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PERFECT 3-COLORINGS OF THE LINE GRAPHS OF THE CONNECTED BICUBIC GRAPHS OF ORDER AT MOST 12

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Abstract. Let $G = (V_G, E_G)$ be a graph and let I be a finite set of size $m \geq 1$. A mapping $T : V_G \rightarrow I$ is called a perfect m-coloring with a parameter matrix $A = (a_{ij})_{i,j\in I}$ of G if it is surjective and for all i, j, every vertex of color i has a_{ij} neighbors of color j. In this paper, we classify all the realizable parameter matrices of perfect 3-colorings of the line graphs of the connected bicubic graphs of order at most 12.

1. Introduction

The notion of a perfect m-coloring (also known as an equitable partition into m parts) of a graph arises naturally in graph theory, algebraic combinatorics and coding theory (completely regular codes). Studies of perfect colorings start usually with the case of two colors. This case is the simplest and also of the most interest, since it possesses great potential for generalization. In recent years, perfect 3-colorings of graphs have been extensively studied. For instance, perfect 3-colorings of the prism graphs, the Möbius ladder graphs, infinite circulant graph with distances 1 and 2, infinite multipath graphs, paths divisible by a matching, the cubic graphs of order $n = 8, 10$, the generalized Petersen graphs including $GP(5, 2)$, $GP(6, 2)$ and $GP(7, 2)$, the Platonic graphs, 6-regular graphs of order 9, the Johnson graph $J(6, 3)$ and the Heawood graph have been investigated (see [12, 15, 13, 14, 3, 5, 4, 6, 16, 2, 1]).

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A bicubic graph is a bipartite cubic graph. There are only 9 connected bicubic graphs of order at most 12 [7] (see Figure 1).

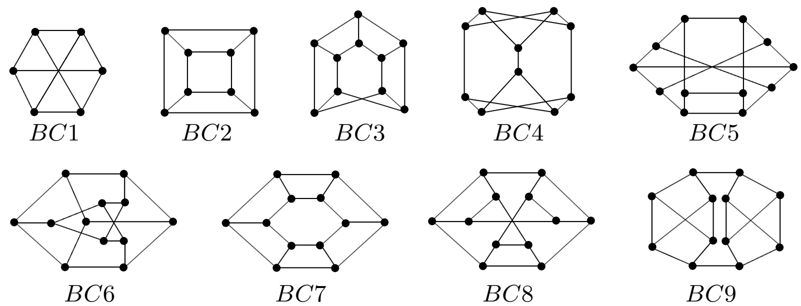

FIGURE 1. The connected bicubic graphs of order at most 12

In this paper, we study perfect 3-colorings of the line graphs of the connected bicubic graphs of order at most 12. In particular, we classify all the realizable parameter matrices of perfect 3-colorings of these graphs.

2. Preliminaries

In the following, we first briefly review some definitions and terminologies related to graphs. For concepts not defined here, we refer to [18]. Throughout this paper, all graphs are finite, simple and undirected. For a graph G, we denote by V_G and E_G the vertex set and the edge set of G, respectively. For $v \in V_G$, denote by $N(v)$ the neighborhood of the vertex v in G.

The line graph $L(G)$ of a graph G is constructed by taking the edges of G as vertices of $L(G)$ and joining two vertices in $L(G)$ whenever the corresponding edges in G have a common vertex.

For a graph G, we refer to the eigenvalues of its adjacency matrix as the eigenvalues of G. Let G be a connected k-regular graph of order n, size m and with the eigenvalues $\lambda_1 = k^1$, $\lambda_2^{m(\lambda_2)}$, $\ldots$ and $\lambda_s^{m(\lambda_s)}$, where a superscript denotes the multiplicity of the respective eigenvalue. Then, the eigenvalues of $L(G)$ are $(2k-2)^1$, $(k-2+\lambda_2)^{m(\lambda_2)}$, $\ldots$, $(k-2+\lambda_s)^{m(\lambda_s)}$ and $(-2)^{m-n}$ [9].

Let I be a finite set of size $m \geq 1$. A mapping $T : V_G \to I$ is called a perfect m-coloring with a parameter matrix $A = (a_{ij})_{i,j \in I}$ of a graph G if it is surjective and for all i, j, every vertex of color i has a_{ij} neighbors of color j. If $T : V_G \to \{c_1, \ldots, c_m\}$ is a perfect m-coloring with the parameter matrix $A = (a_{ij})_{i,j \in \{1,\ldots,m\}}$ then we assume that the rows and columns of A correspond to the colors in the listed order. It is easy to see that the parameter matrix A is symmetric with respect to 0; i.e., $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Let T be a perfect m-coloring with the parameter matrix A of a graph G. A number λ is called an eigenvalue of T if it is an eigenvalue of A.

Theorem 2.1 ([11]). *Let T be a perfect m-coloring of a graph G. Then any eigenvalue of T is an eigenvalue of G.*

Let $T : V_G \to \{0, 1, 2\}$ be a perfect 3-coloring of a graph G with the parameter matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. In this case, the colors 0, 1 and 2 symbolize white, black and red colors, respectively. Denote by W, B and R the set of white, black and red vertices of G, respectively. Hereafter, without loss of generality, assume that $|W| \leq |B| \leq |R|$. Obviously, $|W| + |B| + |R| = |V_G|$. Also [8]

$$a_{12}|W| = a_{21}|B|, \quad a_{13}|W| = a_{31}|R|, \quad a_{23}|B| = a_{32}|R|.$$

Moreover, if G is a connected graph then $a_{12}, a_{13} \neq 0$, $a_{12}, a_{23} \neq 0$ or $a_{13}, a_{23} \neq 0$. Also, if G is a k-regular graph then

$$a_{11} + a_{12} + a_{13} = a_{21} + a_{22} + a_{23} = a_{31} + a_{32} + a_{33} = k.$$

In this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e., we identify perfect 3-coloring with the following parameter matrices

$$\begin{pmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & a_{31} & a_{33} \end{pmatrix}, \begin{pmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{pmatrix}, \begin{pmatrix} a_{11} & a_{13} & a_{12} \\ a_{31} & a_{33} & a_{32} \\ a_{21} & a_{23} & a_{22} \end{pmatrix},$$

$$\begin{pmatrix} a_{33} & a_{31} & a_{32} \\ a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \\ a_{12} & a_{13} & a_{11} \end{pmatrix},$$

obtained by switching the colors with the original coloring. We call the above matrices equivalent to the matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

3. Main results

In this section, we enumerate the parameter matrices for perfect 3-colorings of the line graphs of the connected bicubic graphs of order at most 12. The line graphs of these graphs are shown in Figure 2 [17].

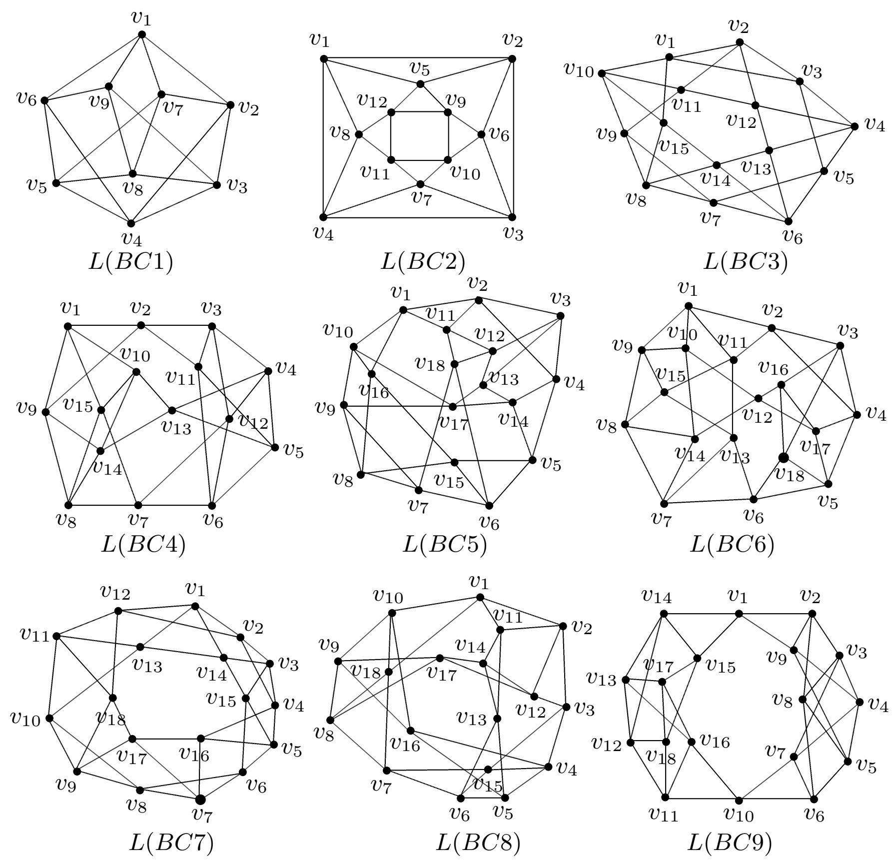

FIGURE 2. The line graphs of the connected bicubic graphs of order at most 12

First, let G be a connected 4-regular graph of order 9 and T be a parameter matrix of a perfect 3-coloring of G with parameter matrix $A = (a_{ij})_{3\times 3}$. Since G is connected, according to the conditions

$$\begin{aligned}
&|W| \leq |B| \leq |R|,\\
&|W| + |B| + |R| = 9,\\
&a_{12}|W| = a_{21}|B|, \quad a_{13}|W| = a_{31}|R|, \quad a_{23}|B| = a_{32}|R|,\\
&a_{11} + a_{12} + a_{13} = a_{21} + a_{22} + a_{23} = a_{31} + a_{32} + a_{33} = 4,\\
&a_{ij} \in \{0, 1, 2, 3, 4\}, \qquad \text{for all } i, j = 1, 2, 3,
\end{aligned}$$

we obtain the following matrices:

- $(|W|, |B|, |R|) = (1, 2, 6)$: $\begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$

- $(|W|,|B|,|R|) = (1,4,4)$: $\begin{pmatrix} 0 & 4 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$,

$\begin{pmatrix} 0 & 4 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 4 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 3 \\ 1 & 3 & 0 \end{pmatrix}$

- $(|W|,|B|,|R|) = (2,3,4)$: $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 4 \\ 1 & 3 & 0 \end{pmatrix}$

- $(|W|,|B|,|R|) = (3,3,3)$: $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}$,

$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & 1 \\ 3 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 0 \end{pmatrix}$,

$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 3 & 0 \\ 3 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$,

$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$,

$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$,

$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$,

$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$,

$\begin{pmatrix} 0 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Note that some of the above matrices may be equivalent. Now, since the line graph of $BC1$ is a connected 4-regular graph of order 9, by Theorem 2.1, up to equivalence, a parameter matrix of a perfect 3-coloring of $L(BC1)$ may be one of the following matrices:

- $(|W|,|B|,|R|) = (1,4,4)$: $\begin{pmatrix} 0 & 4 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$

- $(|W|,|B|,|R|) = (3,3,3)$: $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

By similar procedure for $L(BCi)$ $(i = 2, \ldots, 9)$, we obtain some matrices. All obtained non-equivalent matrices are listed in the following:

$$A_1 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 1 & 2 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \ A_3 = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \ A_5 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \ A_6 = \begin{pmatrix} 0 & 4 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix},$$

$$A_7 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \ A_8 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \ A_9 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix},$$

$$A_{10} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \ A_{11} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \ A_{12} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix},$$

$$A_{13} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \ A_{14} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \ A_{15} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix},$$

$$A_{16} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \ A_{17} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 2 \end{pmatrix}, \ A_{18} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix},$$

$$A_{19} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \ A_{20} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}.$$

Theorem 3.1. *Let G be a bicubic graph of order 6 or 8. Then, all parameter matrices A for perfect 3-colorings of the line graph of G are only the ones listed in the following table:*

G	A
(i) $BC1$	$A_5,\ A_6,\ A_{16}$
(ii) $BC2$	$A_1,\ A_4,\ A_5,\ A_{10},\ A_{11},\ A_{13}$

Proof. (i) The eigenvalues of $BC1$ are 3^1, 0^4 and -3^1 [17]. Thus, the eigenvalues of $L(BC1)$ are 4^1, 1^4 and -2^4. According to the mentioned conditions and Theorem 2.1, a parameter matrix of a perfect 3-coloring of $L(BC1)$ may be one of the following matrices:

A_6	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (1, 4, 4)$
$A_5,\ A_{16}$	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 3, 3)$

Define the mappings $T_{1,5}, T_{1,6}, T_{1,16} : V_{L(BC1)} \to \{0, 1, 2\}$ as follows:

$T_{1,5}(v_i) = 0$ if $i = 1, 4, 8$, $\quad T_{1,5}(v_i) = 1$ if $i = 2, 5, 9$, $\quad T_{1,5}(v_i) = 2$ otherwise,
$T_{1,6}(v_i) = 0$ if $i = 1$, $\quad T_{1,6}(v_i) = 1$ if $i = 2, 6, 7, 9$, $\quad T_{1,6}(v_i) = 2$ otherwise,
$T_{1,16}(v_i) = 0$ if $i = 1, 2, 7$, $\quad T_{1,16}(v_i) = 1$ if $i = 4, 5, 6$, $\quad T_{1,16}(v_i) = 2$ otherwise.

It is easy to see that $T_{1,5}$, $T_{1,6}$ and $T_{1,16}$ are the perfect 3-colorings of $L(BC1)$ with the parameter matrices A_5, A_6 and A_{16}, respectively.

(ii) The eigenvalues of $BC2$ are 3^1, 1^3, -1^3 and -3^1 [10]. Thus, the eigenvalues of $L(BC2)$ are 4^1, 2^3, 0^3 and -2^5. According to the mentioned conditions and Theorem 2.1, a parameter matrix of a perfect 3-coloring of $L(BC2)$ may be one of the following matrices:

A_1	$(\|W\|, \|B\|, \|R\|) = (2, 2, 8)$
A_{17}	$(\|W\|, \|B\|, \|R\|) = (2, 4, 6)$
$A_4,\ A_9,\ A_{11}$	$(\|W\|, \|B\|, \|R\|) = (3, 3, 6)$
$A_5,\ A_{10},\ A_{13}$	$(\|W\|, \|B\|, \|R\|) = (4, 4, 4)$

Define the mappings $T_{2,1}, T_{2,4}, T_{2,5}, T_{2,10}, T_{2,11}, T_{2,13} : V_{L(BC2)} \to \{0, 1, 2\}$ as follows:

$T_{2,1}(v_i) = 0$ if $i = 1, 10$, $T_{2,1}(v_i) = 1$ if $i = 3, 12$,
$T_{2,1}(v_i) = 2$ otherwise,
$T_{2,4}(v_i) = 0$ if $i = 1, 7, 9$, $T_{2,4}(v_i) = 1$ if $i = 4, 5, 10$,
$T_{2,4}(v_i) = 2$ otherwise,
$T_{2,5}(v_i) = 0$ if $i = 1, 3, 10, 12$, $T_{2,5}(v_i) = 1$ if $i = 4, 5, 6, 11$,
$T_{2,5}(v_i) = 2$ otherwise,
$T_{2,10}(v_i) = 0$ if $i = 1, 4, 9, 10$, $T_{2,10}(v_i) = 1$ if $i = 2, 3, 11, 12$
$T_{2,10}(v_i) = 2$ otherwise,
$T_{2,11}(v_i) = 0$ if $i = 1, 4, 8$, $T_{2,11}(v_i) = 1$ if $i = 6, 9, 10$,
$T_{2,11}(v_i) = 2$ otherwise,
$T_{2,13}(v_i) = 0$ if $i = 1, 2, 3, 4$, $T_{2,13}(v_i) = 1$ if $i = 9, 10, 11, 12$,
$T_{2,13}(v_i) = 2$ otherwise.

It is easy to see that $T_{2,1}, T_{2,4}, T_{2,5}, T_{2,10}, T_{2,11}$ and $T_{2,13}$ are the perfect 3-colorings of $L(BC2)$ with the parameter matrices A_1, A_4, A_5, A_{10}, A_{11} and A_{13}, respectively.

It suffices to show that there is no perfect 3-coloring of $L(BC2)$ with the parameter matrix A_9 and A_{17}. Let $T : V_{L(BC2)} \to \{0, 1, 2\}$ be a perfect 3-coloring of $L(BC2)$. First, suppose that A_9 is the parameter matrix of T. Without loss of generality, assume that $T(v_1) = 0$. Note that the induced subgraph $L(BC2)[\{v_1\} \cup N(v_1)]$ is isomorphic to the friendship graph F_2. Thus, there is a black or red vertex with two white neighbors, a contradiction.

Finally, suppose that A_{17} is the parameter matrix of T. Then, there are at least three white vertices, a contradiction. Therefore, the proof is complete. □

Theorem 3.2. *Let G be a bicubic graph of order 10. Then, all parameter matrices A for perfect 3-colorings of the line graph of G are only the ones listed in the following table:*

G	A
(i) $BC3$	A_5
(ii) $BC4$	A_5

Proof. (i) The eigenvalues of $BC3$ are 3^1, 1.62^2, 0.62^2, -0.62^2, -1.62^2 and -3^1 [17]. Thus, the eigenvalues of $L(BC3)$ are 4^1, 2.62^2, 1.62^2, 0.38^2, -0.62^2 and -2^6. According to the mentioned conditions and Theorem 2.1, a parameter matrix of a perfect 3-coloring of $L(BC3)$ may be one of the following matrices:

A_{14}	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 6, 6)$
A_5	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (5, 5, 5)$

Define the mapping $T_{3,5} : V_{L(BC3)} \to \{0, 1, 2\}$ by $T_{3,5}(v_i) = 0$ if $i = 1, 5, 8, 11, 13$, $T_{3,5}(v_i) = 1$ if $i = 2, 4, 7, 10, 14$ and $T_{3,5}(v_i) = 2$ otherwise. It is easy to see that $T_{3,5}$ is the perfect 3-coloring of $L(BC3)$ with the parameter matrix A_5.

It suffices to show that there is no perfect 3-coloring of $L(BC3)$ with the parameter matrix A_{14}. Let $T : V_{L(BC3)} \to \{0, 1, 2\}$ be a perfect 3-coloring of $L(BC3)$. Suppose that A_{14} is the parameter matrix of T. Without loss of generality, assume that $T(v_1) = 1$. Note that the induced subgraph $L(BC3)[\{v_1\} \cup N(v_1)]$ is isomorphic to the friendship graph F_2. Thus, there is a red vertex with two black neighbors, a contradiction.

(ii) The eigenvalues of $BC4$ are 3^1, 2^1, 1^2, 0^2, -1^2, -2^1 and -3^1 [17]. Thus, the eigenvalues of $L(BC4)$ are 4^1, 3^1, 2^2, 1^2, 0^2, -1^1 and -2^6. According to the mentioned conditions and Theorem 2.1, a parameter matrix of a perfect 3-coloring of $L(BC4)$ may be one of the following matrices:

A_2, A_3, A_7, A_{12}, A_{20}	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 3, 9)$
A_5, A_{10}, A_{13}, A_{15}, A_{16}, A_{19}	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (5, 5, 5)$

Define the mapping $T_{4,5} : V_{L(BC4)} \to \{0, 1, 2\}$ by $T_{4,5}(v_i) = 0$ if $i = 2, 4, 6, 8, 10$, $T_{4,5}(v_i) = 1$ if $i = 1, 3, 5, 7, 14$ and $T_{4,5}(v_i) = 2$ otherwise. It is easy to see that $T_{4,5}$ is the perfect 3-coloring of $L(BC4)$ with the parameter matrix A_5.

It suffices to show that there is no perfect 3-coloring of $L(BC4)$ with the parameter matrix A_2, A_3, A_7, A_{10}, A_{12}, A_{13}, A_{15}, A_{16}, A_{19} and A_{20}. Let $T : V_{L(BC4)} \to \{0, 1, 2\}$ be a perfect 3-coloring of $L(BC4)$. Then if A_3 is the parameter matrix of T then the subgraph induced by black vertices is a cubic graph of order 3, a contradiction,

if A_7 is the parameter matrix of T then the subgraph induced by white vertices is a 1-regular graph of order 3, a contradiction,

if A_{10} is the parameter matrix of T then the subgraph induced by white vertices is a 1-regular graph of order 5, a contradiction,

if A_{12} is the parameter matrix of T then the subgraph induced by red vertices is a 1-regular graph of order 3, a contradiction,

if A_{15} is the parameter matrix of T then the subgraph induced by black vertices is a 1-regular graph of order 5, a contradiction,

if A_{19} is the parameter matrix of T then the subgraph induced by white vertices is a cubic graph of order 5, a contradiction,

if A_{20} is the parameter matrix of T then the subgraph induced by white vertices is a cubic graph of order 3, a contradiction.

Now, suppose that A_2 is the parameter matrix of T. Then, the subgraph induced by red vertices is a 2-regular graph of order 9, i.e., it is isomorphic to the cycle C_9 or one of the disjoint unions $3C_3$, C_4+C_5 and C_3+C_6. But $L(BC4)$ has no induced subgraph isomorphic to these graphs, a contradiction.

Finally, suppose that A_{13} or A_{16} is the parameter matrix of T. Then, the subgraph induced by white vertices is a 2-regular graph of order 5, i.e., it is isomorphic to the cycle C_5. But $L(BC4)$ has no induced subgraph isomorphic to C_5, a contradiction. Therefore, the proof is complete. □

Theorem 3.3. *Let G be a bicubic graph of order 12. Then, all parameter matrices A for perfect 3-colorings of the line graph of G are only the ones listed in the following table:*

G	A
(i) $BC5$	$A_1,\ A_5,\ A_6,\ A_{13}$
(ii) $BC6$	$A_5,\ A_6,\ A_{16}$
(iii) $BC7$	$A_1,\ A_5,\ A_6,\ A_{13}\ A_{16}$
(iv) $BC8$	$A_1,\ A_5,\ A_{13}$
(v) $BC9$	$A_5,\ A_6,\ A_{16}$

Proof. (i) The eigenvalues of $BC5$ are 3^1, 2^1, 1.41^2, 1^1, 0^2, -1^1, -1.41^2, -2^1 and -3 [17]. Thus, the eigenvalues of $L(BC5)$ are 4^1, 3^1, 2.41^2, 2^1, 1^2, 0^1, -0.41^2, -1^1 and -2^7. According to the mentioned conditions and Theorem 2.1, a parameter matrix of a perfect 3-coloring of $L(BC5)$ may be one of the following matrices:

A_{20}	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (2, 4, 12)$
A_6	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (2, 8, 8)$
A_1	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 3, 12)$
$A_8,\ A_{17}$	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 6, 9)$
$A_5,\ A_{10},\ A_{13},\ A_{15},\ A_{16},\ A_{19}$	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (6, 6, 6)$

Define the mappings $T_{5,1}, T_{5,5}, T_{5,6}, T_{5,13} : V_{L(BC5)} \to \{0, 1, 2\}$ as follows:

$T_{5,1}(v_i) = 0$ if $i = 3, 6, 10$, $T_{5,1}(v_i) = 1$ if $i = 8, 11, 14$,
$T_{5,1}(v_i) = 2$ otherwise,

$T_{5,5}(v_i) = 0$ if $i = 3, 7, 10, 11, 14, 15$, $T_{5,5}(v_i) = 1$ if $i = 2, 5, 9, 13, 16, 18$,
$T_{5,5}(v_i) = 2$ otherwise,

$T_{5,6}(v_i) = 0$ if $i = 3, 8$, $T_{5,6}(v_i) = 1$ if $i = 2, 4, 7, 9, 12, 13, 15, 16$,
$T_{5,6}(v_i) = 2$ otherwise,

$T_{5,13}(v_i) = 0$ if $i = 1, 2, 4, 5, 15, 16$, $T_{5,13}(v_i) = 1$ if $i = 7, 9, 12, 13, 17, 18$,
$T_{5,13}(v_i) = 2$ otherwise.

It is easy to see that $T_{5,1}$, $T_{5,5}$, $T_{5,6}$ and $T_{5,13}$ are the perfect 3-colorings of $L(BC5)$ with the parameter matrices A_1, A_5, A_6 and A_{13}, respectively.

It suffices to show that there is no perfect 3-coloring of $L(BC5)$ with the parameter matrix $A_8, A_{10}, A_{15}, A_{16}, A_{17}, A_{19}$ and A_{20}. Let $T : V_{L(BC5)} \to \{0, 1, 2\}$ be a perfect 3-coloring of $L(BC5)$. Then

if A_8 is the parameter matrix of T then the subgraph induced by white vertices is a 1-regular graph of order 3, a contradiction,

if A_{20} is the parameter matrix of T then the subgraph induced by white vertices is a cubic graph of order 2, a contradiction.

Now, suppose that A_{10} is the parameter matrix of T. Then the subgraph induced by white vertices is a 1-regular graph of order 6, i.e., it is isomorphic to the disjoint union $3C_2$. If we color all vertices of this subgraph by white, then there exists a red vertex such that it has not two white neighbors, a contradiction.

Suppose that A_{15} or A_{16} is the parameter matrix of T. Then the subgraph induced by white vertices is a 2-regular graph of order 6, i.e., it is isomorphic to the cycle C_6 or the disjoint union $2C_3$. If we color all vertices of these subgraphs by white, then there exists a black or red vertex with no white neighbor, a contradiction.

Suppose that A_{17} is the parameter matrix of T. Without loss of generality, assume that $T(v_1) = 1$. Note that the induced subgraph

$L(BC5)[\{v_1\} \cup N(v_1)]$ is isomorphic to the friendship graph F_2. Thus, there is a red vertex with the white neighbor, a contradiction.

Finally, suppose that A_{19} is the parameter matrix of T. Without loss of generality, assume that $T(v_1) = 0$. Note that the induced subgraph $L(BC5)[\{v_1\} \cup N(v_1)]$ is isomorphic to the friendship graph F_2. Thus, there is a red vertex with two white neighbors, a contradiction.

(ii) The eigenvalues of $BC6$ are 3^1, 2.24^1, 1.41^2, 0^4, -1.41^2, -2.24^1 and -3^1 [17]. Thus, the eigenvalues of $L(BC6)$ are 4^1, 3.24^1, 2.41^2, 1^4, -0.41^2, -1.24^1 and -2^7. According to the mentioned conditions and Theorem 2.1, a parameter matrix of a perfect 3-coloring of $L(BC6)$ may be one of the following matrices:

A_6	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (2, 8, 8)$
A_8	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 6, 9)$
A_5, A_{16}	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (6, 6, 6)$

Define the mappings $T_{6,5}, T_{6,6}, T_{6,16} : V_{L(BC6)} \to \{0, 1, 2\}$ as follows:

$T_{6,5}(v_i) = 0$ if $i = 1, 3, 5, 7, 12, 15$, $T_{6,5}(v_i) = 1$ if $i = 2, 9, 13, 14, 17, 18$, $T_{6,5}(v_i) = 2$ otherwise,

$T_{6,6}(v_i) = 0$ if $i = 3, 8$, $T_{6,6}(v_i) = 1$ if $i = 2, 4, 7, 9, 14, 15, 16, 18$, $T_{6,6}(v_i) = 2$ otherwise,

$T_{6,16}(v_i) = 0$ if $i = 1, 5, 6, 9, 10, 18$, $T_{6,16}(v_i) = 1$ if $i = 11, 12, 13, 15, 16, 17$, $T_{6,16}(v_i) = 2$ otherwise.

It is easy to see that $T_{6,5}$, $T_{6,6}$ and $T_{6,16}$ are the perfect 3-colorings of $L(BC6)$ with the parameter matrices A_5, A_6 and A_{16}, respectively.

It suffices to show that there is no perfect 3-coloring of $L(BC6)$ with the parameter matrix A_8. Let $T : V_{L(BC6)} \to \{0, 1, 2\}$ be a perfect 3-coloring of $L(BC6)$. If A_8 is the parameter matrix of T then the subgraph induced by white vertices is a 1-regular graph of order 3, a contradiction.

(iii) The eigenvalues of $BC7$ are 3^1, 2^2, 1^1, 0^4, -1^1, -2^2, and -3^1 [17]. Thus, the eigenvalues of $L(BC7)$ are 4^1, 3^2, 2^1, 1^4, 0^1, -1^2 and -2^7. According to the mentioned conditions and Theorem 2.1, a parameter matrix of a perfect 3-coloring of $L(BC7)$ may be one of the following matrices:

A_{20}	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (2, 4, 12)$
A_6	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (2, 8, 8)$
A_1	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 3, 12)$

$A_8,\ A_{17}$	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 6, 9)$
$A_5,\ A_{10},\ A_{13},\ A_{15},\ A_{16},\ A_{19}$	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (6, 6, 6)$

Define the mappings $T_{7,1}, T_{7,5}, T_{7,6}, T_{7,13}, T_{7,16} : V_{L(BC7)} \to \{0, 1, 2\}$ as follows:

$T_{7,1}(v_i) = 0$ if $i = 1, 5, 9,$ $T_{7,1}(v_i) = 1$ if $i = 3, 7, 11,$
$T_{7,1}(v_i) = 2$ otherwise,

$T_{7,5}(v_i) = 0$ if $i = 1, 3, 5, 7, 9, 11,$ $T_{7,5}(v_i) = 1$ if $i = 4, 8, 12, 13, 15, 17,$
$T_{7,5}(v_i) = 2$ otherwise,

$T_{7,6}(v_i) = 0$ if $i = 1, 7,$ $T_{7,6}(v_i) = 1$ if $i = 2, 6, 8, 12, 13, 14, 16, 17,$
$T_{7,6}(v_i) = 2$ otherwise,

$T_{7,13}(v_i) = 0$ if $i = 2, 4, 12, 16, 17, 18,$ $T_{7,13}(v_i) = 1$ if $i = 6, 8, 10, 13, 14, 15,$
$T_{7,13}(v_i) = 2$ otherwise,

$T_{7,16}(v_i) = 0$ if $i = 1, 2, 6, 7, 8, 12,$ $T_{7,16}(v_i) = 1$ if $i = 4, 5, 10, 11, 13, 16,$
$T_{7,16}(v_i) = 2$ otherwise.

It is easy to see that $T_{7,1}$, $T_{7,5}$, $T_{7,6}$, $T_{7,13}$ and $T_{7,16}$ are the perfect 3-colorings of $L(BC7)$ with the parameter matrices A_1, A_5, A_6, A_{13} and A_{16}, respectively.

It suffices to show that there is no perfect 3-coloring of $L(BC7)$ with the parameter matrix $A_8, A_{10}, A_{15}, A_{17}, A_{19}$ and A_{20}. Let $T : V_{L(BC7)} \to \{0, 1, 2\}$ be a perfect 3-coloring of $L(BC7)$. Then

if A_8 is the parameter matrix of T then the subgraph induced by white vertices is a 1-regular graph of order 3, a contradiction,

if A_{20} is the parameter matrix of T then the subgraph induced by white vertices is a cubic graph of order 2, a contradiction.

Now, suppose that A_{10} is the parameter matrix of T. Then the subgraph induced by white vertices is a 1-regular graph of order 6, i.e., it is isomorphic to the disjoint union $3C_2$. If we color all vertices of this subgraph by white, then there exists a red vertex such that it has not two white neighbors, a contradiction.

Suppose that A_{15} is the parameter matrix of T. Then the subgraph induced by white vertices is a 2-regular graph of order 6, i.e., it is isomorphic to the cycle C_6 or the disjoint union $2C_3$. If we color all vertices of these subgraphs by white, then there exists a black or red vertex with no white neighbor, a contradiction.

Suppose that A_{17} is the parameter matrix of T. Without loss of generality, assume that $T(v_1) = 1$. Note that the induced subgraph $L(BC7)[\{v_1\} \cup N(v_1)]$ is isomorphic to the friendship graph F_2. Thus, there is a red vertex with the white neighbor, a contradiction.

Finally, suppose that A_{19} is the parameter matrix of T. Without loss of generality, assume that $T(v_1) = 0$. Note that the induced subgraph $L(BC7)[\{v_1\} \cup N(v_1)]$ is isomorphic to the friendship graph F_2. Thus, there is a red vertex with two white neighbors, a contradiction.

(iv) The eigenvalues of $BC8$ are 3^1, 1.73^2, 1^3, -1^3, -1.73^2 and -3 [17]. Thus, the eigenvalues of $L(BC8)$ are 4^1, 2.73^2, 2^3, 0^3, -0.73^2 and -2^7. According to the mentioned conditions and Theorem 2.1, a parameter matrix of a perfect 3-coloring of $L(BC8)$ may be one of the following matrices:

A_1	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 3, 12)$
A_{17}	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 6, 9)$
$A_5,\ A_{13},\ A_{18}$	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (6, 6, 6)$

Define the mappings $T_{8,1}, T_{8,5}, T_{8,13} : V_{L(BC8)} \to \{0, 1, 2\}$ as follows:

$T_{8,1}(v_i) = 0$ if $i = 2, 6, 9,$ $T_{8,1}(v_i) = 1$ if $i = 4, 14, 18,$
$T_{8,1}(v_i) = 2$ otherwise,

$T_{8,5}(v_i) = 0$ if $i = 2, 4, 6, 9, 14, 18,$ $T_{8,5}(v_i) = 1$ if $i = 1, 8, 12, 13, 15, 16,$
$T_{8,5}(v_i) = 2$ otherwise,

$T_{8,13}(v_i) = 0$ if $i = 1, 2, 3, 7, 15, 18,$ $T_{8,13}(v_i) = 1$ if $i = 5, 9, 13, 14, 16, 17,$
$T_{8,13}(v_i) = 2$ otherwise.

It is easy to see that $T_{8,1}$, $T_{8,5}$ and $T_{8,13}$ are the perfect 3-colorings of $L(BC8)$ with the parameter matrices A_1, A_5 and A_{13}, respectively.

It suffices to show that there is no perfect 3-coloring of $L(BC8)$ with the parameter matrix A_{17} and A_{18}. Let $T : V_{L(BC8)} \to \{0, 1, 2\}$ be a perfect 3-coloring of $L(BC8)$. First, suppose that A_{17} is the parameter matrix of T. Without loss of generality, assume that $T(v_1) = 1$. Note that the induced subgraph $L(BC8)[\{v_1\} \cup N(v_1)]$ is isomorphic to the friendship graph F_2. Thus, there is a red vertex with the white neighbor, a contradiction.

Finally, suppose that A_{18} is the parameter matrix of T. Without loss of generality, assume that $T(v_1) = 0$. Note that the induced subgraph $L(BC8)[\{v_1\} \cup N(v_1)]$ is isomorphic to the friendship graph F_2. Thus, there is a red vertex with two white neighbors, a contradiction.

(v) The eigenvalues of $BC9$ are $3^1, 2.56^1, 1.56^1, 0^6, -1.56^1, -2.56^1$ and -3^1 [17]. Thus, the eigenvalues of $L(BC9)$ are $4^1, 3.56^1, 2.56^1, 1^6, -0.56^1$, -1.56^1 and -2^7. According to the mentioned conditions and Theorem 2.1, a parameter matrix of a perfect 3-coloring of $L(BC9)$ may be one of the following matrices:

A_6	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (2, 8, 8)$
A_8	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (3, 6, 9)$
$A_5,\ A_{16}$	$(\lvert W\rvert, \lvert B\rvert, \lvert R\rvert) = (6, 6, 6)$

Define the mappings $T_{9,5}, T_{9,6}, T_{9,16} : V_{L(BC9)} \to \{0, 1, 2\}$ as follows:

$T_{9,5}(v_i) = 0$ if $i = 1, 3, 5, 10, 13, 18,$ $T_{9,5}(v_i) = 1$ if $i = 2, 4, 6, 11, 14, 17,$
$T_{9,5}(v_i) = 2$ otherwise,

$T_{9,6}(v_i) = 0$ if $i = 1, 10,$ $T_{9,6}(v_i) = 1$ if $i = 2, 6, 7, 9, 11, 14, 15, 16,$

$T_{9,6}(v_i) = 2$ otherwise,

$T_{9,16}(v_i) = 0$ if $i = 1, 2, 9, 10, 11, 16$, $T_{9,16}(v_i) = 1$ if $i = 3, 4, 7, 12, 13, 14$,
$T_{9,16}(v_i) = 2$ otherwise.

It is easy to see that $T_{9,5}$, $T_{9,6}$ and $T_{9,16}$ are the perfect 3-colorings of $L(BC9)$ with the parameter matrices A_5, A_6 and A_{16}, respectively.

It suffices to show that there is no perfect 3-coloring of $L(BC9)$ with the parameter matrix A_8. Let $T : V_{L(BC9)} \to \{0, 1, 2\}$ be a perfect 3-coloring of $L(BC9)$. Then, if A_8 is the parameter matrix of T then the subgraph induced by white vertices is a 1-regular graph of order 3, a contradiction. Therefore, the proof is complete. □

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