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ON THE MAXIMAL SUBSEMIGROUPS AND RANK PROPERTIES OF CERTAIN SEMIGROUPS OF PARTIAL INJECTIVE CONTRACTIONS OF A FINITE CHAIN

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Abstract. Denote $[n]$ to be a finite n chain $\{1, 2, \ldots, n\}$. Let $\mathcal{CI}_n$ be the semigroup of partial injective contractions on $[n]$. Denote $\mathcal{ODDP}_n, \mathcal{ODCI}_n$ and $\mathcal{OCI}_n$ to be the semigroups of order-preserving order-decreasing partial isometries, order-preserving order-decreasing and order-preserving partial injective contractions, respectively. In this paper, we characterize all the maximal subsemigroups of $\mathcal{ODDP}_n$, $\mathcal{ODCI}_n$ and $\mathcal{OCI}_n$, respectively. We also characterize the regular elements, Green's relations, their Starred analogues and rank properties of $\mathcal{ODCI}_n$.

1. Introduction

Denote $[n]$ to be a finite chain $\{1, 2, \ldots, n\}$ and let $\mathcal{P}_n$ denote the semigroup of partial transformations of $[n]$, and let $\mathcal{I}_n$ be the subsemigroup of $\mathcal{P}_n$ of all injective partial transformation of $[n]$. A transformation $\alpha \in \mathcal{I}_n$ is said to be *order preserving* if (for all $x, y \in \text{Dom } \alpha$, where Dom α means the domain set of α here and elsewhere) $x \leq y$ implies $x\alpha \leq y\alpha$; *order decreasing* if (for all $x \in \text{Dom } \alpha$) $x\alpha \leq x$; an *isometry* (i.e., *distance preserving*) if (for all $x, y \in \text{Dom } \alpha$) $|x\alpha - y\alpha| = |x - y|$; a *contraction* if (for all $x, y \in \text{Dom } \alpha$,) $|x\alpha - y\alpha| \leq |x - y|$. Let

$$\mathcal{CI}_n = \{\alpha \in \mathcal{I}_n : (\text{for all } x, y \in \text{Dom } \alpha)\ \ |x\alpha - y\alpha| \leq |x - y|\}$$

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be the semigroup of partial injective contractions on $[n]$ and

$$\mathcal{DP}_n = \{\alpha \in \mathcal{I}_n : (\text{for all } x, y \in \text{Dom } \alpha) \ \ |x\alpha - y\alpha| = |x - y|\}$$

be the semigroup of partial injective isometries on $[n]$, as such $\mathcal{CI}_n$ and $\mathcal{DP}_n$ are subsemigroups of $\mathcal{I}_n$. Various algebraic and combinatorial properties of $\mathcal{CI}_n$ and $\mathcal{DP}_n$ have been investigated by several authors, for example see [2, 3, 27, 4, 20].

Let

$$\mathcal{DF}_n = \{\alpha \in \mathcal{P}_n : (\text{for all } x \in \text{Dom } \alpha,) \ x\alpha \leq x\},$$

$$\mathcal{CP}_n = \{\alpha \in \mathcal{P}_n : (\text{for all } x, y \in \text{Dom } \alpha) \ |x\alpha - y\alpha| \leq |x - y|\},$$

$$\mathcal{OCI}_n = \{\alpha \in \mathcal{CI}_n : (\text{for all } x, y \in \text{Dom } \alpha) \ x \leq y \text{ implies } x\alpha \leq y\alpha\}, \tag{1.1}$$

$$\mathcal{OI}_n = \{\alpha \in \mathcal{I}_n : (\text{for all } x, y \in \text{Dom } \alpha) \ x \leq y \text{ implies } x\alpha \leq y\alpha\}$$

$$\mathcal{ODI}_n = \{\alpha \in \mathcal{OI}_n : (\text{for all } x \in \text{Dom } \alpha) \ x\alpha \leq x\},$$

$$\mathcal{ODCI}_n = \{\alpha \in \mathcal{OCI}_n : (\text{for all } x \in \text{Dom } \alpha) \ x\alpha \leq x \ \}, \tag{1.2}$$

$$\mathcal{ODP}_n = \{\alpha \in \mathcal{DP}_n : (\text{for all } x, y \in \text{Dom } \alpha) \ x \leq y \text{ implies } x\alpha \leq y\alpha\}$$

and

$$\mathcal{ODDP}_n = \{\alpha \in \mathcal{ODP}_n : (\text{for all } x \in \text{Dom } \alpha) \ x\alpha \leq x \ \}$$

be the subsemigroup of *order decreasing partial transformations*; the subsemigroup of *partial contractions*; the subsemigroup of *order preserving partial injective transformations*; the subsemigroup of *order preserving order decreasing partial injective transformations*; the subsemigroup of *order preserving partial injective contractions*; the subsemigroup of *order preserving and order decreasing partial injective contractions*; the subsemigroup of *order preserving partial isometries* and the subsemigroup of *order preserving and order decreasing partial isometries* on $[n]$, respectively. Then it is obvious that $\mathcal{OCI}_n$ and $\mathcal{ODCI}_n$ are subsemigroups of $\mathcal{CI}_n$, while $\mathcal{ODP}_n$ and $\mathcal{ODDP}_n$ are subsemigroups of $\mathcal{DP}_n$.

For a semigroup S, a subsemigroup $T \subset S$ is called *maximal* provided that $T \neq S$ and for any subsemigroup $X \subset S$ the inclusion $T \subset X$ implies $T = X$, or $X = S$. Let S and P be two semigroups and $\alpha : S \mapsto P$ be any mapping. Denote $\alpha|_X$ to be the *restriction* of α to the subset $X \subseteq S$, that is $\alpha|_X : X \mapsto P$ is defined by $x(\alpha|_X) = x\alpha$ $(x \in X)$.

There are several subsemigroups of transformations whose maximal subsemigroups have been completely characterized, see for example

[5, 7, 22, 23, 29]. Yang [28] classified all the maximal inverse subsemigroups of $\mathcal{I}_n$ of all finite symmetric inverse semigroup on $[n]$. The maximal subsemigroups and maximal inverse subsemigroups of $\mathcal{OI}_n$ of all order-preserving partial bijections on $[n]$ were recorded in Ganyushkin and Marzorchuk [9], while Dimitrova and Koppitz [6] obtained the maximal subsemigroups of the ideals of $\mathcal{OI}_n$ and $\mathcal{IM}_n$ of all isotone (order preserving) partial injections and all monotone (order preserving or order reversing) partial injections on $[n]$.

The study of various semigroups of contractions mappings on chain was recently initiated, as such it is not surprising that the maximal subsemigroups of the semigroups $\mathcal{OCI}_n$ and its subsemigroups $\mathcal{ODCI}_n$ and $\mathcal{ODDP}_n$ are yet to be investigated. However, the rank of the semigroup $\mathcal{ODDP}_n$ have been investigated by Kehinde *et al.,*[20]. In [3], Al-Kharousi *et al.,* studied the rank of the larger semigroup $\mathcal{OCI}_n$. It seems nothing have been done on the study of rank of the semigroup $\mathcal{ODCI}_n$ and its maximal subsemigroups. In this paper, we classify all the maximal subsemigroups of the semigroups $\mathcal{OCI}_n$, $\mathcal{ODCI}_n$ and $\mathcal{ODDP}_n$; and study the rank of the semigroup $\mathcal{ODCI}_n$. We begin by recalling some definitions and notations that will be used in the subsequent sections. For basic concepts in semigroup theory, we will not failed to refer the reader to [17, 18, 10]. Let $\alpha \in \mathcal{I}_n$, we denote Dom α, Im α and $h(\alpha)$(referred to the *height* of α) to be the domain set of α, image set of α and $|\text{Im }\alpha|$, respectively. For $A \subset [n]$, $id_{[n]}$ denotes the partial identity on A, as such $id_{[n]}$ is the identity map on $[n]$. For a semigroup S and $A \subset S$, the notation $\langle A \rangle = S$ means that A generate the semigroup S.

Let $S \in \{\mathcal{OCI}_n,\ \mathcal{ODCI}_n,\ \mathcal{ODDP}_n\}$, then for $0 \leq k \leq n$, let

$$J_k := \{\alpha \in S \mid h(\alpha) = k\} \tag{1.3}$$

be the set of elements in S of height k and

$$I(\,n, k)\ := \{\alpha \in S\ :\ h(\alpha) \leq k\}$$

be the two sided ideal of S. Then it is obvious that $I(\,n, k)$ is the union of the $\mathcal{J}$-classes $J_0, J_1, \cdots, J_k$. We will pay attention to the $\mathcal{J}$-classes precisely to the sets J_{n-1} and J_n. Notice that the set J_n contains only the identity mapping $id_{[n]}$. Moreover, every element $\alpha \in S$ of height k can be expressed as

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix}. \tag{1.4}$$

Let us briefly discuss the presentation of the paper. In section 1, we give a brief introduction, notations and definitions for proper understanding of the content of the subsequent sections. In section 2, we classify all the maximal subsemigroups of the semigroup $\mathcal{OCI}_n$ and proved that it has $n+1$ maximal subsemigroups. Furthermore, in section 3, we proved that the rank of the semigroup $\mathcal{ODCI}_n$ is $2n$. Moreover, in this section we classify all its maximal subsemigroups and proved that it has $2n$ maximal subsemigroups. In section 4, we characterise the Green's relations and their Starred analogues on the semigroup $\mathcal{ODCI}_n$. Finally, in section 5, we classify all the maximal subsemigroups of the semigroup $\mathcal{ODDP}_n$ and proved that it has $n+2$ maximal subsemigroups.

Let S be a finite semigroup and M be a maximal subsemigroup of S. Then by [[16], Proposition 1], $S\backslash M$ is contained in a single $\mathcal{J}$-class of S. We note the following lemma which gives the necessary and sufficient conditions for the existence of a maximal subsemigroup arising from a particular $\mathcal{J}$-class.

Lemma 1.1. [[7], Lemma 2.1]. *Let S be a finite monoid, and let J be a $\mathcal{J}$-class of S. There exists a maximal subsemigroup arising from J if and only if every generating set for S intersects J non-trivially.*

2. Maximal Subsemigroups of $\mathcal{OCI}_n$

In this section, we characterize all the maximal subsemigroups of the semigroup $\mathcal{OCI}_n$. We also compute the total number of all the maximal subsemigroups of the semigroup $\mathcal{OCI}_n$. For $\alpha, \beta \in \mathcal{OCI}_n$ and elsewhere, the composition of α and β is defined as $x(\alpha \circ \beta) = ((x)\alpha)\beta$ for all $x \in \text{Dom } \alpha$. Without ambiguity, we shall be using the notation $\alpha\beta$ to denote the composition of α and β (i.e., $\alpha \circ \beta$). We begin with the following definition.

Definition 2.1. Let S be a semigroup and $a \in S$. We say that an element $a \in S$ is *non-factorizable* in S if there does not exists $b \neq c \in S\backslash\{a\}$ such that $a = bc$. An element $a \in S$ is said to be *generated* by some $a_1, \ldots, a_i \in S$ for some $i > 1$, if $a = a_1 a_2 \cdots a_i$.

Notice that there does not exists $\beta \neq \gamma \in \mathcal{OCI}_n \backslash \{id_{[n]}\}$ such that $id_{[n]} = \beta\gamma$, as such $id_{[n]}$ is non-factorizable in $\mathcal{OCI}_n$.

Definition 2.2. A subset A of J_k $(0 \leq k \leq n)$ is said to be a *generating set* of J_k, if $J_k \subseteq \langle A \rangle$ and if J_k is a semigroup, then $J_k = \langle A \rangle$.

Now, for $i \in [n]$ and $j \in \{2, 3, \ldots, n-1\}$, the elements of the set J_{n-1} in the semigroup $\mathcal{OCI}_n$ are of the following forms:

$$\begin{aligned}
\epsilon_i &= \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}, \\
\beta_j &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix}, \\
\gamma_j &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j & j+1 & \cdots & n \end{pmatrix}, \\
\alpha &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \\
\text{and } \alpha^{-1} &= \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix}.
\end{aligned} \tag{2.1}$$

Notice that ϵ_i is a partial identity on $[n] \setminus \{i\}$ (i.e., $\epsilon_i = id_{[n]\setminus\{i\}}$), and moreover, it is also easy to verify that:

$$\alpha\alpha^{-1} = \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} = \epsilon_1, \quad \alpha^{-1}\alpha = \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} = \epsilon_n;$$

$$\begin{aligned}
\beta_j\alpha^{-1} &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix} \\
&= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j & j+1 & \cdots & n \end{pmatrix} = \gamma_j;
\end{aligned}$$

$$\begin{aligned}
\gamma_j\alpha &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j & j+1 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j & j & \cdots & n-1 \end{pmatrix} = \beta_j;
\end{aligned}$$

$$\begin{aligned}
\alpha\epsilon_n &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha;
\end{aligned}$$

$$\begin{aligned}
\epsilon_1\alpha &= \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha;
\end{aligned}$$

$$\epsilon_n\alpha^{-1} = \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix} = \alpha^{-1};$$

$$\alpha^{-1}\epsilon_1 = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix} \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix} = \alpha^{-1};$$

$$\beta_j\epsilon_n = \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} = \beta_j$$

and

$$\gamma_j\epsilon_1 = \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j & j+1 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j & j+1 & \cdots & n \end{pmatrix} = \gamma_j.$$

Remark 2.3. It is now clear from the foregoing computations that:

(i) ϵ_1, ϵ_n can be generated by α and α^{-1}; γ_j can be generated by β_j and α^{-1}; β_j can be generated by γ_j and α; as such the elements ϵ_1, ϵ_n need not to be in any generating set of J_{n-1}. More so, β_j and γ_j need not to be both in the generating set of J_{n-1} and $h(\epsilon_i\lambda) < n-1$. Similarly, $h(\lambda\epsilon_i) < n-1$ for all $i \in \{2, \ldots, n-1\}$ and $\lambda \in J_{n-1}$. In particular, $\{\alpha,\ \alpha^{-1},\ \beta_j\}$ or $\{\alpha,\ \alpha^{-1},\ \gamma_j\}$ generates J_{n-1}.

(ii) $\alpha \neq \tau\theta$ and $\alpha^{-1} \neq \tau'\theta'$ for all $\tau \neq \theta$ and $\tau' \neq \theta' \in J_{n-1}$, as such $\alpha,\ \alpha^{-1} \in \mathcal{OCI}_n$ are non-factorizable elements.

Remark 2.4. It is known that $\{\epsilon_i, \alpha,\ \alpha^{-1},\ \beta_j :\ i, j \in \{2, \ldots, n-1\}\}$ generates the semigroup $\mathcal{OCI}_n$ (see [3], Lemma 4.3), and $\{\epsilon_i, \alpha,\ \alpha^{-1},\ \beta_j :\ i, j \in \{2, \ldots, n-1\}\} \cap J_{n-1}$ is non-trivial. Therefore by the Lemma 1.1, there exists a maximal subsemigroup arising from J_{n-1}.

Hence we have proved an enhance version of [[3] Lemma 4.3] in the following proposition.

Proposition 2.5. [[3], Lemma 4.3] *Let $S = \mathcal{OCI}_n$ and J_{n-1} be the elements of S of height $n-1$. Then*

(i) $J_{n-1} \subseteq \langle \epsilon_i, \alpha, \alpha^{-1}, \beta_j : i, j \in \{2, 3, ..., n-1\}\rangle$;
(ii) $J_{n-1} \subseteq \langle \epsilon_i, \alpha, \alpha^{-1}, \gamma_j : i, j \in \{2, 3, ..., n-1\}\rangle$.

Proof. The proof of (i) is as in [3], and (ii) is similar to the proof of (i). □

Now let $S = \mathcal{OCI}_n$ and J_k be as defined in equation (1.3). We now have the following proposition.

Proposition 2.6. *For* $0 \leq k \leq n-2$, $J_k \subseteq \langle J_{k+1}\rangle$.

Proof. Let $\alpha \in J_k$ be as expressed in equation (1.4). We will show that there exists $\beta, \gamma \in J_{k+1}$ such that $\alpha = \beta\gamma$. We consider the following cases for the elements in Dom $\alpha = \{a_1, \ldots, a_k\}$ and Im $\alpha = \{b_1, \ldots, b_k\}$.

Case 1. Suppose Im α is not convex, i.e., there exists $i \in \{1, \cdots, k-1\}$ such that $b_{i+1} - b_i \geq 2$ which implies $a_{i+1} - a_i \geq 2$. This means that there exists $v \in [n]\setminus$ Dom α and $u \in [n]\setminus$ Im α such that $a_i < v < a_{i+1}$ and $b_i < u < b_{i+1}$.

Now let $t = \max\{v \in [n]\setminus\text{Dom } \alpha : \ a_i < v < a_{i+1}\}$, $s = \max\{u \in [n]\setminus\text{Im } \alpha : \ b_i < u < b_{i+1}\}$, $s' = \max([n]\setminus\text{Im } \alpha)$ and $s'' = \min([n]\setminus\text{Im } \alpha)$.

Then we consider two subcases:

1.1. If $b_k < n$, define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b_1 & \cdots & b_i & s & b_{i+1} & \cdots & b_k \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} b_1 & \cdots & b_k & s' \\ b_1 & \cdots & b_k & s' \end{pmatrix}.$$

1.2. If $b_k = n$, define $\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b_1 & \cdots & b_i & s & b_{i+1} & \cdots & b_k \end{pmatrix}$ and

$$\gamma = \begin{cases} \begin{pmatrix} b_1 & \cdots & b_i & s'' & b_{i+1} & \cdots & b_k \\ b_1 & \cdots & b_i & s'' & b_{i+1} & \cdots & b_k \end{pmatrix}, \text{ if } b_i < s'' < b_{i+1} \text{ for some } i \in \{1, \ldots, k-1\} \\ \\ \begin{pmatrix} s'' & b_1 & \cdots & b_i & b_{i+1} & \cdots & b_k \\ s'' & b_1 & \cdots & b_i & b_{i+1} & \cdots & b_k \end{pmatrix}, \text{ if } s'' < b_1. \end{cases}$$

Case 2. Suppose that Im α is convex, i.e., Im $\alpha = \{b+1, \ldots, b+k\}$ for some $b \in \mathbb{Z}^+$. Thus, α can be expressed as

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_k \\ b+1 & \cdots & b+k \end{pmatrix} \in J_k.$$

If $a_i < t < a_{i+1}$ then $t \mapsto b+i$ and therefore, $\gamma(b+i)$ is not defined. If $b+k = n$ then take $t = \max([n]\setminus\text{Dom } \alpha)$. We consider the following subcases for the elements in Dom $\alpha = \{a_1, a_2, \ldots, a_k\}$ and Im $\alpha = \{b+1, \ldots, b+k\}$.

Subcase (1):

1.1. Suppose $i \in \{1, \ldots, k-1\}$ such that $a_i < t < a_{i+1}$. Define

$$\beta = \begin{pmatrix} a_1 & a_2 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b & b+1 & \cdots & b+i-1 & b+i & b+i+1 & \cdots & b+k \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} b-1 & b & \cdots & b+i-1 & b+i+1 & \cdots & b+k \\ b & b+1 & \cdots & b+i & b+i+1 & \cdots & b+k \end{pmatrix}.$$

1.2. If $t < a_1$, define $\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ b & b+1 & \cdots & b+k \end{pmatrix}$ and $\gamma = \begin{pmatrix} t & b+1 & \cdots & b+k \\ t & b+1 & \cdots & b+k \end{pmatrix}$.

1.3. If $t > a_k$, define

$$\beta = \begin{pmatrix} a_1 & a_2 & \cdots & a_k & t \\ b & b+1 & \cdots & b+k-1 & b+k \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} b-1 & b & \cdots & b+k-2 & b+k-1 \\ b & b+1 & \cdots & b+k-1 & b+k \end{pmatrix}.$$

Subcase (2): If $b+1 = 1$ then $b = 0$ as such $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ 1 & 2 & \cdots & k \end{pmatrix}$.

2.1. If $t > a_k$, define $\beta = \begin{pmatrix} a_1 & a_2 & \cdots & a_k & t \\ 1 & 2 & \cdots & k & k+1 \end{pmatrix}$ and

$\gamma = \begin{pmatrix} 1 & \cdots & k & k+2 \\ 1 & \cdots & k & k+2 \end{pmatrix}$.

2.2. If $t < a_1$, define $\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ 1 & 2 & \cdots & k+1 \end{pmatrix}$ and

$\gamma = \begin{pmatrix} 2 & 3 & \cdots & k+2 \\ 1 & 2 & \cdots & k+1 \end{pmatrix}$.

2.3. If $a_i < t < a_{i+1}$, for some $i \in \{1, \ldots, k-1\}$. Define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ 1 & \cdots & i & i+1 & i+2 & \cdots & k+1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} 1 & \cdots & i & i+2 & \cdots & k+2 \\ 1 & \cdots & i & i+1 & \cdots & k+1 \end{pmatrix}.$$

Subcase (3): If $1 < b+1 < \ldots < b+k < n$.

3.1. If $t > a_k$, define $\beta = \begin{pmatrix} a_1 & \cdots & a_k & t \\ b+1 & \cdots & b+k & b+k+1 \end{pmatrix}$ and

$$\gamma = \begin{pmatrix} b & b+1 & \cdots & b+k \\ b & b+1 & \cdots & b+k \end{pmatrix}.$$

3.2. If $t < a_1$, define $\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ b & b+1 & \cdots & b+k \end{pmatrix}$ and $\gamma = \begin{pmatrix} b+1 & \cdots & b+k+1 \\ b+1 & \cdots & b+k+1 \end{pmatrix}$.

3.3. If $a_i < t < a_{i+1}$ for some $i \in \{1, \ldots, k-1\}$. Define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b+1 & \cdots & b+i & b+i+1 & b+i+2 & \cdots & b+k+1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} b & b+1 & \cdots & b+i & b+i+2 & \cdots & b+k+2 \\ b & b+1 & \cdots & b+i & b+i+1 & \cdots & b+k+1 \end{pmatrix}.$$

□

Hence in each case, it is clear that $\beta, \gamma \in J_{k+1}$ and $\alpha = \beta\gamma$. As a consequence we have the following corollary.

Corollary 2.7. *For $0 \leq k \leq n-1$, $I(n,k) = \langle J_k \rangle$.*

Proof. Observe that the two sided ideal $I(n,k)$ is the union of $\mathcal{J}-$classes $J_0, J_1, \cdots, J_k$. Thus by Proposition 2.6, $I(n,k) = J_0 \cup J_1 \cup \ldots \cup J_k \subseteq \langle J_0 \rangle \cup \langle J_1 \rangle \cup \ldots \cup \langle J_k \rangle \subseteq \langle J_k \rangle$. This means that $I(n,k) \subseteq \langle J_k \rangle$ and since $J_k \subseteq I(n,k)$ then $\langle J_k \rangle \subseteq I(n,k)$. Hence $I(n,k) = \langle J_k \rangle$, as required. □

Theorem 2.8. *Let $\mathcal{OCI}_n$ be as defined in equation (1.1). Then $\mathcal{OCI}_n = \langle J_{n-1} \cup \{id_{[n]}\} \rangle$.*

Proof. The proof follows directly from Proposition 2.6 and Corollary 2.7. □

Now, we present the classification of the maximal subsemigroups of the semigroup $\mathcal{OCI}_n$.

Lemma 2.9. *Every maximal subsemigroup S of $\mathcal{OCI}_n$ contains the ideal $I(n, n-2)$.*

Proof. Let S be a maximal subsemigroup of $\mathcal{OCI}_n$. Suppose that $J_{n-1} \subseteq S$. Notice that by Corollary 2.7, $I(n, n-2) \subseteq I(n, n-1) \subseteq \langle J_{n-1} \rangle \subseteq S$, it follows that $I(n, n-2) \subseteq S$. Suppose $J_{n-1} \nsubseteq S$. Notice that $J_{n-1} \cap I(n, n-2) = \emptyset$, as such $J_{n-1} \nsubseteq S \cup I(n, n-2) \subseteq \langle S \cup I(n, n-2) \rangle$.

Thus by the maximality of S, it follows that $I(n, n-2) \subseteq S$, and since S is taken arbitrary, in either of the ways, the result follows. $\square$

Theorem 2.10. *Let $\alpha, \alpha^{-1}, \epsilon_i, \beta_j$ and γ_j be as defined in equation (2.1). A subsemigroup S of $\mathcal{OCI}_n$ is maximal if and only if S belongs to one of the following four types:*

(i) $S_{\{id_{[n]}\}} := \mathcal{OCI}_n \backslash \{id_{[n]}\}$;
(ii) $S_\alpha := \mathcal{OCI}_n \backslash \{\alpha\}$;
(iii) $S_{\alpha^{-1}} := \mathcal{OCI}_n \backslash \{\alpha^{-1}\}$;
(iv) $S_{\epsilon_i} := \mathcal{OCI}_n \backslash \{\epsilon_i\},\ i \in \{2, \dots, n-1\}$.

Proof. Suppose S is a maximal subsemigroup of $\mathcal{OCI}_n$. Then by Lemma 2.9, $S = I(n, n-2) \cup K$, where $K \subseteq (J_{n-1} \cup \{id_{[n]}\})$.

If $\{id_{[n]}\} \nsubseteq K$ then $K \subseteq \mathcal{OCI}_n \backslash \{id_{[n]}\}$. Thus, $S = I(n, n-2) \cup K \subseteq I(n, n-2) \cup \mathcal{OCI}_n \backslash \{id_{[n]}\}$. Since S is maximal then $S = \mathcal{OCI}_n \backslash \{id_{[n]}\} = S_{\{id_{[n]}\}}$. As such (i) follows.

Now, if $\{id_{[n]}\} \subseteq K$, then $J_{n-1} \nsubseteq K$ (otherwise $S = \mathcal{OCI}_n$). Then by Proposition 2.5, $J_{n-1} \subseteq \langle \epsilon_i, \alpha, \alpha^{-1}, \beta_j \rangle$ or $J_{n-1} \subseteq \langle \epsilon_i, \alpha, \alpha^{-1}, \gamma_j \rangle$ (for $i, j \in \{2, \dots, n-1\}$). However, $\gamma_j = \beta_j \alpha^{-1}$ and $\beta_j = \gamma_j \alpha$ as such the set K is contained in $J_{n-1} \backslash \{\tau\}$ for some $\tau \in \{\epsilon_i, \alpha, \alpha^{-1}\}$. Therefore, by maximality of S, $S = S_\tau$ and hence $(ii) - (iv)$ follows.

Conversely, suppose $(i) - (iv)$ holds. Notice that $\mathcal{OCI}_n \backslash \{id_{[n]}\} = I(n, n-1)$ and $I(n, n-1) \cup \{id_{[n]}\} = \mathcal{OCI}_n$, hence $S_{\{id_{[n]}\}}$ is maximal. Notice that by Remark 2.3, for all $x \in \{\alpha,\ \alpha^{-1},\ \epsilon_i,\ \beta_j\}$, $x \notin J_{n-1} \backslash \langle \alpha,\ \alpha^{-1},\ \epsilon_i,\ \beta_j \rangle$ and similarly, for all $y \in \{\alpha,\ \alpha^{-1},\ \epsilon_i,\ \gamma_j\}$, $y \notin J_{n-1} \backslash \langle \alpha,\ \alpha^{-1},\ \epsilon_i,\ \gamma_j \rangle$, $i, j \in \{2, \dots, n-1\}$. Notice also that β_j and γ_j cannot be both in the generating set of J_{n-1} and can also be generated. Therefore, since $\mathcal{OCI}_n \backslash \{\epsilon_i\} \cup \{\epsilon_i\} = \mathcal{OCI}_n$, $\mathcal{OCI}_n \backslash \{\alpha\} \cup \{\alpha\} = \mathcal{OCI}_n$ and $\mathcal{OCI}_n \backslash \{\alpha^{-1}\} \cup \{\alpha^{-1}\} = \mathcal{OCI}_n$. It follows that for all $i \in \{2, \dots, n-1\}$, S_{ϵ_i}, S_α and $S_{\alpha^{-1}}$ are maximal. $\square$

The next corollary gives us the total number of maximal subsemigroups of the semigroup $\mathcal{OCI}_n$.

Corollary 2.11. *The semigroup $\mathcal{OCI}_n$ contains exactly $n+1$ maximal subsemigroups.*

Proof. Notice from Theorem 2.10, the semigroups $S_{\{id_{[n]}\}}$, S_α, $S_{\alpha^{-1}}$ and S_{ϵ_i} have 1, 1, 1 and $n-2$ maximal subsemigroups, respectively. Summing all together gives the required result. $\square$

For the definitions of Green's and Starred Green's relations, we refer the reader to Howie [18] and Foutain [8], respectively. A subsemigroup K of a semigroup S is called an *inverse ideal* of S if for all $k \in K$, there exists $k' \in S$ such that $kk'k = k$ and $kk', k'k \in K$.

An element a of a semigroup S is called an *idempotent* if $a^2 = a$, the collection of all idempotent in S is usually denoted by $E(S)$. A semigroup S is called *abundant* if each $\mathcal{L}^*(\mathcal{S}$-class and each $\mathcal{R}^*(\mathcal{S}$-class contains an idempotent. An abundant semigroup in which the set $E(S)$ is a semilattice is called *adequate* [8]. AL-Kharousi *et al.,* deduced that the semigroup $\mathcal{OCI}_n$ is an adequate semigroup see [3]. As in [8] for an element a of an adequate semigroup S, the (unique) idempotent in the $\mathcal{L}^*(\mathcal{S})$-class containing a is denoted by a^* and the (unique) idempotent in the $\mathcal{R}^*(\mathcal{S})$-class containing a is denoted by a^+. An adequate semigroup S is said to be *ample* if $ea = a(ea)^*$ and $ae = (ae)^+a$ for all a in S and all idempotents e in S. An ample semigroup is also called *type A* semigroup. Al-Kharousi *et al.,* [3] characterized the Green's and Starred Green's relations of the semigroup $\mathcal{OCI}_n$ and also proved that $\mathcal{OCI}_n$ is an ample semigroup.

We next give the characterization of starred Green's relations of the maximal subsemigroups of the semigroup $\mathcal{OCI}_n$. Before we give the characterization we first record the following lemma whose proof is similar to that of [[3], Lemma 1.4].

Lemma 2.12. *Let $S \in \{S_{id_{[n]}}, S_\alpha, S_{\alpha^{-1}}\}$ and $\alpha, \beta \in S$. Then S is an inverse ideal of $\mathcal{OI}_n$.*

Theorem 2.13. *Let $S \in \{S_{id_{[n]}}, S_\alpha, S_{\alpha^{-1}}\}$ and $\alpha, \beta \in S$. Then*

(i) $(\alpha, \beta) \in \mathcal{L}^*(\mathcal{S})$ *if and only if* $\text{Im } \alpha = \text{Im } \beta$*;*
(ii) $(\alpha, \beta) \in \mathcal{R}^*(\mathcal{S})$ *if and only if* $\text{Dom } \alpha = \text{Dom } \beta$*.*

Proof. Notice that by Lemma 2.12 S is an inverse ideal, hence by [[25], Lemma 3.1.9] the results follow. □

Remark 2.14. Notice that the semigroup S_{ϵ_i}, $i \in \{2, \ldots n-1\}\}$ is not an inverse ideal of $\mathcal{OI}_n$. This means that the proof of $\mathcal{L}^*(\mathrm{S}_{\epsilon_\mathrm{i}})$ and $\mathcal{R}^*(\mathrm{S}_{\epsilon_\mathrm{i}})$ are different from that of Theorem 2.13. Therefore, we state the characterization of $\mathcal{L}^*(\mathrm{S}_{\epsilon_\mathrm{i}})$ and $\mathcal{R}^*(\mathrm{S}_{\epsilon_\mathrm{i}})$ below. The proof is similar to that of [[26], Theorem 1].

Theorem 2.15. *Let $S = S_{\epsilon_i}$, $i \in \{2, \ldots n-1\}$ and $\alpha, \beta \in S$. Then*

(i) $(\alpha, \beta) \in \mathcal{L}^*(\mathcal{S})$ *if and only if* $\text{Im } \alpha = \text{Im } \beta$*;*
(ii) $(\alpha, \beta) \in \mathcal{R}^*(\mathcal{S})$ *if and only if* $\text{Dom } \alpha = \text{Dom } \beta$*.*

Remark 2.16. Notice that for $\epsilon_i \in \mathcal{OCI}_n$, $i \in \{2, \ldots, n-1\}$, $\text{Im } \epsilon_i \neq \text{Im } \alpha$ for all $\alpha \in \mathcal{OCI}_n$.

We now record the following results whose proof is similar to that of [[26], Lemma 8].

Theorem 2.17. *The semigroup* $S_{\epsilon_i} := \mathcal{OCI}_n \backslash \{\epsilon_i\},\ i \in \{2, \ldots, n-1\}$ *is left abundant.*

Proposition 2.18. *The semigroup* $S_{\epsilon_i} := \mathcal{OCI}_n \backslash \{\epsilon_i\},\ i \in \{2, \ldots, n-1\}$ *is not right abundant for* $n \geq 3$.

Proof. Let $\epsilon_i = \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}$, $i \in \{2, \ldots, n-1\}$. Notice that $\epsilon_i \notin S_{\epsilon_i}$, for $n \geq 3$. Notice also that

$$R^*_{\epsilon_i} = \left\{ \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1+x & \cdots & i-1+x & i+x & \cdots & n-1+x \end{pmatrix} \Bigg|_{0 \leq x \leq 1} \right\}$$ has no idempotent. □

As a consequence we have the following corollary.

Corollary 2.19. *The semigroup* $S_{\epsilon_i} := \mathcal{OCI}_n \backslash \{\epsilon_i\},\ i \in \{2, \ldots, n-1\}$ *is not ample.*

Proof. It follows from Proposition 2.18. □

We now state the following theorem whose proof is similar to that of [[3], Lemma 3.8].

Theorem 2.20. *Let* $S \in \{S_{id_{[n]}}, S_\alpha, S_{\alpha^{-1}}, S_{\beta_j}, S_{\gamma_j}, j \in \{2, \ldots n-1\}\}$. *Then* S *is an ample semigroup.*

Consequently we have the following corollary.

Corollary 2.21. *Let* $S \in \{S_{id_{[n]}}, S_\alpha, S_{\alpha^{-1}}, S_{\beta_j}, S_{\gamma_j}\},\ j \in \{2, \ldots n-1\}$. *Then* S *is ample maximal subsemigroup of* $\mathcal{OCI}_n$.

3. Rank properties of the subsemigroups $\mathcal{ODCI}_n$

For a semigroup S, the *rank* of S (denoted as Rank(S)) is defined as

$$\text{Rank}(S) = \min\{|A| : A \subseteq S \text{ and } \langle A \rangle = S\}.$$

The notation $\langle A \rangle = S$ means that A generate the semigroup S. The rank of several semigroups of transformation were investigated by various authors, see for example [1, 2, 11, 12, 13, 14, 19]. However, there are several subsemigroups of partial contractions which their ranks seems not to have been investigated. In fact the order and the rank of the semigroup $\mathcal{CP}_n$ is still under investigation. In this section, we investigate the rank properties of the semigroup $\mathcal{ODCI}_n$. We begin this section by noting that $id_{[n]}$ is the only order-decreasing map of height

n in $\mathcal{ODCI}_n$ as such $id_{[n]}$ is non-factorizable. Now consider the elements of height $n-1$ in the semigroup $\mathcal{ODCI}_n$ (i.e., elements of J_{n-1}). These elements are of the following forms:

For $i \in \{1, \ldots, n\}$ and $j \in \{2, \ldots, n-1\}$,

$$\begin{aligned} \epsilon_i &= \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}, \\ \beta_j &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} \\ \text{and } \alpha &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix}. \end{aligned} \tag{3.1}$$

$$\text{Therefore, } J_{n-1} = \{\epsilon_i, \beta_j, \alpha \mid i \in \{1, \ldots, n\},\ j \in \{2, \ldots, n-1\}\}. \tag{3.2}$$

Thus, for $j \in \{2, \ldots, n-1\}$, it is worth noting that:

$$\begin{aligned} \beta_j \epsilon_n &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} = \beta_j, \end{aligned}$$

$$\begin{aligned} \epsilon_1 \alpha &= \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha \end{aligned}$$

and

$$\begin{aligned} \alpha \epsilon_n &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha. \end{aligned}$$

Notice that from the foregoing computations $\alpha, \epsilon_1, \epsilon_n$ and β_j cannot be generated by the elements of height $n-1$ in $\mathcal{ODCI}_n$. As such $\alpha, \epsilon_1, \epsilon_n$ and β_j must be part of the generating set of J_{n-1}.

We now have the following lemma.

Lemma 3.1. *Let $\alpha \in \mathcal{ODCI}_n$ be as expressed in equation (1.4). If $b_k = a_k$ then α is a partial identity.*

Proof. Let $\alpha \in \mathcal{ODCI}_n$ be as expressed in equation (1.4) and suppose that $a_k = b_k$. Notice that α is order decreasing as such $b_r \leq a_r$ for all $r \in \{1, \ldots, k-1\}$.

Suppose that $b_r < a_r$, for any $r \in \{1, \ldots, k-1\}$. Since α is order-preserving, thus $a_r < a_k = b_k$, therefore $b_r < a_r < b_k = a_k$. Thus,

$|b_k - b_r| > |b_k - a_r| = |a_k - a_r|$, contradicting the fact that α is a contraction. As such $b_r = a_r$ for all $r \in \{1, \ldots, k-1\}$. □

Corollary 3.2. *Let $\alpha \in \mathcal{ODCI}_n$ be as expressed in equation (1.4). If $b_k = n$ then α is a partial identity.*

Proof. Let $\alpha \in \mathcal{ODCI}_n$ be as expressed in equation (1.4) and suppose that $b_k = n$. Since α is order decreasing, then $b_k \leq a_k$, but since $b_k = n$ then $a_k = b_k$ and by Lemma 3.1, α is a partial identity. □

We now have the following lemma.

Lemma 3.3. *For $0 \leq k \leq n-2$, $J_k \subseteq \langle J_{k+1} \rangle$.*

Proof. Let $\alpha \in J_k$ be as expressed in equation (1.3). Since α is decreasing then $b_k \leq a_k$. Thus, we consider the following two cases:

Case 1. Suppose $b_k = a_k$. Then by Lemma 3.1, α is a partial identity and as such $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}$.

Now, let $t = \max([n]\backslash \text{Dom } \alpha)$ and $s = \min([n]\backslash \text{Dom } \alpha)$.

1.1. If $t < a_1$, then define β and γ, respectively as:

$$\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ t & a_1 & \cdots & a_k \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} s & a_1 & \cdots & a_k \\ s & a_1 & \cdots & a_k \end{pmatrix}.$$

1.2. If $t > a_k$, then define β and γ as: $\beta = \begin{pmatrix} a_1 & \cdots & a_k & t \\ a_1 & \cdots & a_k & t \end{pmatrix}$ and

$$\gamma = \begin{cases} \begin{pmatrix} a_1 & \cdots & a_k & s \\ a_1 & \cdots & a_k & s \end{pmatrix}, \text{ if } s > a_k \\[2ex] \begin{pmatrix} a_1 & \cdots & a_j & s & a_{j+1} & \cdots & a_k \\ a_1 & \cdots & a_j & s & a_{j+1} & \cdots & a_k \end{pmatrix}, \\ \text{if } a_j < s < a_{j+1} \text{ and } 1 \leq j \leq k-1 \\[2ex] \begin{pmatrix} s & a_1 & \cdots & a_k \\ s & a_1 & \cdots & a_k \end{pmatrix}, \text{ if } s < a_1. \end{cases}$$

1.3. If $a_i < t < a_{i+1}$ for some $i \in \{1, \ldots, k-1\}$, then define β and γ as:

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & \cdots & a_{i+1} & a_k \\ a_1 & \cdots & a_i & t & \cdots & a_{i+1} & a_k \end{pmatrix} \text{ and}$$

$$\gamma = \begin{cases} \begin{pmatrix} a_1 & \cdots & a_i & s & a_{i+1} & \cdots & a_k \\ a_1 & \cdots & a_i & s & a_{i+1} & \cdots & a_k \end{pmatrix}, \text{ if } s > a_i \\ \\ \begin{pmatrix} a_1 & \cdots & a_j & s & a_{j+1} & \cdots & a_i & a_{i+1} & \cdots & a_k \\ a_1 & \cdots & a_j & s & a_{j+1} & \cdots & a_i & a_{i+1} & \cdots & a_k \end{pmatrix}, \\ \text{if } a_j < s < a_{j+1} \text{ for some } j \in \{1, \ldots, i-1\} \\ \\ \begin{pmatrix} s & a_1 & \cdots & a_k \\ s & a_1 & \cdots & a_k \end{pmatrix}, \text{ if } s < a_1. \end{cases}$$

Case 2. If $b_k \leq a_k \leq n$. Then either Im α is convex or Im α is not convex.

Subcase (1): Suppose that Im α is convex, i.e., there exists $b \in \mathbb{Z}^+$ such that Im $\alpha = \{b+1, \ldots, b+k\}$. Thus, since α is decreasing then $b + k \leq a_k$ an as such $1 \leq b+1 \leq \cdots \leq b+k \leq a_k \leq n$. Now there are two cases to consider:

1.1. $1 < b+1 < \cdots < b+k < a_k < n$, i. e., $b+1 = 1$ which implies that $b = 0$, as such

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ 1 & 2 & \cdots & k \end{pmatrix}.$$

1.1.1. If $t > a_k$, define

$$\beta = \begin{pmatrix} a_1 & a_2 & \cdots & a_k & t \\ 1 & 2 & \cdots & k & k+1 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 1 & \cdots & k & k+2 \\ 1 & \cdots & k & k+2 \end{pmatrix}.$$

1.1.2. If $t < a_1$, define

$$\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ 1 & 2 & \cdots & k+1 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 2 & 3 & \cdots & k+1 & k+2 \\ 1 & 2 & \cdots & k & k+1 \end{pmatrix}.$$

1.1.3. If $a_i < t < a_{i+1}$, for some $i \in \{1, \ldots, k-1\}$. Define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ 1 & \cdots & i & i+1 & i+2 & \cdots & k+1 \end{pmatrix} \text{ and}$$

$$\gamma = \begin{pmatrix} 1 & \cdots & i & i+2 & \cdots & k & k+1 & k+2 \\ 1 & \cdots & i & i+2 & \cdots & k-1 & k & k+1 \end{pmatrix}.$$

1.2. If $1 < b+1 < \ldots < b+k < n$.

1.2.1. If $t > a_k$, define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_k & t \\ b+1 & \cdots & b+k & b+k+1 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} b & b+1 & \cdots & b+k \\ b & b+1 & \cdots & b+k \end{pmatrix}.$$

1.2.2. If $t < a_1$, define

$$\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ b & b+1 & \cdots & b+k \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} b+1 & b+2 & \cdots & b+k+1 \\ b & b+2 & \cdots & b+k+1 \end{pmatrix}.$$

1.2.3. If $a_i < t < a_{i+1}$ for some $i \in \{1, \dots, k-1\}$. Define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b+1 & \cdots & b+i & b+i+1 & b+i+2 & \cdots & b+k+1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} b & b+1 & \cdots & b+i & b+i+2 & \cdots & b+k+2 \\ b & b+1 & \cdots & b+i & b+i+1 & \cdots & b+k+1 \end{pmatrix}.$$

Subcase (2): Suppose that Im α is not convex, i.e., there exists $i \in \{1, \cdots, k-1\}$ such that $b_{i+1} - b_i \geq 2$ which implies $a_{i+1} - a_i \geq 2$. This means that there exists $v \in [n] \backslash$ Dom α and $u \in [n] \backslash$ Im α such that $a_i < v < a_{i+1}$ and $b_i < u < b_{i+1}$.

Now, let $t = \max\{v \in [n] \backslash \text{Dom } \alpha : \ a_i < v < a_{i+1}\}$, $s = \max\{u \in [n] \backslash \text{Im } \alpha : \ b_i < u < b_{i+1}\}$, $s' = \max([n] \backslash \text{Im } \alpha)$ and $s'' = \min([n] \backslash \text{Im } \alpha)$.

Then we consider two subcases:

2.1. If $b_k < n$, define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b_1 & \cdots & b_i & s & b_{i+1} & \cdots & b_k \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} b_1 & \cdots & b_k & s' \\ b_1 & \cdots & b_k & s' \end{pmatrix}.$$

2.2. If $b_k = n$, then by Corollary 3.2, α is a partial identity. As such the results follows from case 1.

□

Hence in each case, it is clear that $\beta, \gamma \in J_{k+1}$ and $\alpha = \beta\gamma$. We now have the following lemma.

Lemma 3.4. *For* $0 \leq k \leq n-1, \ J_k \subseteq \langle J_{n-1} \rangle$.

Proof. Suppose $0 \leq k \leq n-1$, then by Lemma 3.3, we see that $J_k \subseteq \langle J_{k+1} \rangle$ and similarly, $J_{k+1} \subseteq \langle J_{k+2} \rangle$ i.e., $\langle J_{k+1} \rangle \subseteq \langle J_{k+2} \rangle$ which means that $\langle J_k \rangle \subseteq \langle J_{k+1} \rangle \subseteq \langle J_{k+2} \rangle$. Therefore, $J_k \subseteq \langle J_{k+1} \rangle \subseteq \langle J_{k+2} \rangle$. If we continue in this fashion we see that $\langle J_k \rangle \subseteq \langle J_{k+1} \rangle \subseteq \langle J_{k+2} \rangle \subseteq \ldots \subseteq \langle J_{n-2} \rangle \subseteq \langle J_{n-1} \rangle$ and therefore, $J_k \subseteq \langle J_{n-1} \rangle$ as required. □

As a consequence we have the following corollary.

Corollary 3.5. $\langle J_{n-1} \rangle = I(n, n-1)$.

Remark 3.6. It is obvious that the generating set of J_{n-1} is non-trivial. Therefore by the Lemma 1.1, there exists a maximal subsemigroup arising from J_{n-1}.

We now have the following theorem.

Theorem 3.7. *Let* $\mathcal{ODCI}_n$ *be as defined in equation* (1.2) *and* $I(n, n-1)$ *be two sided ideal of* $\mathcal{ODCI}_n$. *Then, the* rank $(I(n, n-1)) = 2n-1$.

Proof. Clearly, from Corollary 3.5, $\langle J_{n-1}\rangle = I(n,n-1)$. So it is enough to show that J_{n-1} is the minimum generating set of $I(n,n-1)$. i.e., $\langle J_{n-1}\rangle = I(n,n-1)$ and $\langle J_{n-1}\backslash\{\omega\}\rangle \neq I(n,n-1)$ for all $\omega \in J_{n-1}$. Notice that $J_{n-1} = \{\epsilon_i, \alpha, \beta_j : 1 \leq i \leq n,\ 2 \leq j \leq n-1\}$. Notice also that the $h(\epsilon_k\epsilon_m) < n-1$ for all $k, m \in \{1,\ldots,n\}$, moreover, $h(\epsilon_i\beta_j) < n-1$ for all $i \in \{1,\ldots,n\}$ and $j \in \{2,\ldots,n-2\}$, $\beta_j\epsilon_n = \beta_j$, $h(\beta_j\epsilon_i) < n-1$ $i \in \{1,\ldots,n-1\}$ and $h(\beta_j\beta_j) < n-1$ for $i \in \{2,\ldots,n-1\}$. This means that ϵ_i and β_j do not generates α; as such $\langle J_{n-1}\backslash\{\alpha\}\rangle \neq I(n,n-1)$.

Similarly, for all $i \neq 1$ and $i \neq n$ in $\{1,\ldots,n\}$ the $h(\epsilon_i\alpha) < n-1$ and $\epsilon_1\alpha = \alpha$; $h(\alpha\epsilon_i) < n-1$ and $\alpha\epsilon_n = \alpha$; $h(\alpha\alpha) < n-1$. This means that ϵ_i and α do not generates β_j for $j \in \{2,\ldots,n-1\}$. Hence $\langle J_{n-1}\backslash\{\beta_j\}\rangle \neq I(n,n-1)$.

Furthermore, for all $j \in \{2,\ldots,n-1\}$, the $h(\beta_j\alpha) < n-1$, $h(\alpha\beta_j) < n-1$, $h(\alpha\alpha) < n-1$ and $h(\beta_j\beta_j) < n-1$. This means that β_j and α do not generates ϵ_i for all $i \in \{1,\ldots,n\}$. Hence $\langle J_{n-1}\backslash\{\epsilon_i\}\rangle \neq \langle J_{n-1}\rangle$. Thus, J_{n-1} is the minimum generating set of $I(n,n-1)$ and as such, the rank$(I(n,n-1)) = |J_{n-1}| = 2n-1$. □

As a consequence we readily have the following result.

Theorem 3.8. rank$(\mathcal{ODCI}_n) = 2n$.

Proof. Notice that $\mathcal{ODCI}_n = I(n,n-1) \cup \{id_{[n]}\}$, as such rank$(\mathcal{ODCI}_n) =$ rank$(I(n,n-1)) + 1$. The fact that $\mathcal{ODCI}_n$ is a $\mathcal{J}-$trivial semigroup, the results follows directly from Theorem 3.7. □

4. Maximal Subsemigroups of $\mathcal{ODCI}_n$

In this section we present the classification of the maximal subsemigroups of the semigroup $\mathcal{ODCI}_n$. We begin with the following lemma.

Lemma 4.1. *Every maximal subsemigroup S of $\mathcal{ODCI}_n$ contains the ideal $I(\,n, n-2\,)$.*

Proof. The proof is similar to that of Lemma 2.9.

□

We are now going to present the main results of this section, the classification of the maximal subsemigroups of the semigroup $\mathcal{ODCI}_n$.

Theorem 4.2. *Let α, ϵ_i and β_j be as expressed in equation (3.1). A subsemigroup S of $\mathcal{ODCI}_n$ is maximal if and only if it belongs to one of the following four types:*

(i) $S_{\{id_{[n]}\}} := \mathcal{ODCI}_n\backslash\{id_{[n]}\}$;
(ii) $S_\alpha := \mathcal{ODCI}_n\backslash\{\alpha\}$;

(iii) $S_{\epsilon_i} := \mathcal{ODCI}_n\backslash\{\epsilon_i\},\ i \in \{1, \ldots, n\}$;
(iv) $S_{\beta_j} := \mathcal{ODCI}_n\backslash\{\beta_j\},\ j \in \{2, \ldots, n-1\}$.

Proof. Suppose S is a maximal subsemigroup of $\mathcal{ODCI}_n$. Then by Lemma 4.1, $S = I(n, n-2) \cup K$, where $K \subseteq (J_{n-1} \cup \{id_{[n]}\})$.

If $\{id_{[n]}\} \nsubseteq K$, then $K \subseteq \mathcal{ODCI}_n\backslash\{id_{[n]}\}$. Thus $S = I(n, n-2) \cup K \subseteq I(n, n-2) \cup \mathcal{ODCI}_n\backslash\{id_{[n]}\}$. Since S is maximal then $S = \mathcal{ODCI}_n\backslash\{id_{[n]}\} = S_{\{id_{[n]}\}}$, as such (i) follows.

Now, if $\{id_{[n]}\} \subseteq K$, then $J_{n-1} \nsubseteq K$ (otherwise $S = \mathcal{ODCI}_n$). Since by the proof of Theorem 3.7, the set $\{\epsilon_i,\ \alpha,\ \gamma_j : i \in \{1, \ldots, n\}$ and $j \in \{2, \ldots, n-1\}\}$ is the minimum generating set for J_{n-1}, then the set K is contain in $J_{n-1}\backslash\{\tau\}$ for some $\tau \in \{\epsilon_i, \alpha, \beta_j\}$. Therefore, by maximality of S, $S = S_\tau$ for some $\tau \in \{\epsilon_i, \alpha, \beta_j, \}$ and hence $(ii) - (iv)$ follows.

Conversely, suppose $(i) - (iv)$ holds. Notice that $\mathcal{ODCI}_n\backslash\{id_{[n]}\} = I(n, n-1)$ and $I(n, n-1) \cup \{id_{[n]}\} = \mathcal{ODCI}_n$, hence $S_{\{id_{[n]}\}}$ is maximal. Notice also that from the proof of Theorem 3.7, the elements ϵ_i, α and β_j for all $i \in \{1, \ldots, n\}$ and $j \in \{2, \ldots, n-1\}$ cannot be generated by the elements of the set J_{n-1}.Therefore, since $\mathcal{OCI}_n\backslash\{\epsilon_i\} \cup \{\epsilon_i\} = \mathcal{OCI}_n$, $\mathcal{OCI}_n\backslash\{\alpha\} \cup \{\alpha\} = \mathcal{OCI}_n$, and $\mathcal{OCI}_n\backslash\{\beta_j\} \cup \{\beta_j\} = \mathcal{OCI}_n$. It follows that for all $i \in \{1, \ldots, n\}$, and $j \in \{2, \ldots, n\}$ S_{ϵ_i}, S_α and S_{β_j} are maximal. □

The next corollary gives us the total number of the maximal subsemigroups of the semigroup $\mathcal{ODCI}_n$.

Corollary 4.3. *The semigroup $\mathcal{ODCI}_n$ contains exactly $2n$ maximal subsemigroups*

Proof. The results follows by counting the number of maximal subsemigroups (i) to (iv) in Theorem 4.2. □

5. Green's and Starred Green's relations

Using Lemma 5.1, one can see that the semigroup $\mathcal{ODI}_n$ is the intersection of the semigroups $\mathcal{I}_n$, $\mathcal{OI}_n$ and $\mathcal{PF}_n$, respectively. Therefore, the characterization of Green's relations on the semigroup $\mathcal{ODI}_n$ is the intersection of the corresponding Green's, relations on $\mathcal{I}_n$, $\mathcal{OI}_n$ and $\mathcal{PF}_n$ (for the characterization of Green's relations on the semigroups $\mathcal{I}_n$, $\mathcal{OI}_n$ and $\mathcal{PF}_n$ see [25, 9, 10]).

We now record the following lemma.

Lemma 5.1 ([10], Pp 252)**.** *The semigroup $\mathcal{ODI}_n = \mathcal{I}_n \cap \mathcal{OI}_n \cap \mathcal{PF}_n$.*

Theorem 5.2. *Let $\alpha, \beta \in \mathcal{ODI}_n$. Then*

(i) $(\alpha, \beta) \in \mathcal{L}(\mathcal{ODI}_n)$ *if and only if* $\alpha = \beta$;
(ii) $(\alpha, \beta) \in \mathcal{R}(\mathcal{ODI}_n)$ *if and only if* $\alpha = \beta$;
(iii) $(\alpha, \beta) \in \mathcal{D}(\mathcal{ODI}_n)$ *if and only if* $\alpha = \beta$;
(iv) $(\alpha, \beta) \in \mathcal{H}(\mathcal{ODI}_n)$ *if and only if* $\alpha = \beta$.

For $1 \leq m, r, k \leq n$ denote:

$$\begin{aligned} \alpha &= \begin{pmatrix} a_1 & \cdots & a_m \\ a_1\alpha & \cdots & a_m\alpha \end{pmatrix}, \beta = \begin{pmatrix} b_1 & \cdots & b_m \\ b_1\beta & \cdots & b_m\beta \end{pmatrix}, \\ \sigma &= \begin{pmatrix} c_1 & \cdots & c_r \\ c_1\sigma & \cdots & c_r\sigma \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} d_1 & \cdots & d_k \\ d_1\gamma & \cdots & d_k\gamma \end{pmatrix}. \end{aligned} \tag{5.1}$$

We now give the characterizations of Green's relations on the semigroup $\mathcal{ODCI}_n$ in the following theorem.

Theorem 5.3. *The semigroup* $\mathcal{ODCI}_n$ *is* $\mathcal{J}$ *trivial.*

Proof.

(i) Let α, β, σ and $\gamma \in \mathcal{ODCI}_n$ be as expressed in (5.1) and suppose that $(\alpha, \beta) \in \mathcal{L}(\mathcal{ODCI}_n)$. This means that $\alpha = \sigma\beta$ and $\beta = \gamma\alpha$ for $\sigma, \gamma \in \mathcal{ODCI}_n^1$. This implies that $\text{Im } \alpha = \text{Im } \beta$.

Notice that $\text{Dom } \alpha \subseteq \text{Dom } \sigma$, as such $a_m \leq c_r$ meaning that there exists $c_1 \leq c^* \leq c_r$ such that $a_m = c^*$ and moreover, since σ is order-decreasing then $c^*\sigma \leq c^*$.

Claim: $c^*\sigma = c^*$.

Now, either $\text{Im } \alpha = \emptyset$ or $\text{Im } \alpha \neq \emptyset$. If $\text{Im } \alpha = \emptyset$, then since $\alpha = \sigma\beta$ then $\text{Im } \beta = \emptyset$ (since $\beta = \gamma\alpha$) and trivially $\alpha = \beta$. Now suppose $\text{Im } \alpha \neq \emptyset$ (which also implies $\text{Im } \beta \neq \emptyset$).

Suppose that $a_m = c^*$ and $c^*\sigma < c^*$ since $\text{Im } \alpha = \text{Im } \beta$ and α and β are both order-preserving then $a_i\alpha = b_i\beta$ $(i = 1 \ldots m)$. In particular $a_m\alpha = b_m\beta$ ——(1). Since $a_m = c^*$ then $c^*\sigma = a_m\sigma$. Now $(a_m\sigma)\beta = a_m\alpha = b_m\beta$. Since β is injective then $a_m\sigma = b_m$. But σ is order-decreasing means that $c^* > c^*\sigma = a_m\sigma = b_m$. *i.e.*, $c^* > b_m$ *i.e.*, $a_m > b_m$.

Now since $\text{Dom } \beta \subseteq \text{Dom } \gamma$, then $b_m = d^*$, where $d_1 \leq d_* \leq d_k$ using equation (1) we see that $(d^*\gamma)\alpha = d^*\beta = b_m\beta = a_m\alpha$, and by injectivity of α implies that $d^*\gamma = a_m > b_m = d^*$, *i.e.*, $d^*\gamma > d^*$ contradicting the fact that γ is order-decreasing, it follows that $c^*\sigma = c^*$. Now, since $\text{Dom } \alpha \subseteq \text{Dom } \sigma$ and $\text{Dom } \beta \subseteq \text{Dom } \gamma$, define $[c^*] = \{a \in \text{Dom } \alpha \mid a \leq c^*\} = \text{Dom } \alpha$ and $[d^*] = \{b \in \text{Dom } \beta \mid b \leq d^*\} = \text{Dom } \beta$. Now by Lemma 3.1, $\sigma|_{[c^*]}$ is a partial identity and as such $\alpha = \sigma\beta = \sigma|_{[c^*]}\beta = \beta$. Similarly, since $\text{Dom } \beta \subseteq \text{Dom } \gamma$, it means that there exists $d_1 \leq d^* \leq d_k$ such that $d^*\gamma = d^*$, then we can show similarly that $\gamma|_{[d^*]}$ is a partial identity by Lemma 3.1, as such $\beta = \gamma\alpha = \gamma|_{[d^*]}\alpha = \alpha$. Therefore, $\alpha = \beta$.

Conversely, suppose that $\alpha = \beta$ then since $\mathcal{L}$ is reflexive, it follows that $(\alpha, \beta) \in \mathcal{L}(\mathcal{ODCI}_n)$.

(ii) Let $\alpha, \beta \in \mathcal{ODCI}_n$ be as defined above and suppose that $(\alpha, \beta) \in \mathcal{R}(\mathcal{ODCI}_n)$. This means that $\alpha = \beta\sigma$ and $\beta = \alpha\gamma$ for $\sigma, \gamma \in \mathcal{ODCI}_n^1$. This implies that Dom α = Dom β.

Notice that Im $\alpha \subseteq$ Im σ, as such there exists $c_1 \leq c^* \leq c_r$ such that $a_m\alpha = c^*\sigma$ and moreover, since σ is order-decreasing then $c^*\sigma \leq c^*$.

Claim: $c^*\sigma = c^*$.

Now, either Im $\alpha = \emptyset$ or Im $\alpha \neq \emptyset$. If Im $\alpha = \emptyset$, since $\alpha = \beta\sigma$ then Im $\beta = \emptyset$ (since $\beta = \alpha\gamma$) and trivially $\alpha = \beta$. Now suppose Im $\alpha \neq \emptyset$ (which also implies Im $\beta \neq \emptyset$).

Suppose that $a_m\alpha = c^*\sigma$ and $c^*\sigma < c^*$ since Dom α = Dom β and α and β are both order-preserving then $a_i = b_i$(i=1…m). In particular $a_m = b_m$ ——(1). Notice that $(a_m\beta)\sigma = a_m\alpha = c^*\sigma$ and since σ is injective then $a_m\beta = c^* > c^*\sigma$. *i.e.*, $c^*\sigma < a_m\beta = b_m\beta$. *i.e.*, $a_m\alpha = c^*\sigma < b_m\beta$. *i.e.*, $a_m\alpha < b_m\beta$. Now, since Im $\beta \subseteq$ Im γ, then there exists $d^* \in$ Dom γ such that $b_m\beta = d^*\gamma$. Now using (1), $(a_m\alpha)\gamma = a_m\beta = b_m\beta = d^*\gamma$ and since γ is injective then $a_m\alpha = d^*$. *i.e.*, $d^* = a_m\alpha < b_m\beta = d^*\gamma$. *i.e.*, $d^* < d^*\gamma$ contradicting the fact that γ is order-decreasing. It follows that $c^*\sigma = c^*$. Now, since Im $\alpha \subseteq$ Im σ and Im $\beta \subseteq$ Im γ, then define $[c^*] = \{a\alpha \in \text{Im } \alpha \mid a\alpha \leq c^*\sigma\} = \text{Im } \alpha$ and $[d^*] = \{b\beta \in \text{Im } \beta \mid b\beta \leq d^*\gamma\} = \text{Im } \beta$. Now by Lemma 3.1, $\sigma|_{[c^*]}$ is a partial identity and as such $\alpha = \beta\sigma = \beta\sigma|_{[c^*]} = \beta$. Similarly, since Im $\beta \subseteq$ Im γ, it means that there exists $d^* \in$ Dom γ such that $d^*\gamma = d^*$, then we can show similarly that $\gamma|_{[d^*]}$ is a partial identity by Lemma 3.1, as such $\beta = \alpha\gamma = \alpha\gamma|_{[d^*]} = \alpha$. Therefore, $\alpha = \beta$.

Conversely, suppose that $\alpha = \beta$ then since $\mathcal{R}$ is reflexive, it follows that $(\alpha, \beta) \in \mathcal{R}(\mathcal{ODCI}_n)$.

(iii) Suppose $(\alpha, \beta) \in \mathcal{D}$. It means that there exists $\gamma \in \mathcal{ODCI}_n$ such that $(\alpha, \gamma) \in \mathcal{L}$ and $(\gamma, \beta) \in \mathcal{R}$. Then by (i) and (ii) $\alpha = \gamma$ and $\gamma = \beta$. Therefore, by the transitivity of $\mathcal{R}$, $\alpha = \beta$.

Conversely, suppose $\alpha = \beta$. Then since $\mathcal{D}$ is reflexive, then it follows that $(\alpha, \beta) \in \mathcal{D}$.

(iv) It follows from (i) and (ii).

□

5.1. **Regularity and starred Green's relations on $\mathcal{ODCI}_n$.** Recall that an element a in S is called regular if $a = aba$ for some $b \in S$. The semigroup S is called regular semigroup if all its elements are regular.

The following lemma gives us a necessary and sufficient conditions for an element in the semigroup $\mathcal{ODCI}_n$ to be regular.

Lemma 5.4. *Let $\alpha \in \mathcal{ODCI}_n$. Then α is regular if and only if α is a partial identity.*

Proof. Let $\alpha \in \mathcal{ODCI}_n$ be as expressed in equation (5.1) and suppose that α is regular. This means that there exists $\beta \in \mathcal{ODCI}_n$ such that $\alpha\beta\alpha = \alpha$. Notice that $\text{Im } \alpha \subseteq \text{Dom } \beta$ as such there exists $b^* \in \text{Dom } \beta$ such that $a_n\alpha = b^*$ for any $a_r \in \text{Dom } \alpha$ and since α and β are both order decreasing, then $a_r\alpha \leq a_r$ for all $r \in \{1, \ldots, k\}$ and $b_m\beta \leq b_m$ for all $m \in \{1, \ldots, k\}$. In particular, $a_k\alpha \leq a_k$. There are two cases to consider:
Case (i): Suppose $a_k\alpha = b^*$ and $a_k\alpha = a_k$. Then by Lemma 3.1, α is a partial identity.
Case (ii): Suppose that $a_k\alpha = b^*$ and $a_k\alpha < a_k$. Then $(a_k\alpha\beta)\alpha = a_k\alpha$. Since α is injective then $a_k\alpha\beta = a_k > a_k\alpha$ i.e., $b^*\beta > b^*$ contradicting the fact that β is order decreasing. This means that $a_k\alpha = a_k$ and by Lemma 3.1, α is a partial identity. The converse is obvious as such α is idempotent and hence regular. □

We immediately deduce the following result.

Corollary 5.5. *For $n \geq 3$, the semigroup $\mathcal{ODCI}_n$ is not regular.*

Theorem 5.6. *Let $\alpha, \beta \in \mathcal{ODCI}_n$. Then*

(i) $(\alpha, \beta) \in \mathcal{L}^*(\mathcal{ODCI}_n)$ *if and only if* $\alpha = \beta$;
(ii) $(\alpha, \beta) \in \mathcal{R}^*(\mathcal{ODCI}_n)$ *if and only if* $\alpha = \beta$;
(iii) $(\alpha, \beta) \in \mathcal{D}^*(\mathcal{ODCI}_n)$ *if and only if* $\alpha = \beta$;
(iv) $(\alpha, \beta) \in \mathcal{H}^*(\mathcal{ODCI}_n)$ *if and only if* $\alpha = \beta$.

Proof. It follows from Theorem 5.2. □

Remark 5.7. For $1 \leq n \leq 2$, $\mathcal{ODCI}_n$ is a regular semigroup.

Corollary 5.8. *For $n \geq 3$, the semigroup $\mathcal{ODCI}_n$ is neither left nor right abundant.*

Proof. Let $n \geq 3$ and consider $\alpha = \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix}$. It is obvious that $\alpha \in \mathcal{ODCI}_n$ and
$L^*_\alpha = \left\{ \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix} \right\}$ has no idempotent. Similarly,
$R^*_\alpha = \left\{ \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix} \right\}$ has no idempotent either.
□

6. Maximal subsemigroups of $\mathcal{ODDP}_n$

In this section we classify all the maximal subsemigroups of the semigroup $\mathcal{ODDP}_n$ and later proved that the semigroup has $n+2$ maximal subsemigroups. The rank of the semigroup $\mathcal{ODDP}_n$ was investigated by Kehinde *et al.*, see ([20], Theorem 3.3) and it has been proved to be $n+2$. Moreover, it was also proved that the semigroup has a minimum generating set. Before we begin our investigation we notice that $id_{[n]}$ is the only order-decreasing map of height n, as such $id_{[n]} \in \mathcal{ODCI}_n$ is non-factorizable. Now consider the elements of height $n-1$ in the semigroup $\mathcal{ODCI}_n$ (i.e., elements of J_{n-1}). These elements are of the following forms:

For $i \in \{1, \ldots, n\}$,

$$\epsilon_i = \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix} \text{ and } \alpha = \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix}. \tag{6.1}$$

Consider the possible product of all the elements of height $(n-1)$ in $\mathcal{ODDP}_n$ as follows:

$$\begin{aligned} \epsilon_1\alpha &= \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha \end{aligned}$$

and

$$\begin{aligned} \alpha\epsilon_n &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha. \end{aligned}$$

It is not difficult to see that α is the only non idempotent element of height $n-1$ in the set J_{n-1} and $\epsilon_i, \alpha \in \mathcal{ODDP}_n$ are non-factorizable, for $i \in \{1, \ldots, n\}$.

Remark 6.1. Notice that the set $\{\epsilon_i, \alpha, id_{[n]} : \ i \in \{1, \ldots, n\}\}$ generates the semigroup $\mathcal{ODDP}_n$ and $\{\epsilon_i, \alpha, id_{[n]} : \ i \in \{1, \ldots, n\}\} \cap J_{n-1}$ is non-trivial. Therefore by the Lemma 1.1, there exists a maximal subsemigroup arising from J_{n-1}.

We now record the following result needed in our subsequent discussions.

Lemma 6.2. [[20], Lemma 3.5]. *Let α, β, $\alpha\beta \in \mathcal{ODDP}_n$, be each of height $n-1$. Then $\alpha\beta$ is a partial identity if and only if $\alpha = \beta = \alpha\beta$.*

As a consequence we readily have the following proposition.

Proposition 6.3. *Let* $0 \leq k \leq n-2$. *Then* $J_k \subseteq \langle J_{k+1} \rangle$

Now we prove the following corollary.

Corollary 6.4. *Let* $0 \leq k \leq n-1$. *Then* $I(n,k) = \langle J_k \rangle$ *and* $\mathcal{ODDP}_n = \langle J_{n-1} \cup \{id_{[n]}\} \rangle$.

Proof. The prove is similar to that of Corollary 2.7. □

Now, we present the classification of the maximal subsemigroups of the semigroup $\mathcal{ODDP}_n$.

Lemma 6.5. *Every maximal subsemigroup* S *of* $\mathcal{ODDP}_n$ *contains the ideal* $I(n, n-2)$.

Proof. The proof is similar to that of Lemma 2.9. □

We now present the main results of this section.

Theorem 6.6. *Let* α *and* ϵ_i *be as defined in equation* (6.1). *A subsemigroup* S *of* $\mathcal{ODDP}_n$ *is maximal if and only if it belongs to one of the following three types:*

(i) $S_{\{id_{[n]}\}} := \mathcal{ODDP}_n \backslash \{id_{[n]}\}$;
(ii) $S_\alpha := \mathcal{ODDP}_n \backslash \{\alpha\}$;
(iii) $S_{\epsilon_i} := \mathcal{ODDP}_n \backslash \{\epsilon_i\},\ i \in \{1, \ldots, n\}$.

Proof. Suppose S is a maximal subsemigroup of $\mathcal{ODDP}_n$. Then by Lemma 6.5, $S = I(n, n-2) \cup K$, where $K \subseteq (J_{n-1} \cup \{id_{[n]}\})$. If $\{id_{[n]}\} \nsubseteq K$ then $K \subseteq \mathcal{ODDP}_n \backslash \{id_{[n]}\}$. Thus $S = I(n, n-2) \cup K \subseteq I(n, n-2) \cup \mathcal{ODDP}_n \backslash \{id_{[n]}\}$. Since S is maximal then $S = \mathcal{ODDP}_n \backslash \{id_{[n]}\} = S_{\{id_{[n]}\}}$, as such (i) follows.

Now, if $\{id_{[n]}\} \subseteq K$. Then $J_{n-1} \nsubseteq K$ (otherwise $S = \mathcal{ODDP}_n$). By ([20], Theorem 3.3), the set $\{\epsilon_i,\ \alpha : i \in \{1, \ldots, n\}$ is the minimum generating set for J_{n-1}, and as such K is contain in $J_{n-1} \backslash \{\tau\}$ for some $\tau \in \{\epsilon_i, \alpha\}$. Therefore by maximality of S, $S = S_\tau$ for some $\tau \in \{\epsilon_i, \alpha\}$ and hence (ii) and (iii) follows.

Conversely, suppose (i)-(iii) holds. Notice that $\mathcal{ODDP}_n \backslash \{id_{[n]}\} = I(n, n-1)$ and $I(n, n-1) \cup \{id_{[n]}\} = \mathcal{ODDP}_n$, hence $S_{\{id_{[n]}\}}$ is maximal. Since $\{\epsilon_i, \alpha : i \in \{1, \ldots, n\}$ is the unique minimum generating set for J_{n-1}, then $J_{n-1} \subseteq \langle \epsilon_i, \alpha, i \in \{1, \ldots, n\} \rangle$. Notice also that $\mathcal{ODDP}_n \backslash \{\epsilon_i\} \cup \{\epsilon_i\} = \mathcal{ODDP}_n$ and $\mathcal{ODDP}_n \backslash \{\alpha\} \cup \{\alpha\} = \mathcal{ODDP}_n$. Thus, it follows that for all $i \in \{1, \ldots, n\}$, S_{ϵ_i} and S_α are maximal. □

The next corollary gives us the number of the maximal subsemigroups of the semigroup $\mathcal{ODDP}_n$.

Corollary 6.7. *The semigroup $\mathcal{ODDP}_n$ contains exactly $n+2$ maximal subsemigroups.*

Proof. The results follows directly by counting the number of maximal subsemigroups in (i), (ii) and (iii) of Theorem 6.6. $\square$

Recall that a semigroup S is said to be $0-E-$*unitary* if $E' = E\backslash\{0\}$ and $(\forall e \in E')(\forall s \in S)$ $es \in E' \Rightarrow s \in E$. This means that, a *full* subsemigroup of $\mathcal{I}_n$ is $0-E-$ unitary if and only if only idempotents have fixed points [20]. The structure theory for $0-E-$ unitary inverse semigroups was given by Gomes and Howie [15] and Lawson [21]. Kehinde *et al.,* [20], deduced that the semigroup $\mathcal{ODDP}_n$ is a $0-E$-unitary ample subsemigroup of $\mathcal{I}_n$.

Throughout this section $S_{id_{[n]}}, S_\alpha$ and $S_{\epsilon_i},\ i \in \{1, \ldots, n\}$, denote the maximal subsemigroups of the semigroup $\mathcal{ODDP}_n$.

Theorem 6.8. *Let $S \in \{S_{id_{[n]}}, S_\alpha\}$ of $\mathcal{ODDP}_n$ and $\alpha, \beta \in S$. Then*

(i) *$\alpha \leq_{\mathcal{L}^*} \beta$ if and only if* $\mathrm{Im}\ \alpha \subseteq \mathrm{Im}\ \beta$;
(ii) *$\alpha \leq_{\mathcal{R}^*} \beta$ if and only if* $\mathrm{Dom}\ \alpha \subseteq \mathrm{Dom}\ \beta$.

Proof. The proof is similar to that of ([20], Theorem 2.3). $\square$

Remark 6.9. Notice that $E(S_{\epsilon_i}) \neq E(\mathcal{I}_n)$ as such the maximal subsemigroup S_{ϵ_i}, $(i \in \{1, \ldots n\})$ of $\mathcal{ODDP}_n$ is not an inverse ideal of $\mathcal{I}_n$. This means that the proof of $\mathcal{L}^*(\mathrm{S}_{\epsilon_\mathrm{i}})$ and $\mathcal{R}^*(\mathrm{S}_{\epsilon_\mathrm{i}})$ are different from that of Theorem 6.8 but similar to the proof of Theorem 2.15. Therefore, we state the characterization of $\mathcal{L}^*(\mathrm{S}_{\epsilon_\mathrm{i}})$ and $\mathcal{R}^*(\mathrm{S}_{\epsilon_\mathrm{i}})$ in the theorem below.

Theorem 6.10. *Let $S = S_{\epsilon_i},\ i \in \{1, \ldots n\}$ and $\alpha, \beta \in S$. Then*

(i) *$(\alpha, \beta) \in \mathcal{L}^*(\mathcal{S})$ if and only if* $\mathrm{Im}\ \alpha = \mathrm{Im}\ \beta$;
(ii) *$(\alpha, \beta) \in \mathcal{R}^*(\mathcal{S})$ if and only if* $\mathrm{Dom}\ \alpha = \mathrm{Dom}\ \beta$.

Proof. The proof is similar to that of Theorem 2.15. $\square$

Remark 6.11. Notice that the elements of height $n-1$ in the semigroup $\mathcal{ODDP}_n$ are as in the following set:

$$K := \left\{ \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix}, \begin{pmatrix} 2 & \cdots & n \\ 1 & \cdots & n-1 \end{pmatrix}, \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix}, \right.$$
$$\begin{pmatrix} 1 & 3 & \cdots & n \\ 1 & 3 & \cdots & n \end{pmatrix}, \begin{pmatrix} 1 & 2 & 4 & \cdots & n \\ 1 & 2 & 4 & \cdots & n \end{pmatrix}, \ldots,$$
$$\left. \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}, \ldots, \begin{pmatrix} 1 & \cdots & n-2 & n \\ 1 & \cdots & n-2 & n \end{pmatrix} \right\}.$$

Notice also that for each $\epsilon_i \in \mathcal{ODDP}_n,\ i \in \{2, \ldots, n-1\}$, $\mathrm{Dom}\ \epsilon_i \neq \mathrm{Dom}\ \alpha$ or $\mathrm{Im}\ \epsilon_i \neq \mathrm{Im}\ \alpha$, for all $\alpha \in \mathcal{ODDP}_n \backslash \{\epsilon_i\}$.

Now we have the following corollary.

Corollary 6.12. *The semigroup* $S_{\epsilon_i} := \mathcal{ODDP}_n \backslash \{\epsilon_i\},\ i \in \{1, \ldots, n\}$ *is:*

(i) *abundant if* $i \in \{2, \ldots, n-1\}$*;*
(ii) *left abundant if* $i = 1$*;*
(iii) *right abundant if* $i = n$*.*

Proof. (i) It follows from Theorem 6.10 and Remark 6.11.

(ii) Let $S = S_{\epsilon_i} := \mathcal{ODDP}_n \backslash \{\epsilon_i\},\ i \in \{1, \ldots, n\}$ and suppose that $i = 1$, i.e., $\epsilon_1 = \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix}$. Let K be as defined in Remark 6.11, notice that $\text{Im}\ \epsilon_1 \neq \text{Im}\ \alpha$ for all $\alpha \in K \subseteq S = \mathcal{ODDP}_n \backslash \{\epsilon_i\}$ as such $\text{Im}\ \epsilon_1 \neq \text{Im}\ \alpha$ for all $\alpha \in S$. Thus,

$$L^*_{\epsilon_1} = \emptyset \text{ and } R^*_{\epsilon_1} = \left\{ \begin{pmatrix} 2 & \cdots & n \\ 1 & \cdots & n-1 \end{pmatrix} \right\}$$

has no idempotent. Hence, the results follows.

(iii) Let $S = S_{\epsilon_i} := \mathcal{ODDP}_n \backslash \{\epsilon_i\},\ i \in \{1, \ldots, n\}$ and suppose that $i = n$. I.e., $\epsilon_n = \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix}$. Let K be as defined in Remark 6.11, notice that $\text{Dom}\ \epsilon_1 \neq \text{Dom}\ \alpha$ for all $\alpha \in K \subseteq S = \mathcal{ODDP}_n \backslash \{\epsilon_i\}$ as such $\text{Dom}\ \epsilon_1 \neq \text{Dom}\ \alpha$ for all $\alpha \in S$. Thus,

$$L^*_{\epsilon_n} = \left\{ \begin{pmatrix} 2 & \cdots & n \\ 1 & \cdots & n-1 \end{pmatrix} \right\}$$

has no idempotent and $R^*_{\epsilon_n} = \emptyset$. The result now follows. □

Remark 6.13. Let $S \in \{S_{id_{[n]}}, S_\alpha, S_{\epsilon_i} :\ i \in \{2, \ldots, n-1\}\}$. Then S is a 0-E-unitary subsemigroup of $\mathcal{I}_n$.

Theorem 6.14. *Let* $S \in \{S_{id_{[n]}}, S_\alpha, S_{\epsilon_i} :\ i \in \{2, \ldots, n-1\}\}$*. Then* S *is ample.*

Proof. The proof is similar to that of Theorem 2.20. □

Theorem 6.15. *Let* $S \in \{S_{id_{[n]}}, S_\alpha, S_{\epsilon_i} :\ i \in \{2, \ldots, n-1\}\}$*. Then* S *is a* 0*-*E*-unitary ample subsemigroup of* $\mathcal{I}_n$*.*

Proof. The results follows from Remark 6.13 and Theorem 6.14. □

Corollary 6.16. *Let* $S \in \{S_{id_{[n]}}, S_\alpha, S_{\epsilon_i},\ i \in \{2, \ldots, n-1\}\}$*. Then* S *is a* 0*-*E*- unitary ample maximal subsemigroup of* $\mathcal{ODDP}_n$*.*

Proof. The proof follows directly from Theorem 6.6 and Theorem 6.15. □

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References

1. B. Ali, M. A. Jada, and M. M. Zubairu, *On the ranks of certain semigroups of order preserving partial isometries of a finite chain.* J. Algebra and Rel. Top., (2) **6** (2018), 15–33.
2. F. Al-Kharousi, R. Kehinde and A. Umar, *On the semigroup of partial isometries of a finite chain.* Comm. Algebra, **44** (2016), 639–647.
3. F. Al-Kharousi, G. U. Garba, M. J. Ibrahium, A. T. Imam and A. Umar, *On the semigroup of finite order-preserving partial injective contraction mappings of a finite chain.* (Accepted, Afrika Mathematika).
4. L. Brucci, and L. E. Picasso, *Representations of semigroups of partial isometries.* Bull. Lond. Math. Soc., **39** (2007), 792–802.
5. I. Dimitrova and J. Koppitz, *The maximal subsemigroups of the ideals of some semigroup of partial injections.* Discuss. Math.- Gen Algebra Appl., **29** (2009), 153–167.
6. I. Dimitrova and J. Koppitz, *On the maximal subsemigroups of some transformation semigroups.* Asian-Eur. J. Math., **1** (2008), 189–202.
7. J. East, J. Kumar, D. M. 2. James and A. W. Wilf, *Maximal subsemigroups of finite transformation and diagram monoids.* J. Algebra, **504** (2018), 176–216.
8. J. B. Fountain, *Adequate semigroups.* Proc. Edinb. Math. Soc., **22** (1979), 113–125.
9. O. Ganyushkin and V. Mazorchuck, *On the structure of $\mathcal{IO}_n$.* Semigroup Forum, **66** (2003), 455–483.
10. O. Ganyushkin and V. Mazorchuck, *Classical Finite Transformation Semigroups.* Springer−Verlag: London Limited, 2009.
11. G. U. Garba, *Nilpotents in semigroup of partial injective order preserving mappings.* Semigroup Forum, **48** (1994), 37–49.
12. G. U. Garba, *Nilpotents in semigroups of partial order-preserving transformations.* Proc. Edinb. Math. Soc., **37** (1994), 361–377.
13. G. U. Garba, *On the nilpotents rank of partial transformation semigroup.* Port. Math., **51** (1994), 163–172.
14. G. M. S. Gomes and J. M. Howie, *On the ranks of certain finite semigroups of transformations.* Math. Proc. Camb. Philos. Soc., **101** (1987), 395–403.
15. G. M. S. Gomes and J. M. Howie, *A $P-$ theorem for inverse semigroups with zero.* Port. Math., **53** (1996), 257–278.
16. N. Graham, R. Graham and J. Rhodes, *Maximal subsemigroup of finite semigroups.* J. Comb. Theory, **4** (1968) 203–209.
17. P. M. Higgins, *Techniques of semigroup theory.* Oxford University Press, 1992.
18. J. M. Howie, *Fundamental of semigroup theory*, London Mathematical Society, New series 12. The Clarendon Press, Oxford University Press, 1995.
19. J. M. Howie, M. Paula, O. and M. Smith, *Inverse semigroups generated by nilpotent transformations.* Proc. R. Soc. Edinb. A: Math., **99** (1984), 153–162.

20. R. Kehinde, S. O. Makanjuola and A. Umar, *On the semigroup of order-decreasing partial isometries of a finite chain.* J. Algebra its Appl., (2015), 1–11.
21. M. V. Lawson, *The structure of* $0-E$*-unitary inverse semigroups* I*: the monoid case.* Proc. Edinb. Math. Soc., **42** (1999), 497–520.
22. J. W. Nicholas, *A class of maximal inverse subsemigroups of full transformation semigroup.* Semigroup Forum, **13** (1976), 187–188.
23. N. R. Reilly, *Maximal inverse subsemigroups of full transformation semigroup.* Semigroup Forum, **15** (1978), 319–326.
24. A. Umar, *On the semigroups of partial one-to-one order-decreading finite transformations.* Proc. R. Soc. Edinb. A: Math., **123** (1993), 355–363.
25. A. Umar, *Semigroups of order-decreading transformations, (PhD thesis).* St. Andrews Research Repository (1993). http://hdl.handle.net/10023/2834.
26. A. Umar and M. M. Zubairu, *On certain semigroups of contraction mappings of a finite chain.* Algebra Discret. Math., (2) **32** (2021), 299–320.
27. J. Wallen, Lawrence, *Semigroup of partial isometries.* Bull. Amer. Math. Soc., **75** (1969), 763–764.
28. X. Yang, *A classificatrion of maximal inverse subsemigroups of the finite symmetric inverse semigroups.* Comm. Algebra, (8) **27** (1999), 4089–4096.
29. X. Yang, *A classificatrion of maximal subsemigroups of the finite order-preserving transformation semigroups.* Comm. Algebra, (3) **28** (2000), 1503–1513.

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