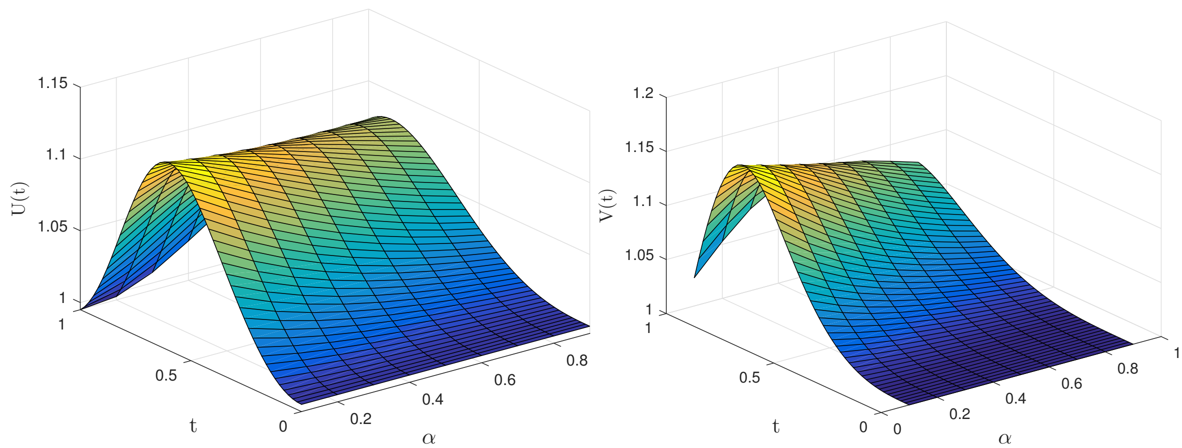
where $\tilde{z}_j$ is the interpolating solution obtained using the computed solutions with N and $2N$ number of partitions. The corresponding order of convergence (ρ_N) is estimated as

$$\rho_N = \log_2 \left(\frac{\Sigma_N}{\Sigma_{2N}} \right).$$

Figure 1 shows the numerical solutions profile with different values of σ for Example 1. The surface plot of the approximate solutions is presented in Figure 2. The log-log plots of the numerical error and the error bounds are displayed in Figure 3. Absolute errors (Σ_N) and order of convergences (ρ_N) are reported in Tables 1 and 2. With these results, it is apparent that the proposed technique provides an acceptable solution.

Example 2. Consider the test problem with $T = 1$:

$$\begin{cases} D_t^\sigma X(t) = -X(t) Y(t) + f(t), \\ D_t^\sigma Y(t) = X(t) Y(t) - Y(t) + g(t), \\ X(0) = Y(0) = 0. \end{cases}$$



(a) $X(t)$ **(b)** $Y(t)$

Figure 2: Surface plot of the numerical solutions with $N = 50$ for Example 1.

Table 1: Σ_N and ρ_N of $X(t)$ for Example 1.

σ/M	32	64	128	256	512	1024	2048
0.1	1.464E-04	4.456E-05	1.324E-05	3.875E-06	1.120E-06	3.208E-07	9.113E-08
	1.725	1.752	1.774	1.790	1.804	1.815	
0.3	6.513E-04	2.174E-04	7.099E-05	2.285E-05	7.283E-06	2.304E-06	7.249E-07
	1.581	1.584	1.583	1.614	1.635	1.649	
0.5	1.225E-03	4.694E-04	1.747E-04	6.391E-05	2.311E-05	8.297E-06	2.964E-06
	1.418	1.451	1.469	1.479	1.486	1.490	
0.7	8.239E-04	3.537E-04	1.497E-04	6.255E-05	2.587E-05	1.063E-05	4.348E-06
	1.237	1.266	1.281	1.289	1.294	1.297	
0.9	1.009E-03	4.733E-04	2.213E-04	1.033E-04	4.823E-05	2.250E-05	1.050E-05
	1.093	1.093	1.094	1.096	1.097	1.098	

We choose $f(t)$ and $g(t)$, so that the exact solution is $X(t) = t^{2+\sigma}, Y(t) = -t^{3+\sigma}$. The approximate solutions with different values of σ are presented graphically in Figure 5 for Example 2. In Figure 6, the log-log plot of the computational error indicates a sharp convergence rate. The computed values Σ_N and ρ_N are displayed in Table 3 and Table 4, indicating the sharp convergence with optimal convergence rate $\mathscr{O}(2-\sigma)$.

Table 2: Σ_N and ρ_N of $Y(t)$ for Example 1.

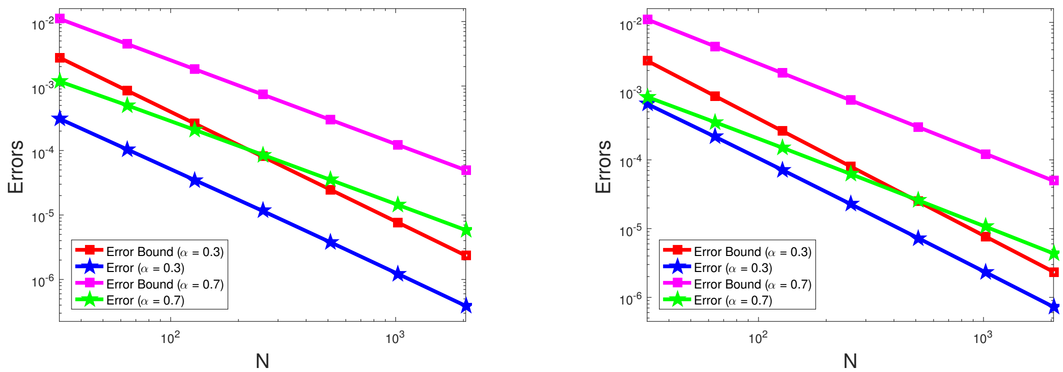
σ/M	32	64	128	256	512	1024	2048
0.2	3.546E-04	1.130E-04	3.511E-05	1.074E-05	3.251E-06	9.753E-07	2.905E-07
	1.650	1.686	1.708	1.724	1.737	1.747	
0.4	1.030E-03	3.673E-04	1.275E-04	4.355E-05	1.473E-05	4.949E-06	1.655E-06
	1.488	1.527	1.549	1.564	1.574	1.581	
0.6	1.137E-03	4.631E-04	1.841E-04	7.198E-05	2.783E-05	1.068E-05	4.082E-06
	1.295	1.331	1.355	1.371	1.381	1.388	
0.8	7.764E-04	3.409E-04	1.491E-04	6.507E-05	2.836E-05	1.235E-05	5.377E-06
	1.187	1.193	1.196	1.198	1.199	1.200	

Table 3: Σ_N and ρ_N of $X(t)$ for Example 2.

σ/M	32	64	128	256	512	1024	2048
0.2	9.677E-03	2.685E-03	7.887E-04	2.344E-04	6.961E-05	2.060E-05	6.070E-06
	1.849	1.767	1.751	1.751	1.757	1.763	
0.4	9.000E-03	3.002E-03	1.008E-03	3.379E-04	1.130E-04	3.765E-05	1.252E-05
	1.584	1.575	1.576	1.581	1.585	1.589	
0.6	1.638E-02	6.198E-03	2.360E-03	8.989E-04	3.421E-04	1.300E-04	4.938E-05
	1.402	1.393	1.392	1.394	1.395	1.397	
0.8	3.287E-02	1.410E-02	6.108E-03	2.656E-03	1.156E-03	5.033E-04	2.192E-04
	1.221	1.207	1.202	1.200	1.200	1.200	

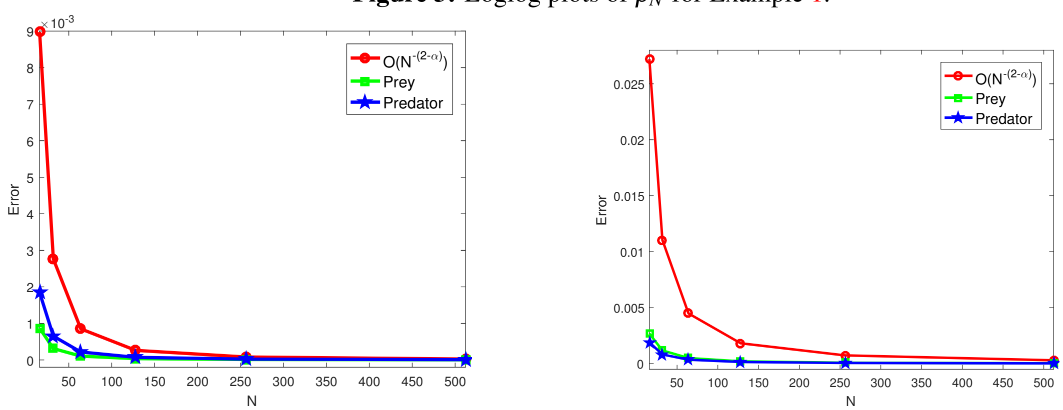
Table 4: Σ_N and ρ_N of $Y(t)$ for Example 2.

σ/M	32	64	128	256	512	1024	2048
0.1	2.704E-01	3.517E-01	9.919E-02	6.840E-03	1.569E-03	4.177E-04	1.151E-04
	-0.379	1.826	3.858	2.124	1.909	1.860	
0.3	5.541E-03	1.737E-03	5.500E-04	1.739E-04	5.475E-05	1.717E-05	5.364E-06
	1.673	1.659	1.661	1.667	1.673	1.678	
0.5	9.990E-03	3.575E-03	1.282E-03	4.585E-04	1.636E-04	5.820E-05	2.067E-05
	1.482	1.480	1.483	1.487	1.491	1.493	
0.7	2.133E-02	8.663E-03	3.531E-03	1.439E-03	5.862E-04	2.385E-04	9.700E-05
	1.300	1.295	1.295	1.296	1.297	1.298	
0.9	4.607E-02	2.119E-02	9.831E-03	4.577E-03	2.134E-03	9.952E-04	4.643E-04
	1.121	1.108	1.103	1.101	1.100	1.100	



(a) $X(t)$ **(b)** $Y(t)$

Figure 3: Loglog plots of ρ_N for Example 1.



(a) $X(t)$ **(b)** $Y(t)$

Figure 4: Error plots of ρ_N for Example 1.

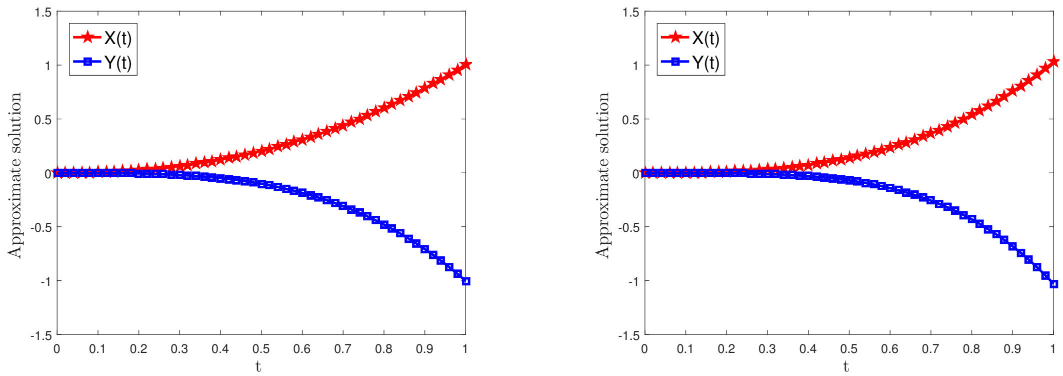
5 Conclusion

This paper presented an efficient numerical algorithm to approximate a system of time-fractional differential equations. Firstly, the L1 scheme is applied to discretize the differential operator. The Newton-Raphson method solves the corresponding nonlinear algebraic system. The scheme is shown to $(2-\sigma)$ order of accuracy. A detailed error analysis is provided. Tables and graphs show that the proposed scheme converges uniformly.

References

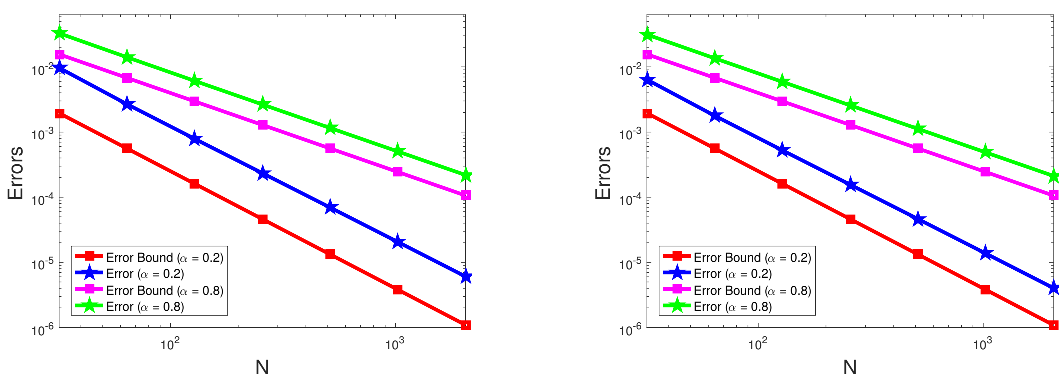
[1] M.A. Alqudah, T. Abdeljawad, K. Shah, K. Jarad, K. Al-Mdallal, *Existence theory and approximate solution to prey-predator coupled system involving nonsingular kernel type derivative*, Adv. Differ. Equ. **2020** (2020) 520.

[2] R.L. Bagley, R.A. Calico, *Fractional order state equations for the control of viscoelastic structures*, J. Guid. Control Dyn.**14** (1999) 2.



(a) $\sigma = 0.3$ **(b)** $\sigma = 0.9$

Figure 5: Numerical solutions with $N = 50$ for Example 2.



(a) $X(t)$ **(b)** $Y(t)$

Figure 6: Loglog plots of ρ_N for Example 2.

[3] S. Das, P.K. Gupta, *A mathematical model on fractional lotkavolterra equations*, J. Theor. Biol. **277** (2011) 1–6.

[4] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, Berlin, 2010.

[5] B. Ghosh, J. Mohapatra, *Numerical simulation for two species time fractional weakly singular model arising in population dynamics*, Int. J. Simul. Model. (2023) 1–14.

[6] B. Ghosh, J. Mohapatra, *An iterative difference scheme for solving arbitrary order nonlinear volterra integro-differential population growth model*, J. Anal. **32** (2024) 57–72.

[7] H. Jafari, R.M. Ganji, N.S. Nkomo, Y.P. Lv, *A numerical study of fractional order population dynamics model*, Results Phys. **27** (2021) 104456.

[8] A.A. Kilbas, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 2006.

[9] R.C. Koeller, *Applications of fractional calculus to the theory of viscoelasticity*, J. Appl. Mech. **51** (1984) 299–307.

[10] W.E. Lamb, *Theory of an optical maser*, Phys. Rev. **134** (1964) A1429.

[11] C. Li, Q. Yi, A. Chen, *Finite difference methods with non-uniform meshes for nonlinear fractional differential equations*, J. Comput. Phys. **316** (2016) 614–631.

[12] C. Li, F. Zeng, *The finite difference methods for fractional ordinary differential equations*, Numer. Funct. Anal. Optim. **34** (2013) 149–179.

[13] R. Lin, F. Liu, *Fractional high order methods for the nonlinear fractional ordinary differential equation*, Nonlinear Anal. Theory Methods Appl. **66** (2007) 856–869.

[14] Y. Lin, C. Xu, *Finite difference/spectral approximations for the time-fractional diffusion equation*, J. Comput. Phys. **225** (2007) 1533–1552.

[15] P. Linz, *Analytical and Numerical Methods for Volterra Equations*, SIAM, Philadelphia, 1985.

[16] A.J. Lotka, *Elements of Physical Biology*, Williams and Wilkins, 1925.

[17] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, Academic Press, San Diego, 1998.

[18] V. Volterra, *Variazioni e fluttuazioni del numero dindividui in specie animali conviventi*, Societa anonima tipografica "Leonardo da Vinc", 1926.

[19] Y. Yan, K. Pal, N.J. Ford, *Higher order numerical methods for solving fractional differential equations*, BIT Numer. Math. **54** (2014) 555–584.