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A NEW APPROACH TO ISOMORPHISM THEOREMS IN HILBERT ALGEBRAS

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Abstract. In this paper, we embark on an in-depth exploration of the profound connections between congruences, ideals, and homomorphisms in Hilbert algebras. Our research unveils a groundbreaking theorem that seamlessly integrates the principles of the first, second, and third isomorphism theorems within this algebraic structure. The study of these isomorphism theorems is crucial, as they provide fundamental insights into the structure and behaviour of algebraic systems, facilitating a deeper understanding and broader applications. This pivotal discovery not only enhances our comprehension of Hilbert algebras but also sets the stage for the development of new and innovative isomorphism theorems, promising to significantly enrich the field.

1. Introduction

Isomorphism theorems are foundational tools in algebra, offering deep insights into the structure of algebraic systems by elucidating how they can be decomposed and reconstructed. These theorems are crucial in various fields such as group theory, ring theory, and lattice theory, aiding in the classification of groups, analysis of ideals, and understanding of lattice structures. The First Isomorphism Theorem, for instance, highlights the relationship between a homomorphism and its kernel, revealing the quotient structure. The Second and Third Isomorphism Theorems further refine our understanding of substructures

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and quotients. Recent research has extended these classical theorems to more abstract systems like Hilbert algebras, demonstrating their versatility and importance in modern algebraic research. This foundational role ensures their continued relevance, paving the way for new discoveries and applications across mathematical disciplines, as detailed in works such as "Abstract Algebra" by Dummit and Foote [11], "Introduction to Commutative Algebra" by Atiyah and Macdonald [2], and recent studies like "Isomorphism Theorems in Hilbert Algebras" by Iampan et al. [13].

Among the diverse array of algebraic structures, algebras of logic stand out as particularly significant. The concept of Hilbert algebras, introduced by Diego [8], marked a pivotal advancement in this field. Diego demonstrated that Hilbert algebras constitute a locally finite variety, a finding that underscores their unique properties [8]. The exploration of Hilbert algebras was further advanced by Busneag [5, 6], who provided critical insights into their structure. Jun [14] identified certain filters within these algebras that form deductive systems, expanding our understanding of their logical applications. Building on this foundation, Jun et al. [15] introduced the innovative concept of stabilizers in Hilbert algebras and established several key propositions. Additionally, Dudek [10] explored the fuzzification of subalgebras, ideals, and deductive systems, further enriching the theoretical landscape of Hilbert algebras.

The study of isomorphism theorems in algebraic logic systems has been a continuous endeavor. For instance, in 2008, Ding et al. [9] proved the fundamental theorem of homomorphism, the first and second isomorphism theorems in BCI-algebras. In 2018, Mosrijai et al. [16] constructed a new fundamental theorem for UP-algebras, focusing on the congruence determined by a UP-homomorphism. They have also applied this theorem to establish the first, second, and third UP-isomorphism theorems, showcasing its practical significance in the study of UP-algebras. In 2019, Iampan [12] constructed the fundamental theorem of UP-homomorphisms within UP-algebras. Furthermore, we have applied this theorem to establish the first, second, third, and fourth UP-isomorphism theorems, demonstrating its broad applicability and significance in the study of UP-algebras. Bejarasco and Gonzaga [3] have introduced the notion of AB-homomorphism for AB-algebras and have obtained several of its properties. Additionally, they have investigated the first and third isomorphism theorems for AB-algebras, providing deeper insights into these algebraic structures. In 2020, Sassanapitax [17] constructed the first isomorphism theorem for QI-homomorphisms in QI-algebras. They have also thoroughly investigated the concepts of normal QI-subalgebras and quotient QI-algebras, providing a comprehensive understanding of these fundamental structures. Abed [1] introduced homomorphisms for BZ-algebras and investigated their properties. He has also explored the relationships between quotient BZ-algebras and isomorphisms, providing valuable insights into their interconnections. In 2022, Sriponpaew and Sassanapitax [18] introduced weak AB-algebras, a generalization of BCC-algebras. They demonstrated congruences and quotient formation and proved the fundamental isomorphism theorems for weak AB-algebras. In 2023, Bolima and Fuentes [4] have presented various properties of the dual B-homomorphism, introduced the concept of the natural dual B-homomorphism, and established the fundamental theorem of dual B-homomorphisms for dual B-algebras. Additionally, they have provided the first and third isomorphism theorems for the dual B-algebra.

In this paper, we present a groundbreaking fundamental theorem for Hilbert algebras, focusing on the congruence defined by a homomorphism. This new theorem not only advances our theoretical understanding but also offers practical applications, particularly in deriving the first, second, and third isomorphism theorems within the framework of Hilbert algebras. By elucidating these key isomorphism principles, our work provides valuable insights and tools that enhance the study and application of Hilbert algebras, paving the way for further research and discoveries in this field.

2. Preliminaries

Let's go through the idea of Hilbert algebras as it was introduced by Diego [8] in 1966 before we start.

Definition 2.1. [8] A *Hilbert algebra* is a triplet with the formula $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of X that is true according to the axioms stated below:

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

In [10], the following conclusion was established.

Lemma 2.2. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1)$,
(2) $(\forall x \in X)(1 \cdot x = x)$,

(3) $(\forall x \in X)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$,
(5) $(\forall x, y, z \in X)((x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1)$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.3. [19] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a *subalgebra* of X if $x \cdot y \in D$ for all $x, y \in D$.

Definition 2.4. [7] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called an *ideal* of X if the following conditions hold:
(1) $1 \in D$,
(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,
(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

Definition 2.5. [7] Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Define the binary relation $\sim_B$ on X as follows:

$$(\forall x, y \in X)(x \sim_B y \Leftrightarrow x \cdot y, y \cdot x \in B). \tag{2.1}$$

Definition 2.6. [7] Let $X = (X, \cdot, 1)$ be a Hilbert algebra. An equivalence relation ρ on X is called a *congruence* if

$$(\forall x, y, z \in X)(x \rho y \Rightarrow x \cdot z \rho y \cdot z, z \cdot x \rho z \cdot y). \tag{2.2}$$

Lemma 2.7. [7] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. An equivalence relation ρ on X is a congruence if and only if*

$$(\forall x, y, u, v \in X)(x \rho y, u \rho v \Rightarrow x \cdot u \rho y \cdot v). \tag{2.3}$$

Let $X = (X, \cdot, 1)$ be a Hilbert algebra and ρ a congruence on X. If $x \in X$, then the ρ-class of x is the $(x)_{\sim_\rho}$ defined as follows: $(x)_{\sim_\rho} = \{y \in X : y \rho x\}$. Then the set of all ρ-classes is called the quotient set of X by ρ and is denoted by X/ρ. That is, $X/\rho = \{(x)_{\sim_\rho} : x \in X\}$.

Theorem 2.8. [7] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Then $(X/\sim_B, *, (1)_{\sim_B})$ is a Hilbert algebra under the $*$ multiplication defined by $(x)_{\sim_B} * (y)_{\sim_B} = (x \cdot y)_{\sim_B}$ for all $x, y \in X$, called the quotient Hilbert algebra of X induced by the congruence $\sim_B$.*

Theorem 2.9. [7] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Then the mapping $\pi_B : X \to X/\sim_B$ defined by $\pi_B(x) = (x)_{\sim_B}$ for all $x \in X$ is an epimorphism, called the natural projection from X to $X/\sim_B$.*

Proposition 2.10. [7] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Then $\sim_B$ is a congruence on X.*

Theorem 2.11. [7] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Then the following statements hold:*

(1) the $\sim_B$-class $(1)_{\sim_B}$ is an ideal and a subalgebra of X which $B = (1)_{\sim_B}$,
(2) a $\sim_B$-class $(x)_{\sim_B}$ is an ideal of X if and only if $x \in B$,
(3) a $\sim_B$-class $(x)_{\sim_B}$ is a subalgebra of X if and only if $x \in B$,
(4) $(X/\sim_B, *, (1)_{\sim_B})$ is a Hilbert algebra under the $*$ multiplication defined by $(x)_{\sim_B} * (y)_{\sim_B} = (x \cdot y)_{\sim_B}$ for all $x, y \in X$, called the quotient Hilbert algebra of X induced by the congruence $\sim_B$.

Theorem 2.12. [13] *Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and let $f : A \to B$ be a homomorphism. Then the following statements hold:*

(1) $f(1_A) = 1_B$,
(2) for any $x, y \in A$, if $x \leq y$, then $f(x) \leq f(y)$,
(3) if C is a subalgebra of A, then the image $f(C)$ is a subalgebra of B. In particular, $\mathrm{Im}(f)$ is a subalgebra of B,
(4) if D is a subalgebra of B, then the inverse image $f^{-1}(D)$ is a subalgebra of A. In particular, $\mathrm{Ker}(f)$ is a subalgebra of A,
(5) if C is an ideal of A, then the image $f(C)$ is an ideal of $f(A)$,
(6) if D is an ideal of B, then the inverse image $f^{-1}(D)$ is an ideal of A. In particular, $\mathrm{Ker}(f)$ is an ideal of A,
(7) $\mathrm{Ker}(f) = \{1_A\}$ if and only if f is injective.

3. Isomorphism Theorems

In this section, we embark on an exploration of a crucial concept in Hilbert algebras: the congruence defined by a homomorphism. This concept serves as a gateway to a deeper understanding of algebraic structures, allowing us to unravel their inherent symmetries and relationships. We will unveil a groundbreaking fundamental theorem that redefines our approach to Hilbert algebras through the lens of this congruence. Furthermore, we will demonstrate how this theorem facilitates the derivation of the first, second, and third isomorphism theorems specifically tailored to Hilbert algebras. By delving into these isomorphism theorems, we aim to provide new insights and tools that enhance our theoretical framework and practical applications within this fascinating area of mathematics.

Definition 3.1. Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and $f : A \to B$ a homomorphism. Define the binary relation

$\sim_f$ on A as follows:

$$(\forall x, y \in A)(x \sim_f y \Leftrightarrow f(x) = f(y)). \tag{3.1}$$

Theorem 3.2. *Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and $f : A \to B$ a homomorphism. Then $\sim_f$ is a congruence on A, called the congruence determined by f.*

Proof. For all $x \in A$, we have $f(x) = f(x)$. Thus $x \sim_f x$. Hence, $\sim_f$ is reflexive. Let $x, y \in A$ be such that $x \sim_f y$. Then $f(x) = f(y)$, so $f(y) = f(x)$. Thus $y \sim_f x$. Hence, $\sim_f$ is symmetric. Let x, y, z be such that $x \sim_f y$ and $y \sim_f z$. Then $f(x) = f(y)$ and $f(y) = f(z)$, so $f(x) = f(z)$. Thus $x \sim_f z$. Hence, $\sim_f$ is transitive. Therefore, $\sim_f$ is an equivalence relation on A. Finally, let $x, y, u, v \in A$ be such that $x \sim_f u$ and $y \sim_f v$. Then $f(x) = f(u)$ and $f(y) = f(v)$. Since f is a homomorphism, we get $f(x \cdot y) = f(x) \star f(y) = f(u) \star f(v) = f(u \cdot v)$. Thus $x \cdot y \sim_f u \cdot v$. By Lemma 2.7, we have $\sim_f$ is a congruence on A. □

Theorem 3.3. *Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and $f : A \to B$ a homomorphism. Then $(A/\sim_f, *, (1_A)_{\sim_f})$ is a Hilbert algebra under the $*$ multiplication defined by $(x)_{\sim_f} * (y)_{\sim_f} = (x \cdot y)_{\sim_f}$ for all $x, y \in A$, called the quotient Hilbert algebra of A induced by the congruence $\sim_f$.*

Proof. Let $x, y, u, v \in A$ be such that $(x)_{\sim_f} = (y)_{\sim_f}$ and $(u)_{\sim_f} = (v)_{\sim_f}$. Since $\sim_f$ is an equivalence relation on A, we get $x \sim_f y$ and $u \sim_f v$. By Lemma 2.7, we have $x \cdot u \sim_f y \cdot v$. Hence, $(x)_{\sim_f} * (u)_{\sim_f} = (x \cdot u)_{\sim_f} = (y \cdot v)_{\sim_f} = (y)_{\sim_f} * (v)_{\sim_f}$, showing $*$ is well-defined. By routine calculation, we have $(A/\sim_f, *, (1_A)_{\sim_f})$ is a Hilbert algebra. □

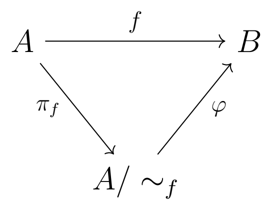
Theorem 3.4. *Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and $f : A \to B$ a homomorphism. Then the mapping $\pi_f : A \to A/\sim_f$ defined by $\pi_f(x) = (x)_{\sim_f}$ for all $x \in A$ is an epimorphism, called the natural projection from A to $A/\sim_f$.*

Proof. Let $x, y \in A$ be such that $x = y$. Then $(x)_{\sim_f} = (y)_{\sim_f}$, so $\pi_f(x) = \pi_f(y)$. Thus π_f is well-defined. Note that by the definition of π_f, we have π_f is surjective. Let $x, y \in A$. Then $\pi_f(x \cdot y) = (x \cdot y)_{\sim_f} = (x)_{\sim_f} * (y)_{\sim_f} = \pi_f(x) * \pi_f(y)$. Thus π_f is a homomorphism. Hence, π_f is an epimorphism. □

Theorem 3.5. *(Fundamental Theorem of homomorphisms) Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and $f : A \to B$ a homomorphism. Then there exists uniquely a homomorphism φ from $A/\sim_f$ to B such that $f = \varphi \circ \pi_f$. Moreover,*

(1) π_f *is an epimorphism and* φ *a monomorphism,*
(2) f *is an epimorphism if and only if* φ *is an isomorphism.*

As f *makes the following diagram commute,*



Proof. By Theorem 3.3, we have $(A/\sim_f, *, (1_A)_{\sim_f})$ is a Hilbert algebra. Define a mapping $\varphi : A/\sim_f \to B$ by $\varphi((x)_{\sim_f}) = f(x)$ for all $(x)_{\sim_f} \in A/\sim_f$. Indeed, let $(x)_{\sim_f}, (y)_{\sim_f} \in A/\sim_f$ be such that $(x)_{\sim_f} = (y)_{\sim_f}$. Then $x \sim_f y$, so

$$\varphi((x)_{\sim_f}) = f(x) = f(y) = \varphi((y)_{\sim_f}).$$

For any $x, y \in A$, we see that

$$\begin{aligned} \varphi((x)_{\sim_f} * (y)_{\sim_f}) &= \varphi((x \cdot y)_{\sim_f}) \\ &= f(x \cdot y) \\ &= f(x) \star f(y) \\ &= \varphi((x)_{\sim_f}) * \varphi((y)_{\sim_f}). \end{aligned}$$

Thus φ is a homomorphism. Also, since

$$(\varphi \circ \pi_f)(x) = \varphi(\pi_f(x)) = \varphi((x)_{\sim_f}) = f(x) \text{ for all } x \in A,$$

we obtain $f = \varphi \circ \pi_f$. We have shown the existence. Let φ' be a mapping from $A/\sim_f$ to B such that $f = \varphi' \circ \pi_f$. Then for any $(x)_{\sim_f} \in A/\sim_f$, we have

$$\begin{aligned} \varphi'((x)_{\sim_f}) &= \varphi'(\pi_f(x)) \\ &= (\varphi' \circ \pi_f)(x) \\ &= f(x) \\ &= (\varphi \circ \pi_f)(x) \\ &= \varphi(\pi_f(x)) \\ &= \varphi((x)_{\sim_f}). \end{aligned}$$

Hence, $\varphi = \varphi'$, showing the uniqueness.

(1) By Theorem 3.4, we have π_f is an epimorphism. Also, let $(x)_{\sim_f}, (y)_{\sim_f} \in A/\sim_f$ be such that $\varphi((x)_{\sim_f}) = \varphi((y)_{\sim_f})$. Then $f(x) = f(y)$, so $x \sim_f y$. Thus $(x)_{\sim_f} = (y)_{\sim_f}$. Therefore, φ is a monomorphism.

(2) Assume that f is an epimorphism. By (1), it suffices to prove that φ is surjective. Let $y \in B$. Then there exists $x \in A$ such that

$f(x) = y$. Thus $y = f(x) = \varphi((x)_{\sim_f})$, so φ is surjective. Hence, φ is an isomorphism.

Conversely, assume that φ is an isomorphism. Then φ is surjective. Let $y \in B$. Then there exists $(x)_{\sim_f} \in A/\sim_f$ such that $\varphi((x)_{\sim_f}) = y$. Thus $f(x) = \varphi((x)_{\sim_f}) = y$, so f is surjective. Hence, f is an epimorphism. □

Theorem 3.6. *(First isomorphism Theorem) Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and $f : A \to B$ a homomorphism. Then $A/\sim_f \cong \mathrm{Im}(f)$.*

Proof. By Theorem 2.12 (3), we have $\mathrm{Im}(f)$ is a subalgebra of B. Thus $f : A \to \mathrm{Im}(f)$ is an epimorphism. Applying Theorem 3.5 (2), we obtain $A/\sim_f \cong \mathrm{Im}(f)$. □

Lemma 3.7. *Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras, $f : A \to B$ a homomorphism, and H a subalgebra of A. Denote $H_{\sim_f} = \bigcup_{h \in H} (h)_{\sim_f}$. Then $H_{\sim_f}$ is a subalgebra of A.*

Proof. Clearly, $\emptyset \neq H_{\sim_f} \subseteq A$. Let $a, b \in H_{\sim_f}$. Then $a \in (x)_{\sim_f}$ and $b \in (y)_{\sim_f}$ for some $x, y \in H$, so $(a)_{\sim_f} = (x)_{\sim_f}$ and $(b)_{\sim_f} = (y)_{\sim_f}$. Theorem 3.3 gives $(A/\sim_f, *, (1_A)_{\sim_f})$ is a Hilbert algebra, so $(a \cdot b)_{\sim_f} = (a)_{\sim_f} * (b)_{\sim_f} = (x)_{\sim_f} * (y)_{\sim_f} = (x \cdot y)_{\sim_f}$. Thus $a \cdot b \in (x \cdot y)_{\sim_f}$. Since $x, y \in H$ and H is a subalgebra of A, we have $x \cdot y \in H$. Thus $a \cdot b \in (x \cdot y)_{\sim_f} \subseteq H_{\sim_f}$. Hence, $H_{\sim_f}$ is a subalgebra of A. □

Theorem 3.8. *(Second isomorphism Theorem) Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras, $f : A \to B$ a homomorphism, and H a subalgebra of A. Denote $H_{\sim_f}/\sim_f = \{(x)_{\sim_f} : x \in H_{\sim_f}\}$. Then $H/\sim_{\pi_f|H} \cong H_{\sim_f}/\sim_f$.*

Proof. By Lemma 3.7, we have $H_{\sim_f}$ is a subalgebra of A. Then it is easy to check that $H_{\sim_f}$ is a subalgebra of A, thus $(H_{\sim_f}/\sim_f, *, (1_A)_{\sim_f})$ itself is a Hilbert algebra. Also, it is obvious that $H \subseteq H_{\sim_f}$, then $(\pi_f|H =)g : H \to H_{\sim_f}/\sim_f, x \mapsto (x)_{\sim_f}$ is a mapping. Indeed, g is the restriction of π_f to H. Thus g is an epimorphism. Indeed, $H_{\sim_f}/\sim_f = H/\sim_f$. Theorem 3.6 gives $H/\sim_{\pi_f|H} \cong H_{\sim_f}/\sim_f$. □

Theorem 3.9. *Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and $f : A \to B$ and $g : A \Rightarrow B$ homomorphisms with $\sim_f \subseteq \sim_g$. Define the binary relation $\sim_g / \sim_f$ on $A/\sim_f$ as follows:*

$$(\forall x, y \in A)((x)_{\sim_f} \sim_g / \sim_f (y)_{\sim_f} \Leftrightarrow x \sim_g y). \tag{3.2}$$

Then $\sim_g / \sim_f$ is a congruence on $A/\sim_f$.

Proof. By Theorem 3.3, we have $(A/\sim_f, *, (1_A)_{\sim_f})$ is a Hilbert algebra.

Reflexive: For all $x \in A$, we have $x \sim_g x$. Thus $(x)_{\sim_f} \sim_g / \sim_f (x)_{\sim_f}$.

Symmetric: Let $x, y \in A$ be such that $(x)_{\sim_f} \sim_g / \sim_f (y)_{\sim_f}$. Then $x \sim_g y$, so $y \sim_g x$. Thus $(y)_{\sim_f} \sim_g / \sim_f (x)_{\sim_f}$.

Transitive: Let x, y, z be such that $(x)_{\sim_f} \sim_g / \sim_f (y)_{\sim_f}$ and $(y)_{\sim_f} \sim_g / \sim_f (z)_{\sim_f}$. Then $x \sim_g y$ and $y \sim_g z$, so $x \sim_g z$. Thus $(x)_{\sim_f} \sim_g / \sim_f (z)_{\sim_f}$.

Therefore, $\sim_g / \sim_f$ is an equivalence relation on $A/\sim_f$. Finally, let $x, y, u, v \in A$ be such that $(x)_{\sim_f} \sim_g / \sim_f (u)_{\sim_f}$ and $(y)_{\sim_f} \sim_g / \sim_f (v)_{\sim_f}$. Then $x \sim_g u$ and $y \sim_g v$. The binary relation $\sim_g$ is a congruence on A by Theorem 3.2, that is, $x \cdot y \sim_g u \cdot v$. Thus $(x \cdot y)_{\sim_f} \sim_g / \sim_f (u \cdot v)_{\sim_f}$, so $(x)_{\sim_f} * (y)_{\sim_f} \sim_g / \sim_f (u)_{\sim_f} * (v)_{\sim_f}$. Hence, $\sim_g / \sim_f$ is a congruence on $A/\sim_f$. □

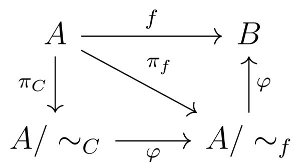
Theorem 3.10. *(Third isomorphism Theorem) Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and $f : A \to B$ and $g : A \to B$ homomorphisms with $\sim_f \subseteq \sim_g$. Then $(A/\sim_f)/(\sim_g / \sim_f) \cong A/\sim_g$.*

Proof. By Theorem 3.3, we obtain $(A/\sim_f, *, (1_A)_{\sim_f})$ and $(A/\sim_g, *, (1_A)\sim_g)$ are Hilbert algebras. By Theorem 3.4, we obtain $\pi_f : A \to A/\sim_f, x \mapsto (x)_{\sim_f}$ and $\pi_g : A \to A/\sim_g, x \mapsto (x)_{\sim_g}$ are epimorphisms. Applying Theorem 3.5 (2), there exists an isomorphism $g/f : A/\sim_f \to A/\sim_g, (x)_{\sim_f} \mapsto (x)\sim_g$. Indeed, $A/\sim_f \cong A/\sim_g$. By Theorems 3.9 and 3.3, we have $(A/\sim_f)/\sim_{g/f}$ is a Hilbert algebra. By Theorem 3.4, we obtain $\pi_{g/f} : A/\sim_f \to (A/\sim_f)/\sim_{g/f}, (x)_{\sim_f} \mapsto ((x)_{\sim_f})\sim_{g/f}$ is an epimorphism. Applying Theorem 3.5 (2), there exists an isomorphism $\varphi : (A/\sim_f)/\sim_{g/f} \to A/\sim_g, ((x)_{\sim_f})_{\sim_{g/f}} \mapsto (x)_{\sim_g}$. That is, $(A/\sim_f)/\sim_{g/f} \cong A/\sim_g$. We shall show that $\sim_{g/f} = \sim_g / \sim_f$. For any $(x)_{\sim_f}, (y)_{\sim_f} \in A/\sim_f$, we have

$$\begin{aligned}(x)_{\sim_f} \sim_{g/f} (y)_{\sim_f} &\Leftrightarrow (g/f)((x)_{\sim_f}) = (g/f)((y)_{\sim_f}) \\ &\Leftrightarrow (x)_{\sim_f} = (y)_{\sim_g} \\ &\Leftrightarrow x \sim_g y \\ &\Leftrightarrow (x)_{\sim_f} \sim_g / \sim_f (y)_{\sim_f}\end{aligned}$$

by (3.1) and (3.2). Thus $\sim_{g/f} = \sim_g / \sim_f$. Hence, $(A/\sim_f)/(\sim_g / \sim_f) \cong A/\sim_g$. □

Corollary 3.11. *Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras, $f : A \to B$ a homomorphism, and C an ideal of A. Then $A/\sim_C \cong A/\sim_f$. As π_f makes the following diagram commute,*



Proof. It is straightforward by Theorems 2.8, 2.9, 3.4, and 3.5 (2). □

4. Conclusion

Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras with $f : A \to B$ denoting a homomorphism. It is well-established that $\sim_f$, the congruence defined by f, acts on A. In this paper, we have developed a pivotal fundamental theorem concerning homomorphisms characterized by $\sim_f$ within Hilbert algebras. This theorem not only deepens our theoretical understanding but also extends to provide the first, second, and third isomorphism theorems specifically tailored to Hilbert algebras. By establishing these foundational results, our work offers new insights into the structure and behaviour of Hilbert algebras, enriching both theoretical perspectives and practical applications in the field.

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