Journal of Mathematical Modeling
Vol. 13, No. 3, 2025, pp. 497-518. Research Article

JMM

Portfolio optimization under regime-switching with market path-dependent returns

Reza Keykhaei*

Department of Mathematics, Khansar Campus, University of Isfahan, Isfahan, Iran
Email(s): r.keykhaei@khc.ui.ac.ir

Abstract. Asset prices typically follow significant trends influenced by the economic environment or overall investor sentiment. Regime-switching is commonly employed to capture asset price dynamics, as it effectively describes significant trends and reflects the changing correlations of asset returns over various periods. This paper explores multi-period mean-variance portfolio optimization under regime-switching with path-dependent returns. Unlike conventional models, this paper assumes that asset returns depend on the entire path of market states rather than just the current one. Consequently, investors base their decisions on all observed states up to the current moment. Utilizing dynamic programming techniques, we derive the path-dependent optimal portfolio strategy and the mean-variance efficient frontier in closed form. Furthermore, we demonstrate that the results from the traditional regime-switching model, can be viewed as specific cases of our proposed model.

Keywords: Portfolio optimization, mean-variance model, regime-switching, market path-dependent, dynamic programming.
AMS Subject Classification 2010: 90C39, 91G10.

1 Introduction

The mean-variance (M-V) portfolio selection problem was first formulated by Markowitz [21], who introduced a static model aimed at minimizing the portfolio return variance for a specified level of expected return. Alternatively, this problem can also be approached by maximizing the portfolio's expected return for a fixed level of return variance. Merton [22] provided the first analytical solution for the static version when negative weights are allowed. However, dynamic versions faced challenges due to the non-separability of variance in dynamic programming. Li and Ng [18] and Zhou and Li [32] overcame this with an embedding technique, providing closed-form solutions for multi-period and continuous-

*Corresponding author
Received: 6 November 2024 / Revised: 31 December 2024 / Accepted: 1 January 2025
DOI: 10.22124/jmm.2025.28912.2571

time settings. Later works expanded on this, exploring various constraints, such as no-shorting [11, 19], bankruptcy prevention [4, 34], and asset-liability management [9, 17].

The studies mentioned above assume that risky asset returns are independent across time periods. However, numerous empirical analyses suggest that returns often exhibit intertemporal dependency. Therefore, it is reasonable to consider M-V portfolio models with correlated returns. General correlation structures have been explored in the portfolio selection models, see [12, 14, 15, 27, 28, 30].

A practical approach to capture return correlations is through Markov chains. In Markov regime-switching models, asset returns depend on the market regime, which is modeled as a finite-state Markov chain. These regimes represent underlying economic conditions, investor sentiment, and other factors. Many portfolio selection models incorporate regime-switching, such as M-V portfolio selection [5, 33], expected utility maximization [1], and M-V asset-liability management [7, 8]. Other studies include no bankruptcy restriction [10], time-consistent M-V portfolio selection [23], and the optimal R&D investment problem [24]. Models with uncertain exit-times under regime-switching were explored in [6, 13, 16, 26, 29]. Hidden Markov regime-switching models were also studied in [3, 25, 31].

In the traditional regime-switching model, asset returns and investor decisions depend on the current market state. We extend this assumption by considering that asset returns in each period depend on all market states from the beginning to the present, making returns path-dependent. This means that past market conditions influence the current performance of assets. For instance, one of the primary indicators of a bull market is the sustained increase in asset prices. This price growth typically results from high investor demand, driven by confidence in the market's upward trend and a stronger inclination to purchase assets. To illustrate this trend, we can analyze the path of market states over time. Recently, Keykhaei [16] applied the concept of path-dependency to model the random exit-time from the market. He assumed that the investor makes a definitive decision to exit and end investment period based on information about the market state up to the current moment, meaning that the exit-time is a stopping time with respect to the market filtration. This model effectively examines the impact of the market path on the investment duration and the decision-making process of investors.

To the best of our knowledge, the multi-period portfolio optimization problem in the M-V framework under path-dependency for asset returns has not been studied. To address this gap, we model key market parameters as path-dependent variables, capturing richer market dynamics compared to the traditional state-dependent approaches. Using a path-dependent value function technique via dynamic programming, we introduce a generalized Bellman equation that incorporates path-dependent state variables. Furthermore, we employ Lagrange multipliers and duality theory to derive the optimal portfolio strategy analytically. These developments allow us to compute the M-V efficient frontier explicitly and to determine optimal strategies and risks by modeling key parameters recursively and providing their explicit forms. Our results demonstrate that the optimal portfolio strategy is inherently dependent on the market path rather than solely on the current state. As expected, the results from Cakmak and Ozekici [5] for the traditional model are recovered as special cases of our model.

The rest of the paper is organized as follows: Section 2 introduces the basic notations, model assumptions, and M-V portfolio formulation. Section 3 provides the analytical solutions and the M-V efficient frontier. In Section 4, we examine portfolio selection with state-dependent returns. An illustrative example is presented in Section 5, and Section 6 discusses an empirical study. Finally, conclusions are drawn in Section 7.

2 Problem formulation

Consider an investor entering a financial market at time 0 with initial wealth w_0 and planning investments over T periods. Let the market state at time n ($n = 0,1,\dots,T$) be denoted by X_n. The market consists of $N+1$ risky assets, where the returns over period n ($n = 0,1,\dots,T-1$) depend on the market path $(X_0,X_1,\dots,X_n)$. For $k = 0,1,\dots,N$, denote the return of the k-th asset over period n by $R_n^k(X_0,\dots,X_n)$. We assume $\{X_n;\ n = 0,1,\dots,T\}$ is a Markov chain with state space $\mathbb{S} = \{1,2,\dots,L\}$ and time-dependent transition matrices Q_n such that $Q_n(i,j) = \Pr(X_n = j|X_{n-1} = i)$, where $i,j \in \mathbb{S}$.

We make the following assumption about the Markov chain:

(A1) $Q_n(i,j) > 0$ for all $i,j \in \mathbb{S}$.

For the given market path $(X_0,\dots,X_n) = (i_0,\dots,i_n)$, let

$$R_n(i_0,\dots,i_n) = \big(R_n^0(i_0,\dots,i_n), R_n^1(i_0,\dots,i_n),\dots,R_n^N(i_0,\dots,i_n)\big)',$$

and

$$R_n^e(i_0,\dots,i_n) = \big(R_n^1(i_0,\dots,i_n) - R_n^0(i_0,\dots,i_n),\dots,R_n^N(i_0,\dots,i_n) - R_n^0(i_0,\dots,i_n)\big)',$$

be the vectors of asset returns and excess returns, respectively, where A' denotes the transpose of a matrix or vector A.

Let $\mathbb{E}\left[\cdot\right]$ represent the expectation operator. We make the following assumptions about asset returns for all $m,n = 0,1,\dots,T-1$ and market paths $(i_0,\dots,i_n)$ and $(j_0,\dots,j_m)$:

(A2) $\mathbb{E}\big[R_n^e(i_0,\dots,i_n)\big] \neq \mathbf{0}$;

(A3) $\mathbb{E}\big[R_n(i_0,\dots,i_n)R_n(i_0,\dots,i_n)'\big]$ is positive definite;

(A4) $R_n(i_0,\dots,i_n)$ is independent of $R_m(j_0,\dots,j_m)$ for all $n \neq m$;

(A5) $R_n(i_0,\dots,i_n)$ is independent of the market state $X_{n+1} = i_{n+1}$.

For $n = 0,1,\dots,T-1$ and a given market path $(i_0,\dots,i_n)$, we define

$$\begin{aligned}
h_n(i_0,\dots,i_n) &= r_n^e(i_0,\dots,i_n)'V_n(i_0,\dots,i_n)^{-1}r_n^e(i_0,\dots,i_n),\\
g_n(i_0,\dots,i_n) &= r_n(i_0,\dots,i_n) - r_n^e(i_0,\dots,i_n)'V_n(i_0,\dots,i_n)^{-1}U_n(i_0,\dots,i_n),\\
f_n(i_0,\dots,i_n) &= r_n^2(i_0,\dots,i_n) - U_n(i_0,\dots,i_n)'V_n(i_0,\dots,i_n)^{-1}U_n(i_0,\dots,i_n),
\end{aligned}$$

where

$$\begin{aligned}
&V_n(i_0,\dots,i_n) = \mathbb{E}\big[R_n^e(i_0,\dots,i_n)R_n^e(i_0,\dots,i_n)'\big], U_n(i_0,\dots,i_n) = \mathbb{E}\big[R_n^0(i_0,\dots,i_n)R_n^e(i_0,\dots,i_n)\big],\\
&r_n^e(i_0,\dots,i_n) = \mathbb{E}\big[R_n^e(i_0,\dots,i_n)\big], r_n(i_0,\dots,i_n) = \mathbb{E}\big[R_n^0(i_0,\dots,i_n)\big], r_n^2(i_0,\dots,i_n) = \mathbb{E}\big[(R_n^0(i_0,\dots,i_n))^2\big].
\end{aligned}$$

We use the following lemma to ensure the existence of an optimal solution. The proof is similar to [26, Lemmas 2 and 3] and is omitted here for brevity.

Lemma 1. *$V_n(i_0,\dots,i_n)$ is positive definite, and $h_n(i_0,\dots,i_n)$, $f_n(i_0,\dots,i_n) > 0$.*

Let π_n^k represent the capital allocated to asset k, and $\pi_n = (\pi_n^1, \pi_n^2, \dots, \pi_n^N)' \in \mathbb{R}^N$ as the investor's portfolio at time n. The investor's portfolio strategy is the investment sequence $\pi = \{\pi_0, \pi_1, \dots, \pi_{T-1}\}$. Additionally, W_n^π signifies the investor's wealth level at time n given the portfolio strategy π. The investment in asset 0 at time n is $\pi_n^0 = W_n^\pi - \sum_{k=1}^N \pi_n^k$. Consequently, within the self-financing framework, the wealth evolution under π can be described as:

$$W_{n+1}^\pi = W_n^\pi R_n^0(X_0, \dots, X_n) + \pi_n' R_n^e(X_0, \dots, X_n), \quad n = 0, 1, \dots, T-1.$$

In this context, we make the following assumptions:

(A6) The investor's current wealth $W_n^\pi = w_n$ is assumed not to influence the vector return $R_n(i_0, \dots, i_n)$ and the market state X_{n+1};

(A7) Short selling is permitted, and there are no associated transaction costs.

Let $\mathbb{E}_0[\cdot] = \mathbb{E}[\cdot \mid X_0 = i_0; W_0^\pi = w_0]$ and $\mathbb{V}_0[\cdot] = \mathbb{E}_0[\cdot]^2 - \mathbb{E}_0^2[\cdot]$ represent the conditional expectation and variance under initial conditions i_0 and w_0. Using the M-V criterion, the multi-period portfolio selection problem is formulated through the following three optimization problems:

$$MV1 : \begin{cases} \max\limits_{\pi} \ \mathbb{E}_0\left[W_T^\pi\right] \\ \text{s.t. } \mathbb{V}_0\left[W_T^\pi\right] = \sigma^2, \\ \qquad W_{n+1}^\pi = W_n^\pi R_n^0(X_0, \dots, X_n) + \pi_n' R_n^e(X_0, \dots, X_n), \quad n = 0, 1, \dots, T-1, \end{cases}$$

$$MV2 : \begin{cases} \min\limits_{\pi} \ \mathbb{V}_0\left[W_T^\pi\right] \\ \text{s.t. } \mathbb{E}_0\left[W_T^\pi\right] = \mu, \\ \qquad W_{n+1}^\pi = W_n^\pi R_n^0(X_0, \dots, X_n) + \pi_n' R_n^e(X_0, \dots, X_n), \quad n = 0, 1, \dots, T-1, \end{cases}$$

$$MV3 : \begin{cases} \max\limits_{\pi} \ \mathbb{E}_0\left[W_T^\pi\right] - \omega \mathbb{V}_0\left[W_T^\pi\right] \ (\omega > 0) \\ \text{s.t. } W_{n+1}^\pi = W_n^\pi R_n^0(X_0, \dots, X_n) + \pi_n' R_n^e(X_0, \dots, X_n), \quad n = 0, 1, \dots, T-1. \end{cases}$$

In problem $MV1$, an investor seeks the optimal investment strategy to maximize the expected terminal wealth while maintaining the variance of the terminal wealth at a preselected risk level σ^2. In problem $MV2$, the investor aims to minimize the variance of the terminal wealth, ensuring that the expected terminal wealth equals a preselected level μ. In problem $MV3$, the focus is on the investor's risk aversion, represented by the parameter ω. The larger the value of ω, the more risk-averse the investor is, reflecting their preference for safer investments with lower variance.

Remark 1. *It is worth noting that the problem $MV1$ (or equivalently $MV2$ and $MV3$) can be interpreted as a specific case of the Bolza form commonly used in investment-consumption problems. Here, the absence of consumption simplifies the model to the Mayer form, focusing only on terminal wealth optimization.*

Remark 2. *It is well known that the three optimization problems above are equivalent in the sense that they yield the same optimal solution for some specific values of the parameters μ, σ^2, and ω. More precisely, if the efficient strategy π^* (to be defined later) is the optimal solution of $MV2$, then it also is* *the optimal solution of $MV1$ with $\sigma^2 = \mathbb{V}_0\left[W_T^{\pi^*}\right]$. Similarly, if π^* is the optimal solution of $MV1$, it also is the optimal solution of $MV2$ with $\mu = \mathbb{E}_0\left[W_T^{\pi^*}\right]$. On the other hand, if π^* is the optimal solution of $MV3$, it also is the optimal solution of $MV2$ with $\mu = \mathbb{E}_0\left[W_T^{\pi^*}\right]$ and the optimal solution of $MV1$ with $\sigma^2 = \mathbb{V}_0\left[W_T^{\pi^*}\right]$. Furthermore, at the optimal solution of $MV3$, the following relation holds:*

$$\frac{\mathrm{d}\mathbb{V}_0\left[W_T^{\pi}\right]}{\mathrm{d}\mathbb{E}_0\left[W_T^{\pi}\right]} = \frac{1}{\omega}. \tag{1}$$

3 Analytical solutions

Initially, we will tackle the optimization problem $MV2$ by employing the Lagrangian dual method alongside a dynamic programming framework. Subsequently, the optimal solutions for the problems $MV1$ and $MV3$ will be derived from the interrelations outlined in Remark 2.

Now, let us examine the optimization scenario $MV2$. By considering the constraint $\mathbb{E}_0\left[W_T^{\pi}\right] = \mu$ and the variance definition, we can reformulate the problem as follows:

$$MV2: \begin{cases} \min\limits_{\pi} \ \mathbb{E}_0\left[(W_T^{\pi} - \mu)^2\right] \\ \text{s.t. } \mathbb{E}_0\left[W_T^{\pi} - \mu\right] = 0, \\ \qquad W_{n+1}^{\pi} = W_n^{\pi} R_n^0(X_0, \ldots, X_n) + \pi_n' R_n^e(X_0, \ldots, X_n), \qquad n = 0, 1, \ldots, T-1. \end{cases}$$

$MV2$ is identified as a convex problem. To solve it, we employ a Lagrange multiplier $2\lambda \in \mathbb{R}$ and reformulate $MV2$ into an unconstrained auxiliary problem, expressed as follows:

$$MV4: \begin{cases} \min\limits_{\pi} \ \mathbb{E}_0\left[(W_T^{\pi} - \mu)^2\right] + 2\lambda \mathbb{E}_0\left[W_T^{\pi} - \mu\right] \\ \text{s.t. } W_{n+1}^{\pi} = W_n^{\pi} R_n^0(X_0, \ldots, X_n) + \pi_n' R_n^e(X_0, \ldots, X_n), \qquad n = 0, 1, \ldots, T-1. \end{cases}$$

By letting $d = \lambda - \mu$ and $d_0 = \mu^2 - 2\lambda\mu$, we can recast the problem as:

$$MV4: \begin{cases} \min\limits_{\pi} \ \mathbb{E}_0\left[(W_T^{\pi})^2 + 2dW_T^{\pi} + d_0\right] \\ \text{s.t. } W_{n+1}^{\pi} = W_n^{\pi} R_n^0(X_0, \ldots, X_n) + \pi_n' R_n^e(X_0, \ldots, X_n), \qquad n = 0, 1, \ldots, T-1. \end{cases}$$

By disregarding the constant term d_0 in the objective function of $MV4$, we formulate the following equivalent optimization problem:

$$MV5: \begin{cases} \min\limits_{\pi} \ \mathbb{E}_0\left[(W_T^{\pi})^2 + 2dW_T^{\pi}\right] \\ \text{s.t. } W_{n+1}^{\pi} = W_n^{\pi} R_n^0(X_0, \ldots, X_n) + \pi_n' R_n^e(X_0, \ldots, X_n), \qquad n = 0, 1, \ldots, T-1. \end{cases}$$

Prior to solving this problem, we will introduce some definitions and notations, along with auxiliary results that will be referenced throughout the paper.

3.1 Matrix expressions overview

In the subsequent section, we will perform intricate matrix computations to derive the explicit optimal solution. To facilitate these calculations, we will use some notations introduced in [16], which will simplify the forthcoming expressions.

Let k,m, and n be non-negative integers. For $2 \leq m \leq n$ we define matrices A and B as follows: A is an $\underbrace{L \times L \times \cdots \times L}_{m \text{ times}}$ matrix, and B is an $\underbrace{L \times L \times \cdots \times L}_{n \text{ times}}$ matrix. We denote $(A \bullet B)$ as an $\underbrace{L \times L \times \cdots \times L}_{n \text{ times}}$ matrix and $\overline{B}$ as an $\underbrace{L \times L \times \cdots \times L}_{(n-1) \text{ times}}$ matrix defined by the equations:

$$
\begin{aligned}
&(A \bullet B)(i_1,\ldots,i_n) = A(i_{n-m+1},\ldots,i_n)B(i_1,\ldots,i_n),\\
&\overline{B}(i_1,\ldots,i_{n-1}) = \sum_{i_n \in \mathbb{S}} B(i_1,\ldots,i_{n-1},i_n).
\end{aligned}
$$

For $1 \leq n < k$, let $A_{k-n},\ldots,A_k$ represent a sequence of $L \times L$ matrices, while B_{k-j} $(j = 1,\ldots,n)$ and α_k denote an $\underbrace{L \times L \times \cdots \times L}_{(k-j+1) \text{ times}}$ matrix and an $\underbrace{L \times L \times \cdots \times L}_{(k+1) \text{ times}}$ matrix, respectively. Using the notation established earlier, we define

$$
\overline{\prod_{j=k-n}^{k-1} (A_j \bullet B_j) \bullet (A_k \bullet \alpha_k)} := \overline{((A_{k-n} \bullet B_{k-n}) \bullet \overline{(\ldots \bullet \overline{((A_{k-1} \bullet B_{k-1}) \bullet \overline{(A_k \bullet \alpha_k)})} \ldots)})} \tag{2}
$$

as an $\underbrace{L \times L \times \cdots \times L}_{(k-n) \text{ times}}$ matrix. To facilitate our calculations, we define $\overline{\prod_{\emptyset}(A_j \bullet B_j) \bullet (A_k \bullet \alpha_k)} = \overline{(A_k \bullet \alpha_k)}$. Additionally, we have $\sum_{\emptyset}(.) = \mathbf{0}$ and $\prod_{\emptyset}(.) = I$ where I denotes the identity matrix.

The subsequent lemma provides a more detailed characterization of the structure represented in (2). The proofs of Lemmas 2 and 3 are similar to those presented in [16] and have therefore been omitted.

Lemma 2. *For $n \geq 0$,*

$$
\begin{aligned}
&\overline{\prod_{j=k-n}^{k-1} (A_j \bullet B_j) \bullet (A_k \bullet \alpha_k)}(i_0,\ldots,i_{k-n-1}) = \\
&\quad \sum_{i_{k-n} \in \mathbb{S}} \cdots \sum_{i_{k-1} \in \mathbb{S}} \sum_{i_k \in \mathbb{S}} A_{k-n}(i_{k-n-1},i_{k-n})B_{k-n}(i_0,\ldots,i_{k-n}) \ldots A_{k-1}(i_{k-2},i_{k-1})B_{k-1}(i_0,\ldots,i_{k-1})\\
&\quad \times A_k(i_{k-1},i_k)\alpha_k(i_0,\ldots,i_k).
\end{aligned}
$$

Lemma 3. *Let $\{A_n\}_{n=1}^{T}$ represent a sequence of $L \times L$ matrices, while $\{\beta_n\}_{n=0}^{T}$ and $\{\theta_n\}_{n=0}^{T-1}$ denote sequences of $\underbrace{L \times L \times \cdots \times L}_{(n+1) \text{ times}}$ matrices. Define $\{\alpha_n\}_{n=0}^{T}$ as a recursive sequence of $\underbrace{L \times L \times \cdots \times L}_{(n+1) \text{ times}}$ matrices with backward-defined components given by*

$$
\alpha_n(i_0,\ldots,i_n) = \beta_n(i_0,\ldots,i_n) + \theta_n(i_0,\ldots,i_n)\overline{(A_{n+1} \bullet \alpha_{n+1})}(i_0,\ldots,i_n),
$$

with the terminal condition $\alpha_T = \beta_T$. Then, explicitly we have

$$
\alpha_n(i_0,\ldots,i_n) = \beta_n(i_0,\ldots,i_n) + \theta_n(i_0,\ldots,i_n) \sum_{k=n+1}^{T} \overline{\prod_{j=n+1}^{k-1} (A_j \bullet \theta_j) \bullet (A_k \bullet \beta_k)}(i_0,\ldots,i_n).
$$

3.2 Solution to problem $MV5$

The dynamic programming approach is used to solve stochastic optimization problems. This approach is particularly useful when the decision-making process is multi-stage, as it allows for the decomposition of the problem into smaller and more manageable subproblems. In dynamic programming, the value function $v_n(X_n)$ typically represents the minimum (or maximum) cost (or reward) from time n onwards under the random state variable X_n. The problem is formulated by considering the optimization objective starting from time 0, expressed as

$$v_0(x_0) = \min_{u_0,\ldots,u_{T-1}} \mathbb{E}\left[\sum_{n=0}^{T-1} c_n(X_n,u_n) + g(X_T)\right],$$

under the given initial state $X_0 = x_0$ where $c_n(X_n,u_n)$ is the cost at time n, $g(X_T)$ is the terminal cost, and u_n represents the decision variables. The value function $v_n(x_n)$ represents the optimal value starting from time n under the current state $X_n = x_n$, i.e.,

$$v_n(x_n) = \min_{u_n,\ldots,u_{T-1}} \mathbb{E}\left[\sum_{k=n}^{T-1} c_k(X_k,u_k) + g(X_T) \mid X_n = x_n\right],$$

and can be expressed as

$$v_n(x_n) = \min_{u_n} \mathbb{E}\left[c_n(x_n,u_n) + v_{n+1}(X_{n+1}) \mid X_n = x_n\right].$$

This recursive relationship, known as the Bellman equation, with the terminal condition $v_T(x_T) = g(x_T)$, provides the connection between the value functions at consecutive time steps and is a key feature of dynamic programming in stochastic optimization problems. Note that, if the running cost $c_n(X_n,u_n)$ is zero for all n, the problem reduces to the Mayer form, focusing solely on the terminal cost $g(X_T)$. Conversely, if the terminal cost $g(X_T)$ is zero, the formulation corresponds to the Lagrange form, minimizing the cumulative running cost. When both $g(X_T)$ and $c_n(X_n,u_n)$ are non-zero, the problem represents the Bolza form, combining terminal and running costs. In this study, the Mayer form is adopted, as the objective focuses exclusively on optimizing terminal wealth. For further details, see [2].

We utilize dynamic programming approach to address the problem $MV5$. To achieve this, we introduce a path-dependent value function method that relies on observations of market states. Let the value function of $MV5$ at time n be defined based on the market state path $(i_0,\ldots,i_n)$ and the wealth amount w_n as follows

$$v_n(i_0,\ldots,i_n;w_n) = \min_{\pi_n,\ldots,\pi_{T-1}} \mathbb{E}_{i_0,\ldots,i_n;w_n}\left[(W_T^\pi)^2 + 2dW_T^\pi\right],$$

where $\mathbb{E}_{i_0,\ldots,i_n;w_n}\left[\,\cdot\,\right] = \mathbb{E}\left[\,\cdot \mid X_0 = i_0,\ldots,X_n = i_n; W_n^\pi = w_n\right]$.

Remark 3. *The key distinction between the current model and other multi-period portfolio selection frameworks involving regime-switching lies in how value functions and optimal portfolios are formulated. In our approach, at each time n, both the value function and the resulting optimal portfolio are influenced by the market state path $(i_0,\ldots,i_n)$ from the start until time n. In contrast, previous models typically assume that value functions and optimal portfolios are determined solely by the present market state i_n.*

Based on our assumptions and the dynamic programming principle, the Bellman equation for solving problem $MV5$ is given as follows

$$\begin{aligned} v_n(i_0,\dots,i_n;w_n) &= \min_{\pi_n} \mathbb{E}\big[v_{n+1}(i_0,\dots,i_n,X_{n+1};W_{n+1}^{\pi})|X_0=i_0,\dots,X_n=i_n,W_n^{\pi}=w_n\big] \\ &= \min_{\pi_n} \sum_{i_{n+1}\in\mathbb{S}} Q_{n+1}(i_n,i_{n+1})\mathbb{E}\big[v_{n+1}(i_0,\dots,i_{n+1};w_nR_n^0(i_0,\dots,i_n)+\pi_n'R_n^e(i_0,\dots,i_n))\big], \end{aligned}$$

with the terminal condition $v_T(i_0,\dots,i_T;w_T)=w_T^2+2dw_T$. The subsequent theorem provides a more comprehensive description of the structure of the value functions.

Theorem 1. *The value functions for problem $MV5$ and respective optimal policies can be expressed as*

$$\begin{aligned} v_n(i_0,\dots,i_n;w_n) &= a_n(i_0,\dots,i_n)w_n^2+b_n(i_0,\dots,i_n)w_n+c_n(i_0,\dots,i_n), \\ \pi_n^*(i_0,\dots,i_n;w_n) &= -V_n(i_0,\dots,i_n)^{-1}\Big[w_nU_n(i_0,\dots,i_n)+\frac{\overline{(Q_{n+1}\bullet b_{n+1})}(i_0,\dots,i_n)}{2\overline{(Q_{n+1}\bullet a_{n+1})}(i_0,\dots,i_n)}r_n^e(i_0,\dots,i_n)\Big], \end{aligned} \tag{3}$$

where the coefficients a_n, b_n, and c_n satisfy the following recursive relationships

$$\begin{aligned} a_n(i_0,\dots,i_n) &= f_n(i_0,\dots,i_n)\overline{(Q_{n+1}\bullet a_{n+1})}(i_0,\dots,i_n), \\ b_n(i_0,\dots,i_n) &= g_n(i_0,\dots,i_n)\overline{(Q_{n+1}\bullet b_{n+1})}(i_0,\dots,i_n), \\ c_n(i_0,\dots,i_n) &= e_n(i_0,\dots,i_n)+\overline{(Q_{n+1}\bullet c_{n+1})}(i_0,\dots,i_n), \\ e_n(i_0,\dots,i_n) &= -\frac{(\overline{(Q_{n+1}\bullet b_{n+1})}(i_0,\dots,i_n))^2}{4\overline{(Q_{n+1}\bullet a_{n+1})}(i_0,\dots,i_n)}h_n(i_0,\dots,i_n). \end{aligned}$$

Proof. To establish the result, we employ backward induction on n. For the case $n=T$, Eq. (3) holds directly by setting $a_T(i_0,\dots,i_T)=1, b_T(i_0,\dots,i_T)=2d$, and $c_T(i_0,\dots,i_T)=0$. Now, consider $n=T-1$ for an arbitrary market path $(i_0,\dots,i_{T-1})$ and a wealth level of w_{T-1}. Then

$$\begin{aligned} &v_{T-1}(i_0,\dots,i_{T-1};w_{T-1}) \\ &\quad= \min_{\pi_{T-1}} \mathbb{E}\Big\{\sum_{i_T\in\mathbb{S}} Q_T(i_{T-1},i_T)v_T\big(i_0,\dots,i_T;w_{T-1}R_{T-1}^0(i_0,\dots,i_{T-1})+\pi_{T-1}'R_{T-1}^e(i_0,\dots,i_{T-1})\big)\Big\} \\ &\quad= \min_{\pi_{T-1}} \mathbb{E}\Big\{\sum_{i_T\in\mathbb{S}} Q_T(i_{T-1},i_T)a_T(i_0,\dots,i_T)\big[w_{T-1}R_{T-1}^0(i_0,\dots,i_{T-1})+\pi_{T-1}'R_{T-1}^e(i_0,\dots,i_{T-1})\big]^2 \\ &\qquad+\sum_{i_T\in\mathbb{S}} Q_T(i_{T-1},i_T)b_T(i_0,\dots,i_T)\big[w_{T-1}R_{T-1}^0(i_0,\dots,i_{T-1})+\pi_{T-1}'R_{T-1}^e(i_0,\dots,i_{T-1})\big] \\ &\qquad+\sum_{i_T\in\mathbb{S}} Q_T(i_{T-1},i_T)c_T(i_0,\dots,i_T)\Big\} \\ &\quad= \min_{\pi_{T-1}} \mathbb{E}\Big\{\overline{(Q_T\bullet a_T)}(i_0,\dots,i_{T-1})\big[w_{T-1}R_{T-1}^0(i_0,\dots,i_{T-1})+\pi_{T-1}'R_{T-1}^e(i_0,\dots,i_{T-1})\big]^2 \\ &\qquad+\overline{(Q_T\bullet b_T)}(i_0,\dots,i_{T-1})\big[w_{T-1}R_{T-1}^0(i_0,\dots,i_{T-1})+\pi_{T-1}'R_{T-1}^e(i_0,\dots,i_{T-1})\big] \\ &\qquad+\overline{(Q_T\bullet c_T)}(i_0,\dots,i_{T-1})\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \overline{(Q_T \bullet a_T)}(i_0,\ldots,i_{T-1}) r_{T-1}^2(i_0,\ldots,i_{T-1}) w_{T-1}^2 + \overline{(Q_T \bullet b_T)}(i_0,\ldots,i_{T-1}) r_{T-1}(i_0,\ldots,i_{T-1}) w_{T-1} \\
&\quad + \overline{(Q_T \bullet c_T)}(i_0,\ldots,i_{T-1}) + \min_{\pi_{T-1}} \Big\{ \overline{(Q_T \bullet a_T)}(i_0,\ldots,i_{T-1}) \pi_{T-1}' V_{T-1}(i_0,\ldots,i_{T-1}) \pi_{T-1} \\
&\quad + \pi_{T-1}' \big[2\overline{(Q_T \bullet a_T)}(i_0,\ldots,i_{T-1}) w_{T-1} U_{T-1}(i_0,\ldots,i_{T-1}) \\
&\quad + \overline{(Q_T \bullet b_T)}(i_0,\ldots,i_{T-1}) r_{T-1}^e(i_0,\ldots,i_{T-1}) \big] \Big\}.
\end{aligned} \tag{4}
$$

Given that $\overline{(Q_T \bullet a_T)}(i_0,\ldots,i_{T-1}) > 0$ and that $V_{T-1}(i_0,\ldots,i_{T-1})$ is positive definite, as established by Lemma 1, the optimal policy meets the following necessary and sufficient condition for optimality

$$
\begin{aligned}
&\overline{(Q_T \bullet a_T)}(i_0,\ldots,i_{T-1}) V_{T-1}(i_0,\ldots,i_{T-1}) \pi_{T-1} + \overline{(Q_T \bullet a_T)}(i_0,\ldots,i_{T-1}) w_{T-1} U_{T-1}(i_0,\ldots,i_{T-1}) \\
&\qquad + \frac{1}{2} \overline{(Q_T \bullet b_T)}(i_0,\ldots,i_{T-1}) r_{T-1}^e(i_0,\ldots,i_{T-1}) = \mathbf{0}.
\end{aligned}
$$

Consequently, the optimal policy can be determined as follows

$$
\begin{aligned}
\pi_{T-1}^*(i_0,\ldots,i_{T-1}; w_{T-1}) = &- V_{T-1}(i_0,\ldots,i_{T-1})^{-1} \big[w_{T-1} U_{T-1}(i_0,\ldots,i_{T-1}) \\
&+ \frac{\overline{(Q_T \bullet b_T)}(i_0,\ldots,i_{T-1})}{2\overline{(Q_T \bullet a_T)}(i_0,\ldots,i_{T-1})} r_{T-1}^e(i_0,\ldots,i_{T-1}) \big].
\end{aligned}
$$

By substituting $\pi_{T-1}^*(i_0,\ldots,i_{T-1}; w_{T-1})$ back into (4), we obtain

$$
v_{T-1}(i_0,\ldots,i_{T-1}; w_{T-1}) = a_{T-1}(i_0,\ldots,i_{T-1}) w_{T-1}^2 + b_{T-1}(i_0,\ldots,i_{T-1}) w_{T-1} + c_{T-1}(i_0,\ldots,i_{T-1}),
$$

where

$$
\begin{aligned}
a_{T-1}(i_0,\ldots,i_{T-1}) &= f_{T-1}(i_0,\ldots,i_{T-1}) \overline{(Q_T \bullet a_T)}(i_0,\ldots,i_{T-1}), \\
b_{T-1}(i_0,\ldots,i_{T-1}) &= g_{T-1}(i_0,\ldots,i_{T-1}) \overline{(Q_T \bullet b_T)}(i_0,\ldots,i_{T-1}), \\
c_{T-1}(i_0,\ldots,i_{T-1}) &= -\frac{(\overline{(Q_T \bullet b_T)}(i_0,\ldots,i_{T-1}))^2}{4\overline{(Q_T \bullet a_T)}(i_0,\ldots,i_{T-1})} h_{T-1}(i_0,\ldots,i_{T-1}) + \overline{(Q_T \bullet c_T)}(i_0,\ldots,i_{T-1}).
\end{aligned}
$$

Observe that $a_{T-1}(i_0,\ldots,i_{T-1}) > 0$ by Lemma 1.

Now, assume that (3) is valid for $n+1$, and that $a_{n+1}(i_0,\ldots,i_{n+1}) > 0$ holds for every market path $(i_0,\ldots,i_{n+1})$. We will prove the statement for n under the given market path $(i_0,\ldots,i_n)$ and wealth amount w_n. By employing the induction hypothesis, we obtain

$$
\begin{aligned}
&v_n(i_0,\ldots,i_n; w_n) \\
&\quad = \min_{\pi_n} \mathbb{E} \Big\{ \sum_{i_{n+1} \in \mathbb{S}} Q_{n+1}(i_n, i_{n+1}) v_{n+1} \big(i_0,\ldots,i_{n+1}; w_n R_n^0(i_0,\ldots,i_n) + \pi_n' R_n^e(i_0,\ldots,i_n) \big) \Big\} \\
&\quad = \min_{\pi_n} \mathbb{E} \Big\{ \sum_{i_{n+1} \in \mathbb{S}} Q_{n+1}(i_n, i_{n+1}) a_{n+1}(i_0,\ldots,i_{n+1}) \big[w_n R_n^0(i_0,\ldots,i_n) + \pi_n' R_n^e(i_0,\ldots,i_n) \big]^2 \\
&\qquad + \sum_{i_{n+1} \in \mathbb{S}} Q_{n+1}(i_n, i_{n+1}) b_{n+1}(i_0,\ldots,i_{n+1}) \big[w_n R_n^0(i_0,\ldots,i_n) + \pi_n' R_n^e(i_0,\ldots,i_n) \big] \\
&\qquad + \sum_{i_{n+1} \in \mathbb{S}} Q_{n+1}(i_n, i_{n+1}) c_{n+1}(i_0,\ldots,i_{n+1}) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \min_{\pi_n} \mathbb{E}\Big\{ \overline{(Q_{n+1} \bullet a_{n+1})}(i_0,\ldots,i_n)\big[w_n R_n^0(i_0,\ldots,i_n) + \pi_n' R_n^e(i_0,\ldots,i_n)\big]^2 \\
&\quad + \overline{(Q_{n+1} \bullet b_{n+1})}(i_0,\ldots,i_n)\big[w_n R_n^0(i_0,\ldots,i_n) + \pi_n' R_n^e(i_0,\ldots,i_n)\big] + \overline{(Q_{n+1} \bullet c_{n+1})}(i_0,\ldots,i_n)\Big\} \\
&= \overline{(Q_{n+1} \bullet a_{n+1})}(i_0,\ldots,i_n) r_n^2(i_0,\ldots,i_n) w_n^2 + \overline{(Q_{n+1} \bullet b_{n+1})}(i_0,\ldots,i_n) r_n(i_0,\ldots,i_n) w_n \\
&\quad + \overline{(Q_{n+1} \bullet c_{n+1})}(i_0,\ldots,i_n) + \min_{\pi_n}\Big\{ \overline{(Q_{n+1} \bullet a_{n+1})}(i_0,\ldots,i_n) \pi_n' V_n(i_0,\ldots,i_n) \pi_n \\
&\quad + \pi_n' \big[2\overline{(Q_{n+1} \bullet a_{n+1})}(i_0,\ldots,i_n) w_n U_n(i_0,\ldots,i_n) + \overline{(Q_{n+1} \bullet b_{n+1})}(i_0,\ldots,i_n) r_n^e(i_0,\ldots,i_n)\big]\Big\}. \qquad (5)
\end{aligned}
$$

Given that $a_{n+1}(i_0,\ldots,i_{n+1}) > 0$, it follows that $\overline{(Q_{n+1} \bullet a_{n+1})}(i_0,\ldots,i_n) > 0$. Furthermore, by Lemma 1, $V_n(i_0,\ldots,i_n)$ is positive definite. The minimization problem in (5) mirrors the structure of (4). Using the same reasoning, the optimal policy can be determined as follows

$$
\pi_n^*(i_0,\ldots,i_n; w_n) = -V_n(i_0,\ldots,i_n)^{-1}\Big[w_n U_n(i_0,\ldots,i_n) + \frac{\overline{(Q_{n+1} \bullet b_{n+1})}(i_0,\ldots,i_n)}{2\overline{(Q_{n+1} \bullet a_{n+1})}(i_0,\ldots,i_n)} r_n^e(i_0,\ldots,i_n)\Big].
$$

By substituting $\pi_n^*(i_0,\ldots,i_n; w_n)$ back into (5), we obtain

$$
v_n(i_0,\ldots,i_n; w_n) = a_n(i_0,\ldots,i_n) w_n^2 + b_n(i_0,\ldots,i_n) w_n + c_n(i_0,\ldots,i_n),
$$

where

$$
\begin{aligned}
a_n(i_0,\ldots,i_n) &= f_n(i_0,\ldots,i_n) \overline{(Q_{n+1} \bullet a_{n+1})}(i_0,\ldots,i_n), \\
b_n(i_0,\ldots,i_n) &= g_n(i_0,\ldots,i_n) \overline{(Q_{n+1} \bullet b_{n+1})}(i_0,\ldots,i_n), \\
c_n(i_0,\ldots,i_n) &= -\frac{(\overline{(Q_{n+1} \bullet b_{n+1})}(i_0,\ldots,i_n))^2}{4\overline{(Q_{n+1} \bullet a_{n+1})}(i_0,\ldots,i_n)} h_n(i_0,\ldots,i_n) + \overline{(Q_{n+1} \bullet c_{n+1})}(i_0,\ldots,i_n).
\end{aligned}
$$

As before, we have $a_n(i_0,\ldots,i_n) > 0$. □

Corollary 1. *Let a_n, b_n, and c_n be the components as defined in Theorem 1. Then*

$$
\begin{aligned}
a_n(i_0,\ldots,i_n) &= f_n(i_0,\ldots,i_n) \overline{\prod_{j=n+1}^{T-1} (Q_j \bullet f_j) \bullet (Q_T \bullet \mathbf{1}_T)}(i_0,\ldots,i_n), \\
b_n(i_0,\ldots,i_n) &= 2d g_n(i_0,\ldots,i_n) \overline{\prod_{j=n+1}^{T-1} (Q_j \bullet g_j) \bullet (Q_T \bullet \mathbf{1}_T)}(i_0,\ldots,i_n), \\
c_n(i_0,\ldots,i_n) &= e_n(i_0,\ldots,i_n) + \sum_{k=n+1}^{T} \overline{\prod_{j=n+1}^{k-1} (Q_j \bullet \mathbf{1}_j) \bullet (Q_k \bullet e_k)}(i_0,\ldots,i_n),
\end{aligned}
$$

where we set $\mathbf{1}_n(i_0,\ldots,i_n) = 1$ $(n = 1,2,\ldots,T)$. Also, $e_T(i_0,\ldots,i_T) = c_T(i_0,\ldots,i_T) = 0$ and, for $n = 0,1,\ldots,T-1$,

$$
e_n(i_0,\ldots,i_n) = -d^2 \frac{\Big[\overline{\prod_{j=n+1}^{T-1} (Q_j \bullet g_j) \bullet (Q_T \bullet \mathbf{1}_T)}(i_0,\ldots,i_n)\Big]^2}{\overline{\prod_{j=n+1}^{T-1} (Q_j \bullet f_j) \bullet (Q_T \bullet \mathbf{1}_T)}(i_0,\ldots,i_n)} h_n(i_0,\ldots,i_n).
$$

The optimal policy $\pi_n^*(i_0,\dots,i_n;w_n)$ *is then given by*

$$\begin{aligned}\pi_n^*(i_0,\dots,i_n;w_n) = &-V_n(i_0,\dots,i_n)^{-1}\Big[w_nU_n(i_0,\dots,i_n)\\ &+d\frac{\overline{\prod_{j=n+1}^{T-1}(Q_j\bullet g_j)\bullet(Q_T\bullet\mathbf{1}_T)}(i_0,\dots,i_n)}{\overline{\prod_{j=n+1}^{T-1}(Q_j\bullet f_j)\bullet(Q_T\bullet\mathbf{1}_T)}(i_0,\dots,i_n)}r_n^e(i_0,\dots,i_n)\Big].\end{aligned}\tag{6}$$

Proof. Using Lemmas 3 and 2, the assertions for a_n, b_n, c_n, and consequently for e_n and π_n^*, are derived. To obtain a_n, set $\beta_n(i_0,\dots,i_n)=0$ for $n=0,1,\dots,T-1$ and $\beta_T(i_0,\dots,i_T)=a_T(i_0,\dots,i_T)=1$ according to Lemma 3. For b_n, set $\beta_n(i_0,\dots,i_n)=0$ for $n=0,1,\dots,T-1$ and $\beta_T(i_0,\dots,i_T)=b_T(i_0,\dots,i_T)=2d$. Finally, to obtain c_n, set $\theta_n(i_0,\dots,i_n)=1$. □

3.3 Solutions to problems $MV1$, $MV2$ and $MV3$

The optimal value of problem $MV5$, given the initial conditions $X_0=i_0$ and $W_0^\pi=w_0$, is represented by $v_0(i_0;w_0)$. Therefore, according to Theorem 1 and Corollary 1, the optimal value of problem $MV4$ can be expressed as follows

$$\begin{aligned}v_0(i_0;w_0)+d_0 =&a_0(i_0)w_0^2+b_0(i_0)w_0+c_0(i_0)+d_0\\ =&a_0(i_0)w_0^2+2db(i_0)w_0-d^2c(i_0)+d_0\\ =&a_0(i_0)w_0^2+2(\lambda-\mu)b(i_0)w_0-(\lambda-\mu)^2c(i_0)+\mu^2-2\lambda\mu,\end{aligned}\tag{7}$$

where

$$\begin{aligned}a_0(i_0)&=f_0(i_0)\overline{\prod_{j=1}^{T-1}(Q_j\bullet f_j)\bullet(Q_T\bullet\mathbf{1}_T)}(i_0),\\ b(i_0)&=g_0(i_0)\overline{\prod_{j=1}^{T-1}(Q_j\bullet g_j)\bullet(Q_T\bullet\mathbf{1}_T)}(i_0),\\ c(i_0)&=e_0^*(i_0)+\sum_{k=1}^{T-1}\overline{\prod_{j=1}^{k-1}(Q_j\bullet\mathbf{1}_j)\bullet(Q_k\bullet e_k^*)}(i_0),\end{aligned}$$

and $e_n^*=-(1/d^2)e_n$. To derive $c(i_0)$, observe that for $k=T$ we have $e_T(i_0,\dots,i_T)=0$. By Lemma 2, it follows that $\overline{\prod_{j=1}^{T-1}(Q_j\bullet\mathbf{1}_j)\bullet(Q_T\bullet e_T^*)}(i_0)=0$. As observed, the optimal value of problem $MV4$ is a function of the Lagrange multiplier λ. We define this function as follows

$$L(\lambda):=v_0(i_0;w_0)+d_0.$$

According to the Lagrange duality theorem (refer to [20]), the optimal value for the problem $MV2$, denoted as $\mathbb{V}_0^*(\mu)$, can be determined by maximizing the expression in (7) with respect to $\lambda\in\mathbb{R}$. Thus, we have

$$\mathbb{V}_0^*(\mu)=\max_{\lambda\in\mathbb{R}}L(\lambda).$$

Note that $e_n^*(i_0,\dots,i_n) \geq 0$ and $e_{T-1}^*(i_0,\dots,i_{T-1}) = h_{T-1}(i_0,\dots,i_{T-1}) > 0$ according to Lemma 1. Therefore, the assumption (A1) guarantees that

$$\overline{\prod_{j=1}^{T-2}(Q_j \bullet \mathbf{1}_j) \bullet (Q_{T-1} \bullet e_{T-1}^*)}(i_0) = \sum_{i_1 \in \mathbb{S}} \dots \sum_{i_{T-1} \in \mathbb{S}} Q_1(i_0,i_1)\dots Q_{T-1}(i_{T-2},i_{T-1}) e_{T-1}^*(i_0,\dots,i_{T-1}) > 0.$$

Consequently, we have

$$c(i_0) \geq \overline{\prod_{j=1}^{T-2}(Q_j \bullet \mathbf{1}_j) \bullet (Q_{T-1} \bullet e_{T-1}^*)}(i_0) > 0.$$

Thus, it follows that $L(\lambda)$ attains its maximum at $\lambda^* = \frac{b(i_0)w_0 - \mu}{c(i_0)} + \mu$. Now, it suffices to substitute

$$d = \lambda^* - \mu = \frac{b(i_0)w_0 - \mu}{c(i_0)} \tag{8}$$

into Eq. (6) to derive the optimal strategy for problem $MV2$. By substituting λ^* into (7), we obtain the expression for the minimum variance in problem $MV2$ as follows

$$\mathbb{V}_0^*(\mu) = \frac{1-c(i_0)}{c(i_0)}\Big(\mu - \frac{b(i_0)w_0}{1-c(i_0)}\Big)^2 + \big(a_0(i_0) - \frac{(b(i_0))^2}{1-c(i_0)}\big)w_0^2. \tag{9}$$

We recall that a portfolio strategy π^* is considered M-V *efficient* if there exists no other portfolio strategy π such that $\mathbb{E}_0\left[W_T^{\pi}\right] \geq \mathbb{E}_0\left[W_T^{\pi^*}\right]$ and $\mathbb{V}_0\left[W_T^{\pi}\right] \leq \mathbb{V}_0\left[W_T^{\pi^*}\right]$, with at least one of these inequalities being strict. A point of the form $(\mathbb{E}_0\left[W_T^{\pi^*}\right], \mathbb{V}_0\left[W_T^{\pi^*}\right])$ is referred to as an M-V efficient point in the M-V plane. The collection of all efficient points constitutes the *efficient frontier*. This frontier can be derived by varying the parameter values of μ, σ^2, and ω in the problems $MV2$, $MV1$, and $MV3$, respectively. It is important to note that for any optimal portfolio strategy with a mean and variance of terminal wealth located on the lower branch of the parabola described by (9), there exists an alternative optimal portfolio strategy with the same variance but a higher expected terminal wealth. Therefore, the M-V efficient frontier corresponds to the upper branch of the parabola (9) in the M-V plane, associated with values of $\mu \geq \frac{b(i_0)w_0}{1-c(i_0)}$.

Based on the discussion above, we state the following theorem along with its corollary.

Theorem 2. *Assume that $\mu \geq \frac{b(i_0)w_0}{1-c(i_0)}$. The efficient portfolio strategy for problem $MV2$ is market path-dependent and is provided by* (6)*, where*

$$d = \frac{b(i_0)w_0 - \mu}{c(i_0)}. \tag{10}$$

The efficient frontier is generated by varying μ according to (9).

Corollary 2. *The global minimum variance, $\sigma_{\min}^2 = \big(a_0(i_0) - \frac{(b(i_0))^2}{1-c(i_0)}\big)w_0^2$, corresponding to the expected wealth $\mu_{\min} = \frac{b(i_0)w_0}{1-c(i_0)}$, can be achieved under the portfolio strategy provided by* (6) *with $d = \frac{b(i_0)w_0}{c(i_0)-1}$.*

For determining the optimal portfolio strategy in problem $MV1$, as noted in Remark 2, we can set $\sigma^2 = \mathbb{V}_0^*(\mu)$ and express μ in terms of σ^2 using Eq. (9) as follows

$$\mu = \sqrt{\frac{c(i_0)}{1-c(i_0)}\big[\sigma^2 + \big(\frac{(b(i_0))^2}{1-c(i_0)} - a_0(i_0)\big)w_0^2\big]} + \frac{b(i_0)w_0}{1-c(i_0)}.$$

Substituting μ in (8) yields

$$d = \frac{b(i_0)w_0}{c(i_0)-1} - \sqrt{\frac{1}{c(i_0)(1-c(i_0))}\big[\sigma^2 + \big(\frac{(b(i_0))^2}{1-c(i_0)} - a_0(i_0)\big)w_0^2\big]}. \tag{11}$$

Once again, by substituting the above value of d into (6), we obtain the optimal portfolio strategy for problem $MV1$. To determine the optimal solution for problem $MV3$, we refer to Eq. (1) and use the following relation at the optimal solution

$$\frac{1}{\omega} = \frac{\mathrm{d}\mathbb{V}_0^*(\mu)}{\mathrm{d}\mu} = \frac{2(1-c(i_0))}{c(i_0)}\Big(\mu - \frac{b(i_0)w_0}{1-c(i_0)}\Big).$$

We can express μ in terms of ω and then substitute this value into (8) to obtain

$$d = \frac{1+2\omega b(i_0)w_0}{2\omega(c(i_0)-1)}. \tag{12}$$

Now, substituting this value of d into (6) provides the optimal portfolio strategy for problem $MV3$. The following corollary provides a summary of these discussions.

Corollary 3. *The efficient portfolio strategies for $MV1$ and $MV3$ are market path-dependent and follow* (6)*, with*

$$d = \begin{cases} \frac{b(i_0)w_0}{c(i_0)-1} - \sqrt{\frac{1}{c(i_0)(1-c(i_0))}\left[\sigma^2 + \left(\frac{b(i_0)^2}{1-c(i_0)} - a_0(i_0)\right)w_0^2\right]}, & \text{for } MV1, \\ \frac{1+2\omega b(i_0)w_0}{2\omega(c(i_0)-1)}, & \text{for } MV3. \end{cases}$$

4 Portfolio selection with state-dependent asset returns

In this section, we examine the M-V portfolio selection in a Markovian regime-switching market where asset returns are influenced solely by the current state of the market. Under this framework, we define the vectors of returns and excess returns as $R_n(i_n) = \big(R_n^0(i_n), R_n^1(i_n), \ldots, R_n^N(i_n)\big)'$ and $R_n^e(i_n) = \big(R_n^1(i_n) - R_n^0(i_n), \ldots, R_n^N(i_n) - R_n^0(i_n)\big)'$, where $R_n^k(i_n)$ represents the return of the k-th asset over period n associated with the current state i_n. This model has been analyzed by Cakmak and Ozekici [5], albeit under more restrictive conditions, specifically in the presence of a riskless asset. We demonstrate that the findings in Cakmak and Ozekici [5] are particular cases of our broader results. Importantly, in contrast to their approach, we do not require the Markov chain to be time-homogeneous, we allow for the possibility of the absence of a riskless asset in the market, and we acknowledge that the returns of risky assets are contingent on both the market state and the time period.

Given our assumption that market parameters are dependent on the current state, we redefine the model's key parameters accordingly. The new definitions are as follows

$$
\begin{aligned}
h_n(i_n) &= r_n^e(i_n)' V_n(i_n)^{-1} r_n^e(i_n), \\
g_n(i_n) &= r_n(i_n) - r_n^e(i_n)' V_n(i_n)^{-1} U_n(i_n), \\
f_n(i_n) &= r_n^2(i_n) - U_n(i_n)' V_n(i_n)^{-1} U_n(i_n),
\end{aligned}
$$

where the parameters are defined as

$$
\begin{aligned}
&V_n(i_n) = \mathbb{E}\big[R_n^e(i_n) R_n^e(i_n)'\big], \;\; U_n(i_n) = \mathbb{E}\big[R_n^0(i) R_n^e(i_n)\big], \\
&r_n^e(i_n) = \mathbb{E}\big[R_n^e(i_n)\big], \; r_n(i_n) = \mathbb{E}\big[R_n^0(i_n)\big], \; r_n^2(i_n) = \mathbb{E}\big[(R_n^0(i_n))^2\big].
\end{aligned}
$$

To streamline our calculations, we introduce some notations. Let A and B be $L \times L$ matrices while α and $\mathbf{1} = (1, \ldots, 1)'$ represent L-column vectors. We define A_α as the $L \times L$ matrix where $A_\alpha(i, j) = A(i, j)\alpha(j)$. It is important to note that we can express the vector sum as $\overline{A} = A\mathbf{1}$. Consequently, we have $\overline{AB} = AB\mathbf{1} = A\overline{B}$ and $\overline{A_\alpha} = A\alpha$. Let n be a positive integer. For convenience in notation, we define $A_{\alpha_n} = (A_n)_{\alpha_n}$ for any $L \times L$ matrix A_n and L-column vector α_n. Given that the new model parameters depend on the current state, we have $g_n(i_0, \ldots, i_n) = g_n(i_n)$. By applying Lemma 2 and our notations, we obtain

$$
\begin{aligned}
&\overline{\prod_{j=n+1}^{T-1} (Q_j \bullet g_j) \bullet (Q_T \bullet \mathbf{1}_T)}(i_0, \ldots, i_n) \\
&\quad = \sum_{i_{n+1} \in \mathbb{S}} \cdots \sum_{i_{T-1} \in \mathbb{S}} \sum_{i_T \in \mathbb{S}} Q_{n+1}(i_n, i_{n+1}) g_{n+1}(i_0, \ldots, i_{n+1}) \ldots Q_{T-1}(i_{T-2}, i_{T-1}) g_{T-1}(i_0, \ldots, i_{T-1}) \\
&\qquad \times Q_T(i_{T-1}, i_T) \mathbf{1}_T(i_0, \ldots, i_T) \\
&\quad = \sum_{i_{n+1} \in \mathbb{S}} \cdots \sum_{i_{T-1} \in \mathbb{S}} \sum_{i_T \in \mathbb{S}} Q_{n+1}(i_n, i_{n+1}) g_{n+1}(i_{n+1}) \ldots Q_{T-1}(i_{T-2}, i_{T-1}) g_{T-1}(i_{T-1}) Q_T(i_{T-1}, i_T) \\
&\quad = \sum_{i_{n+1} \in \mathbb{S}} \cdots \sum_{i_{T-1} \in \mathbb{S}} Q_{g_{n+1}}(i_n, i_{n+1}) \ldots Q_{g_{T-1}}(i_{T-2}, i_{T-1}) \sum_{i_T \in \mathbb{S}} Q_T(i_{T-1}, i_T) \\
&\quad = \sum_{i_{n+1} \in \mathbb{S}} \cdots \sum_{i_{T-1} \in \mathbb{S}} Q_{g_{n+1}}(i_n, i_{n+1}) \ldots Q_{g_{T-1}}(i_{T-2}, i_{T-1}) \mathbf{1}(i_{T-1}) \\
&\quad = \big((\prod_{j=n+1}^{T-1} Q_{g_j}) \mathbf{1} \big)(i_n) = (\overline{\prod_{j=n+1}^{T-1} Q_{g_j}})(i_n).
\end{aligned}
$$

Following similar manipulations, we derive $\overline{\prod_{j=n+1}^{T-1} (Q_j \bullet f_j) \bullet (Q_T \bullet \mathbf{1}_T)}(i_0, \ldots, i_n) = \overline{(\prod_{j=n+1}^{T-1} Q_{f_j})}(i_n)$. The parameters outlined here rely solely on the current market state i_n. Under this condition,

$$
\begin{aligned}
e_n^*(i_0, \ldots, i_n) &= e_n^*(i_n) := \frac{\Big[\overline{(\prod_{j=n+1}^{T-1} Q_{g_j})}(i_n) \Big]^2}{\overline{(\prod_{j=n+1}^{T-1} Q_{f_j})}(i_n)} h_n(i_n), \;\; n = 0, \ldots, T-1, \\
e_T^*(i_0, \ldots, i_T) &= e_T^*(i_T) := 0.
\end{aligned}
$$

Using a similar approach, we derive $\overline{\prod_{j=n+1}^{k-1}(Q_j \bullet \mathbf{1}_j) \bullet (Q_k \bullet e_k^*)}(i_0,\ldots,i_n) = \big((\prod_{j=n+1}^{k} Q_j)e_k^*\big)(i_n)$. Consequently, we find

$$
\begin{aligned}
a_0(i_0) &= f_0(i_0)\Big[\overline{(\prod_{j=1}^{T-1} Q_{f_j})(i_0)}\Big],\\
b(i_0) &= g_0(i_0)\Big[\overline{(\prod_{j=1}^{T-1} Q_{g_j})(i_0)}\Big],\\
c(i_0) &= e_0^*(i_0) + \sum_{k=1}^{T-1}\big((\prod_{j=1}^{k} Q_j)e_k^*\big)(i_0) = \sum_{k=0}^{T-1}\big((\prod_{j=1}^{k} Q_j)e_k^*\big)(i_0).
\end{aligned}
$$

Based on Eq. (6), the optimal portfolios are structured as follows

$$
\pi_n^*(i_0,\ldots,i_n;w_n) = \pi_n^*(i_n;w_n) = -V_n(i_n)^{-1}\Big[w_n U_n(i_n) + d\,\frac{\overline{(\prod_{j=n+1}^{T-1} Q_{g_j})(i_n)}}{\overline{(\prod_{j=n+1}^{T-1} Q_{f_j})(i_n)}} r_n^e(i_n)\Big].
$$

Thus, as anticipated, the optimal portfolios depend solely on the current market state.

In Cakmak and Ozekici [5], the market setup includes both risky assets and a risk-free asset, with returns dependent solely on the market state and assuming a time-homogeneous Markov chain. The parameters defined there, i.e., $h(i)$, $g(i)$, $f(i)$, $a_1(i)$, $a_2(i)$, and $b(i)$, align with our parameters as follows

$$
h(i) = h_n(i),\;\; g(i) = g_n(i),\;\; f(i) = f_n(i),\;\; a_1(i) = b(i_0),\;\; a_2(i) = a_0(i_0),\;\; b(i) = c(i_0)/2.
$$

Through straightforward adjustments, it becomes clear that the optimal portfolios and M-V efficient frontier in Corollaries 4 and 5 of Cakmak and Ozekici [5] correspond with our findings.

5 A numerical example

Consider a market comprising two assets: one riskless and one risky. The market shifts between two regimes, labeled as regime 1 (downward or bearish) and regime 2 (upward or bullish), representing unfavorable and favorable economic conditions, respectively. Assume that the log-return of the risky asset follows a normal distribution dependent on the market path as follows

$$
\begin{cases}
\log R_n^1(i_0,\ldots,i_n) \sim N\big(\mu(i_0,\ldots,i_n), \sigma^2(i_0,\ldots,i_n)\big),\\
\mu(i_0,\ldots,i_n) = \sum_{m=0}^{n} (-1)^{i_m} 0.5^{-m+n+1},\;\; \sigma^2(i_0,\ldots,i_n) = 0.05 + n10^{-2}.
\end{cases}
$$

In fact, we assume that the good performance of the economy in the past increases the expected return of the risky asset for the next periods, while the bad performance decreases it. More specifically, these parameters reflect the influence of market paths, particularly the proximity of states to the current state. In particular, states closer to the present have a greater impact on asset returns. Additionally, we suppose that the return of the riskless asset follows the equation

$$
R_n^0(i_0,\ldots,i_n) = 1.02 + \sum_{m=0}^{n} (-1)^{1+i_m} 0.02^{-m+n+1}.
$$

In this case, unlike the risky asset, the return of the riskless asset increases under unfavorable conditions for the risky asset and decreases under favorable conditions.

Suppose an investor enters the financial market in the initial state $i_0 = 1$ with an initial wealth $w_0 = 100$ and plans to invest over the next three periods, i.e., $T = 3$. We examine the time-homogeneous case defined by the transition probability matrix $Q = \left(\begin{smallmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{smallmatrix}\right)$. We then proceed to calculate the necessary parameters to derive the optimal portfolios as follows

$$
\begin{aligned}
&U_0(1) = \begin{pmatrix} -0.4348 \end{pmatrix}, && V_0(1) = \begin{pmatrix} 0.1946 \end{pmatrix}, && r^e_0(1) = \begin{pmatrix} -0.4181 \end{pmatrix}, \\
&U_1(1,:) = \begin{pmatrix} -0.5760 & 0.3229 \end{pmatrix}, && V_1(1,:) = \left(\begin{pmatrix} 0.3212 & 0.2124 \end{pmatrix}\right), && r^e_1(1,:) = \begin{pmatrix} -0.5536 & 0.3227 \end{pmatrix}, \\
&U_2(1,:,:) = \begin{pmatrix} -0.6333 & 0.1732 \\ -0.3408 & 0.9348 \end{pmatrix}, && V_2(1,:,:) = \begin{pmatrix} 0.3840 & 0.1298 \\ 0.1442 & 1.1460 \end{pmatrix}, && r^e_2(1,:,:) = \begin{pmatrix} -0.6087 & 0.1731 \\ -0.3278 & 0.9352 \end{pmatrix}, \\
&h_0(1) = \begin{pmatrix} 0.8981 \end{pmatrix}, && g_0(1) = \begin{pmatrix} 0.1059 \end{pmatrix}, && f_0(1) = \begin{pmatrix} 0.1102 \end{pmatrix}, \\
&h_1(1,:) = \begin{pmatrix} 0.9544 & 0.4903 \end{pmatrix}, && g_1(1,:) = \left(\begin{pmatrix} cc0.0475 & 0.5099 \end{pmatrix}\right), && f_1(1,:) = \begin{pmatrix} 0.0494 & 0.5101 \end{pmatrix}, \\
&h_2(1,:,:) = \begin{pmatrix} 0.9648 & 0.2308 \\ 0.7453 & 0.7632 \end{pmatrix}, && g_2(1,:,:) = \begin{pmatrix} 0.0366 & 0.7695 \\ 0.2648 & 0.2368 \end{pmatrix}, && f_2(1,:,:) = \begin{pmatrix} 0.0381 & 0.7698 \\ 0.2753 & 0.2367 \end{pmatrix}.
\end{aligned}
$$

Additionally, we calculate the following

$$
\begin{aligned}
&\hat{g}_0(1) = \begin{pmatrix} 0.0840 \end{pmatrix}, && \hat{f}_0(1) = \begin{pmatrix} 0.0854 \end{pmatrix}, \\
&\hat{g}_1(1,:) = \begin{pmatrix} 0.4763 & 0.2452 \end{pmatrix}, && \hat{f}_1(1,:) = \begin{pmatrix} 0.4771 & 0.2483 \end{pmatrix}, \\
&\hat{g}_2(1,:,:) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, && \hat{f}_2(1,:,:) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\end{aligned}
$$

where

$$\hat{g}_n(i_0,\dots,i_n) = \overline{\prod_{j=n+1}^{T-1} (Q \bullet g_j) \bullet (Q \bullet \mathbf{1}_T)}(i_0,\dots,i_n),\ \hat{f}_n(i_0,\dots,i_n) = \overline{\prod_{j=n+1}^{T-1} (Q \bullet f_j) \bullet (Q \bullet \mathbf{1}_T)}(i_0,\dots,i_n).$$

The optimal strategies for problems $MV2$, $MV1$, and $MV3$ can be determined using (6) with the values calculated above and an appropriate value for d based on (10), (11), and (12). For example, the optimal portfolio strategy for problem $MV2$ with $\mu = 130$, given the market path $(i_0, i_1, i_2) = (1, 2, 1)$ and wealth levels $(w_0, w_1, w_2) = (100, 115, 120)$, is calculated under $d = -130.2151$ as $(\pi_0^*, \pi_1^*, \pi_2^*) = (-51.8771, 20.5916, -12.4172)$. As observed, in unfavorable market conditions for the risky asset, the investment in it decreases (taking negative values), whereas in favorable conditions, the investment increases (taking positive values).

The M-V efficient frontier is represented by the following equation, illustrated in Figure 1

$$\mathbb{V}_0^*(\mu) = \frac{1-c(1)}{c(1)}\left(\mu - \frac{b(1)100}{1-c(1)}\right)^2 + \left(a_0(1) - \frac{(b(1))^2}{1-c(1)}\right)10000,$$

where, $a_0(1) = 0.0094, b(1) = 0.0089$, and $c(1) = 0.9915$. As shown in Figure 1, according to Corollary 2, the minimum level of risk is $\sigma^2_{\min} = 0.7103$, corresponding to $\mu_{\min} = 104.9$. In fact, achieving zero risk is not possible, even with full investment in the riskless asset. Note that while the return of the riskless asset is certain for a given state, the uncertainty of future market states makes the riskless return uncertain in subsequent stages, leading to the presence of risk.

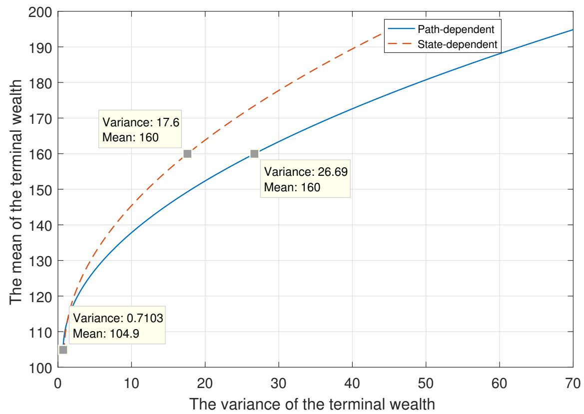

Figure 1: M-V efficient frontiers corresponding to $i_0 = 1$.

To examine the impact of market paths on the results (compared to the impact of the current market state), we studied the optimal investment strategy in problem $MV2$ (for a desired return of $\mu = 130$ under the path $(i_0, i_1, i_2) = (1, 2, 1)$ and wealth levels $(w_0, w_1, w_2) = (100, 115, 120))$ as well as the efficient frontier, under the assumptions of return dependency on the path and the current market state. The results of these two cases are shown in Figures 1 and 2. Regarding the optimal strategies (Figure 2), although they initially suggest almost identical portfolios, they begin to diverge as time progresses and the path develops. In the current model, since the favorable state $i_1 = 2$ following the unfavorable past $i_0 = 1$ is expected to yield a lower return increase (compared to the classical model), the investment in the risky asset increases to compensate for this and achieve the desired final wealth. On the other hand, in the unfavorable state $i_2 = 1$ followed by the favorable past $i_1 = 2$, the return increase for the riskless asset is smaller (compared to the classical model), and as a result, the investment in the riskless asset increases to compensate for this and achieve the desired final wealth (equivalently, the investment in the risky asset decreases).

Regarding the efficient frontiers, as shown in the Figure 1, the efficient frontier in the present model lies below that of the classical model. This indicates that, to achieve a specified expected terminal wealth (e.g., $\mu = 160$), the investor in the current model bears a higher level of risk compared to the classical model ($17.6 < 26.69$). This increased risk stems from the heightened uncertainty in the present model, where parameters depend on additional random variables, namely the market path, as opposed to solely relying on the current market state in the classical model. The greater stochastic nature introduced by this dependency translates into higher risk.

6 An empirical study

In this section, we demonstrate how the daily returns of an asset can be influenced by the historical path or memory of the market. To this end, we utilize the historical data of the Tehran Stock Exchange (TSE) index from November 2019 to June 2024. It is assumed that the market exhibits two states each day: bullish (upward) and bearish (downward). The daily candlestick chart mechanism is employed to identify market states and calculate daily rate of returns, considering both path-dependence and the

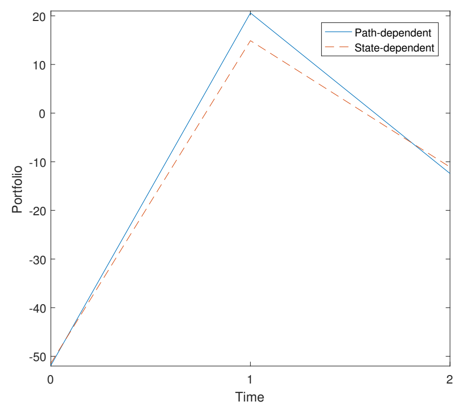

Figure 2: Optimal portfolios corresponding to problem $MV2$ for $\mu = 130$.

current state. For simplicity, it is assumed that returns are time-independent and rely solely on the length and type of the market path. The following approach is employed to determine the daily rate of return in the interval $[n, n+1)$ under the path $(i_0, \ldots, i_n)$. First, the observed $(n+1)$-day paths of the form $(i_0, \ldots, i_n)$ are identified. Next, the rate of return for the last day is calculated using the corresponding candlestick as $P_l/P_f - 1$, where P_l and P_f denote the last and first prices of the day, respectively. The average return of return of these last-day candlesticks under the path $(i_0, \ldots, i_n)$ is then considered as the expected daily return of return of the TSE index for the given path $(i_0, \ldots, i_n)$.

Some of the average daily returns, denoted by $\bar{R}$, corresponding to different paths are presented in Table 1. In this table, the returns associated with single-state paths, such as (1) and (2), represent returns dependent on the current state, while the returns for other paths reflect their dependence on the entire market path. As observed, the return corresponding to each state is significantly influenced by the path leading to that state. For instance, under the traditional model, for an upward day (state (1)), an expected return of return 0.0078 is predicted. However, if the prior day or the two days prior were also upward (paths $(1,1)$ and $(1,1,1)$), the expected rate of returns for an upward day increase to 0.0095 and 0.0097, respectively, which are noticeably higher. This phenomenon is also evident in the reverse for downward states. Overall, the effect of market paths–considering their length, the type of states, or the distance of past states from the current one– on the daily returns is apparent.

Additionally, the long-term impact of market paths on daily returns is examined in Table 2 for consecutive upward and downward paths. The data from Table 2 are visualized in Figure 3, respectively, based on the number of consecutive upward and downward states. As observed in these figures, the trend of increasing or decreasing returns reverses over the long term. For instance, as the number of consecutive downward states increases, the magnitude of negative returns decreases. This phenomenon can be interpreted as follows: in the short term, an increase in consecutive downward states, accompanied by negative sentiment, intensifies the decline in value. However, over the long term, as market sentiment stabilizes and assets become more attractive for purchase, the extent of the value decline diminishes. A similar but inverse phenomenon is observed for consecutive upward paths.

Table 1: Expected daily rate of returns under different market paths.

Path	(1)	(2)	(1,1)	(2,1)	(1,2)	(2,2)		
$\bar{R}$	0.0078	-0.0079	0.0095	0.0056	-0.0063	-0.0088		
Path	(1,1,1)	(2,1,1)	(1,2,1)	(2,2,1)	(1,1,2)	(1,2,2)	(2,1,2)	(2,2,2)
$\bar{R}$	0.0097	0.0094	0.0057	0.0056	-0.0058	-0.0084	-0.0070	-0.0091
Path	(1,1,1,1)	(2,1,1,1)	(1,2,1,1)	(1,1,2,1)	(2,2,1,1)	(2,1,2,1)	(1,2,2,1)	(2,2,2,1)
$\bar{R}$	0.0096	0.0097	0.0102	0.0064	0.0088	0.0041	0.0066	0.0048
Path	(1,1,1,2)	(2,1,1,2)	(1,2,1,2)	(1,1,2,2)	(2,2,1,2)	(2,1,2,2)	(1,2,2,2)	(2,2,2,2)
$\bar{R}$	-0.0053	-0.0065	-0.0079	-0.0073	-0.0067	-0.0094	-0.0086	-0.0093

Table 2: Expected daily rate of returns for consecutive upward and downward paths.

Path	(1)	(1,1)	(1,1,1)	(1,1,1,1)	(1,1,1,1,1)	(1,1,1,1,1,1)	(1,1,1,1,1,1,1)
$\bar{R}$	0.0078	0.0097	0.0095	0.0096	0.0099	0.0106	0.0101
Path	(2)	(2,2)	(2,2,2)	(2,2,2,2)	(2,2,2,2,2)	(2,2,2,2,2,2)	(2,2,2,2,2,2,2)
$\bar{R}$	-0.0079	-0.0088	-0.0091	-0.0093	-0.0101	-0.0089	-0.0092

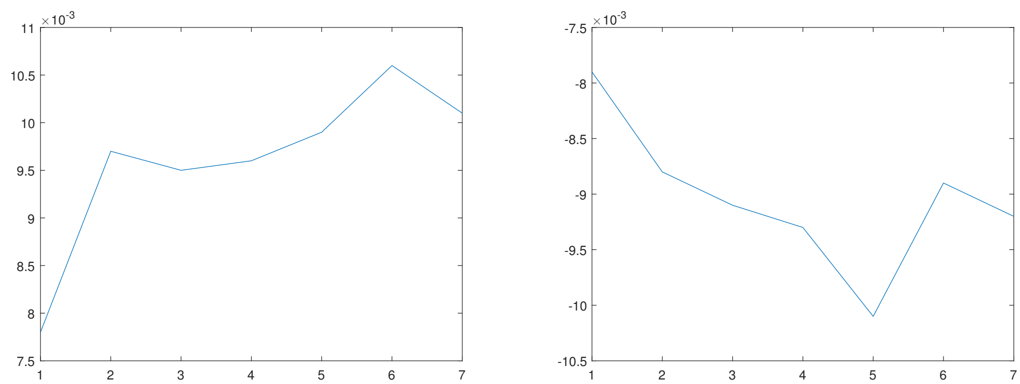

Figure 3: Expected daily returns for consecutive upward (left) and downward (right) paths.

7 Conclusion

In this paper, we explore the multi-period version of Markowitz's mean-variance portfolio selection model within a distinct framework of Markov regime-switching. Unlike traditional regime-switching models, we propose that asset returns are path-dependent, not solely reliant on the current state. Specifically, we assume that asset returns depend on the market states observed from the outset up to the decision point. As time progresses, investors observe more market states and base their decisions on all previously observed conditions. To address this problem, we employ the Lagrange duality method along with a dynamic programming approach. This path-dependent assumption significantly complicates the optimal portfolio selection problem, making it challenging to derive a closed-form solution. To overcome this, we introduce a path-dependent value function method to derive the closed forms of optimal portfolio strategies and the efficient frontier. Our findings reveal that optimal portfolios are path-dependent. Furthermore, we demonstrate that the results from the traditional model, where asset returns are only dependent on the current state, can be viewed as special cases of the outcomes presented in our model.

References

[1] N. Bauerle, U. Rieder, *Portfolio optimization with Markov-modulated stock prices and interest rates*, IEEE Trans. Automat. Control **49** (2004) 442–447.

[2] D.P. Bertsekas, *Dynamic Programming and Optimal Control*, Athena Scientific, Belmont, MA, USA, 2005.

[3] L. Bian, L. Zhang, *Equilibrium multi-period investment strategy for a DC pension plan with incomplete information: Hidden Markov model*, Comm. Statist. Theory Methods **54** (2025) 1702–1728.

[4] T.R. Bielecki, H. Jin, S.R. Pliska, X.Y. Zhou, *Continuous-time mean-variance portfolio selection with bankruptcy prohibition*, Math. Finance **15** (2005) 213–244.

[5] U. Cakmak, S. Ozekici, *Portfolio optimization in stochastic markets*, Math. Methods Oper. Res. **63** (2006) 151–168.

[6] T. Chen, R. Liu, Z. Wu, *Continuous-time mean-variance portfolio selection under non-Markovian regime-switching model with random horizon*, J. Syst. Sci. Complex. **36** (2023) 457–479.

[7] P. Chen, H. Yang, *Markowitz's mean-variance asset-liability management with regime switching: A multi-period model*, Appl. Math. Finance **18** (2011) 29–50.

[8] P. Chen, H. Yang, G. Yin, *Markowitz's mean-variance asset-liability management with regime switching: a continuous-time model*, Insurance Math. Econom. **43** (2008) 456–465.

[9] M.C. Chiu, D. Li, *Asset and liability management under a continuous-time mean-variance optimization framework*, Insurance Math. Econom. **39** (2006) 330–355.

[10] O.L.V. Costa, M.V. Araujo, *A generalized multi-period mean-variance portfolio optimization with Markov switching parameters*, Automatica **44** (2008) 2487–2497.

[11] X. Cui, J. Gao, X. Li, D. Li, *Optimal multi-period mean-variance policy under no-shorting constraint*, European J. Oper. Res. **234** (2014) 459–468.

[12] N. Dokuchaev, *Discrete time market with serial correlations and optimal myopic strategies*, European J. Oper. Res. **177** (2007) 1090–1104.

[13] H. Ge, X. Li, X. Li, Z. Li, *Equilibrium strategy for a multi-period weighted mean-variance portfolio selection in a Markov regime-switching market with uncertain time-horizon and a stochastic cash flow*, Comm. Statist. Theory Methods **52** (2023) 1797–1832.

[14] J. Gao, D. Li, *Multiperiod mean-variance portfolio optimization with general correlated returns*, IFAC Proceedings **47** (2014) 9007–9012.

[15] J. Gao, D. Li, X. Cui, S. Wang, *Time cardinality constrained mean-variance dynamic portfolio selection and market timing: a stochastic control approach*, Automatica **54** (2015) 91–99

[16] R. Keykhaei, *Mean-variance portfolio selection in a Markovian regime-switching market when the uncertain time horizon is a stopping time of market state filtration: a multi-period model*, J. Oper. Res. Soc. China, 2024. https://doi.org/10.1007/s40305-024-00559-8.

[17] M. Leippold, F. Trojani, P. Vanini, *A geometric approach to multiperiod mean variance optimization of assets and liabilities*, J. Econom. Dynam. Control **28** (2004) 1079–1113.

[18] D. Li, W.L. Ng, *Optimal dynamic portfolio selection: multiperiod mean-variance formulation*, Math. Finance **10** (2000) 387–406.

[19] X. Li, X.Y. Zhou, A.E.B. Lim, *Dynamic mean-variance portfolio selection with no-shorting constraints*, SIAM J. Control Optim. **40** (2002) 1540–1555.

[20] D.G. Luenberger, *Optimization by Vector Space Methods*, Wiley, New York, 1968.

[21] H. Markowitz, *Portfolio selection* J. Finance **7** (1952) 77–91.

[22] R.C. Merton, *An analytic derivation of the efficient portfolio*, J. Finan. Quant. Anal. **7** (1972) 1852–1872.

[23] H. Wu, H. Chen, *Nash equilibrium strategy for a multi-period mean-variance portfolio selection problem with regime switching*, Econ. Model. **46** (2015) 79–90.

[24] M.H. Wang, J. Yue, N.J. Huang, *Optimal R&D Investment Problem with Regime-Switching*, J. Optim. Theory Appl. **202** (2024) 878–896.

[25] B. Wu, L. Li, *Reinforcement learning for continuous-time mean-variance portfolio selection in a regime-switching market*, J. Econom. Dynam. Control **158** (2024) 104787.

[26] H. Wu, Y. Zeng, H. Yao, *Multi-period Markowitz's mean-variance portfolio selection with state-dependent exit probability*, Econ. Model. **36** (2014) 69–78.

[27] H. Xiao, Z. Zhou, T. Ren, Y. Bai, W. Liu, *Time-consistent strategies for multi-period mean-variance portfolio optimization with the serially correlated returns*, Comm. Statist. Theory Methods **49** (2020) 2831–2868.

[28] Y. Xu, Z.F. Li, *Dynamic portfolio selection based on serially correlated return-dynamic mean-variance formulation*, Syst. Eng. Theory Pract. **18** (2008) 123–131.

[29] H. Yao, D. Li, H. Wu, *Dynamic trading with uncertain exit time and transaction costs in a general Markov market*, Int. Rev. Financial Anal. **84** (2022) 102371.

[30] L. Zhang, Z. Li, *Multi-period mean-variance portfolio selection with uncertain time horizon when returns are serially correlated*, Math. Probl. Eng. **2012** (2012) 216891.

[31] L. Zhang, Z. Li, Y. Xu, Y. Li, *Multi-period mean variance portfolio selection under incomplete information*, Appl. Stoch. Models Bus. Ind. **32** (2016) 753–774.

[32] X.Y. Zhou, D. Li, *Continuous-time mean-variance portfolio selection: a stochastic LQ framework*, Appl. Math. Optim. **42** (2000) 19–33.

[33] X.Y. Zhou, G. Yin, *Markowitz's mean-variance portfolio selection with regime switching: a continuous-time model*, SIAM J. Control Optim. **42** (2003) 1466–1482.

[34] S.S. Zhu, D. Li, S.Y. Wang, *Risk control over bankruptcy in dynamic portfolio selection A generalized mean-variance formulation*, IEEE Trans. Automat. Control **49** (2004) 447–457.