Journal of Algebra and Related Topics
Vol. 12, No 02, (2024), pp 75-86

DERIVATIONS MAPPING INTO THE JACOBSON RADICAL OF A BANACH ALGEBRA

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Abstract. Let $\mathcal{A}$ be a Banach algebra with Jacobson radical $Rad_{\mathcal{A}}$ and d a continuous derivation of $\mathcal{A}$. The purpose of this article is to investigate some sufficient conditions under which $d(\mathcal{A}) \subseteq Rad_{\mathcal{A}}$ from a topological point of view. Interesting results are established with some applications.

1. Introduction

Throughout this paper, $\mathcal{R}$ denotes an associative unitary ring with unit 1. For any $x, y \in \mathcal{R}$, as usual $[x, y] = xy - yx$ will denote the well-known Lie product. The symbol $Inv(\mathcal{R})$ stand for the set of all units elements of $\mathcal{R}$. The commutant of $a \in \mathcal{R}$ is defined by $\mathrm{comm}_{\mathcal{R}}(a) = \{x \in \mathcal{R} \mid [a, x] = 0\}$. The intersection of all maximal left (right) ideals of $\mathcal{R}$ is said to be the left (right) Jacobson radical of $\mathcal{R}$, which is denoted by $Rad_{\mathcal{R}}$. In particular, when $Rad_{\mathcal{R}} = \{0\}$, the ring $\mathcal{R}$ is said to be semi-simple. As is well known, $Rad_{\mathcal{R}} = \{a \in \mathcal{R} \mid 1 - ax \in Inv(\mathcal{R}) \text{ for all } x \in \mathcal{R}\}$. Due to Harte [5], an element $a \in \mathcal{R}$ is called quasinilpotent if $1 - ax \in Inv(\mathcal{R})$ for every x in $\mathrm{comm}_{\mathcal{R}}(a)$, the set of all quasinilpotents of $\mathcal{R}$ is designated by $Q_{\mathcal{R}}$, it is clear that $Rad_{\mathcal{R}} \subset Q_{\mathcal{R}}$. Recall that a ring $\mathcal{R}$ is local [6] if $\mathcal{R} = Q_{\mathcal{R}} \cup Inv(\mathcal{R})$. An additive mapping $d : \mathcal{R} \longrightarrow \mathcal{R}$ is a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. In particular, for a fixed $a \in \mathcal{R}$, the mapping $d_a : \mathcal{R} \to \mathcal{R}$ given by $d_a(x) = [a, x]$ for all $x \in \mathcal{R}$ is a derivation called the inner derivation of $\mathcal{R}$ associated to a.

MSC(2010): Primary: 13N15; 46E25; 16N204.
Keywords: Derivation, Jacobson radical, Banach algebras.
Received: 10 September 2021, Accepted: 12 April 2024.
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When $\mathcal{A}$ is a unitary Banach algebra with norm $\|.\|$ and center $Z(\mathcal{A})$, then the set $Inv(\mathcal{A})$ is an open subset of $\mathcal{A}$ and $Z(\mathcal{A})$ is a non-void closed subset. According to [5], the subset $Q_{\mathcal{A}}$ is defined as follows : $Q_{\mathcal{A}} = \{a \in \mathcal{A} \mid \lim_{n \to +\infty} \|a^n\|^{\frac{1}{n}} = 0\}$.
In the literature, many authors have introduced sufficient conditions for that a derivation on a Banach algebra maps it in its Jacobson radical, with an algebraic approach. In 1955 Singer and Wermer [8] proved a classical theorem of Banach algebra theory, which says that, every continuous linear derivation on a commutative Banach algebra maps this algebra into its radical and in 1988 Thomas [9] proved that if d is a derivation in a Banach algebra $\mathcal{A}$ such that $[d(x), x] \in Z(\mathcal{A})\ \forall x \in \mathcal{A}$, then d has its range in the radical of $\mathcal{A}$. (This generalization is called the Singer-Wermer conjecture).

In 1992 Mathieu and Runde [7] generalized the Singer-Wermer conjecture by proving that every centralizing derivation on a Banach algebra maps an into its Jacobson radical. On the other hand, Bresar[1] in 1994 has also proved that, if d is a bounded derivation of a Banach algebra $\mathcal{A}$, such that $[d(x), x] \in Q_{\mathcal{A}}$ for every $x \in \mathcal{A}$, then d maps $\mathcal{A}$ into $Rad_{\mathcal{A}}$.

Motivated by this, the main objective in writing this article has been to present some sufficient conditions to conclude a similar result as Bresar's theorem [1], but with other local identities from topological concepts.

We started this paper by presenting some results, from which we have been inspired in this article. Next, we introduced the proof of main theory by using the Baire's theorem and some properties of functional analysis, then we mentioned some immediate results of it. After we presented two applications by which we have deduced that any derivation in a commutative Banach algebra is non-surjective and that his restriction defined on arbitrary Banach algebra on its center is not sujective. Finally, by a counterexample, we proved that the main theorems are false, if we replace derivation by a homomorphism.

We now state the results which present the motivation of this article.

Lemma 1.1. [[1], *Bresar's theorem*] *Let d be a bounded derivation of a Banach algebra $\mathcal{A}$. Suppose that $[d(x), x] \in Q_{\mathcal{A}}$ for every $x \in \mathcal{A}$. Then d maps $\mathcal{A}$ into $Rad_{\mathcal{A}}$.*

Lemma 1.2. [[2], *Theorem 1*] *Let d and g be continuous derivations of a Banach algebra $\mathcal{A}$ such that $[d^2(x)+g(x),x]\in Rad_{\mathcal{A}}$ for all x in $\mathcal{A}$. Then both d and g map $\mathcal{A}$ into $Rad_{\mathcal{A}}$.*

Lemma 1.3. [[2], *Theorem 3*] *Let d be a continuous derivation on a Banach algebra $\mathcal{A}$. If $[d(x),x]^2\in Rad_{\mathcal{A}}$ for all x in $\mathcal{A}$, then d maps $\mathcal{A}$ into $Rad_{\mathcal{A}}$.*

2. Main results

Throughout this section, $\mathcal{A}$ denotes an unital Banach algebra with norm $\|.\|$ and with center $Z(\mathcal{A})$.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{O}_1$ and $\mathcal{O}_2$ two non-void open subsets of $\mathcal{A}$ and M a closed subspace of $\mathcal{A}$ included in $Q_{\mathcal{A}}$. If $\mathcal{A}$ admits a continuous linear derivation d, satisfying*

$$(\forall(x,y)\in\mathcal{O}_1\times\mathcal{O}_2)(\exists(p,q)\in\mathbb{N}^{*2})\ :\ [d(x^p),y^q]\in M.$$

Then $d(\mathcal{A})\subset Rad_{\mathcal{A}}$.

Proof. For all $(p,q)\in{\mathbb{N}^*}^2$, we define the following sets:

$$O_{p,q}=\{(x,y)\in\mathcal{A}^2\mid[d(x^p),y^q]\notin M\}$$

and

$$F_{p,q}=\{(x,y)\in\mathcal{A}^2\mid[d(x^p),y^q]\in M\}.$$

We observe that $(\cap O_{p,q})\cap(\mathcal{O}_1\times\mathcal{O}_2)=\varnothing$.
Now we claim that each $O_{p,q}$ is open in $\mathcal{A}\times\mathcal{A}$. That is, we have to show that $F_{p,q}$ the complement of $O_{p,q}$ is closed. For this, we consider a sequence $((x_k,y_k))_{k\in\mathbb{N}}\subset F_{p,q}$ converge to $(x,y)\in\mathcal{A}\times\mathcal{A}$. Since $((x_k,y_k))_{k\in\mathbb{N}}\subset F_{p,q}$, so

$$[d((x_k)^p),(y_k)^q]\in M\text{ for all }k\in\mathbb{N}.$$

Since d is continuous, we conclude that the sequence $([d((x_k)^p),(y_k)^q])_{k\in\mathbb{N}}$ converges to $[d(x^p),y^q]$, knowing that M is closed, then $[d(x^p),y^q]\in M$. Therefore $(x,y)\in F_{p,q}$ and $F_{p,q}$ is closed (i.e $O_{p,q}$ is open).
If every $O_{p,q}$ is dense, we know that their intersection is also dense by Baire category theorem, which contradict with of $(\cap O_{p,q})\cap(\mathcal{O}_1\times\mathcal{O}_2)=\varnothing$. Hence, there is $(n,m)\in\mathbb{N}^{*2}$ such that $O_{n,m}$ is not a dense set and there exists a nonvoid open subset $\mathcal{H}_1\times\mathcal{H}_2$ in $F_{n,m}$ such that :

$$[d(x^n),y^m]\in M\text{ for all }(x,y)\in\mathcal{H}_1\times\mathcal{H}_2.$$

Fix $y\in\mathcal{H}_2$. Let $x\in\mathcal{H}_1$ and $z\in\mathcal{A}$, we have $x+tz\in\mathcal{H}_1$ for all sufficiently small real t, therefore

$$Q(t)=[d((x+tz)^n),y^m]\in M.$$

Since d is continuous, can be written as

$$Q(t) = A_0 + tA_1 + t^2A_2 + \cdots + t^nA_n.$$

For a $t \neq 0$, we can write

$$Q(t) = A_0 + tA_1 + t^2A_2 + \cdots + t^nA_n.$$

While $A_0 + tA_1 + t^2A_2 + \cdots + t^nA_n \in M$ and $A_0 = [d(x^n), y^m] \in M$, we conclude that:

$$tA_1 + t^2A_2 + \cdots + t^nA_n \in M.$$

Multiplying by t^{-1} (M is a subspace of $\mathcal{A}$), we obtain

$$R(t) = A_1 + tA_2 + ... + t^{n-1}A_n \in M,$$

and take t to zero (R is continuous at 0 and M is closed), we get $A_1 \in M$, we conclude that

$$tA_2 + \cdots + t^{n-1}A_n \in M.$$

Multiplying by t^{-1} and take t to zero, we have $A_2 \in M$. And so on, we conclude that $A_k \in M$ for all $1 \leq k \leq n$. For $k = n$, we have $A_n = [d(z^n), y^m] \in M$.
Consequently, $[d(x^n), y^m] \in M$ for all $x \in \mathcal{A}$. Now, fix x in $\mathcal{A}$, proceeding in the same way, we find that either $[d(x^n), y^m] \in M$ for all $(x, y) \in \mathcal{A}^2$.
Let $(x, y) \in \mathcal{A}^2$, we have $[d((tx + 1)^n), y^m] \in M$ for all $t \in \mathbb{R}$. Since d is continuous, we can write

$$[d((tx+1)^n), y^m] = \sum_{k=0}^{n} \binom{n}{k} t^k [d((x^k.1^{n-k})), y^m] \in M.$$

The first term in this polynomial is $[d(1^n), y^m] = 0$ who belongs to M, we prove as proceeding, we conclude that $\binom{n}{k}[d(1^{n-k}x^k), y^m] \in M$ for all $0 \leq k \leq n$, in particular for $k = 1$, we have $n[d(x), y^m] \in M$, we can simplify by n (because M is a subspace of $\mathcal{A}$), therefore $[d(x), y^m] \in M$. Proceeding in the same way, we have $[d(x), y] \in M$ for all $(x, y) \in \mathcal{A}^2$, and $[d(x), y] \in M$ for all $(x, y) \in \mathcal{A}^2$.
In particular for $y = x$, we have $[d(x), x] \in M \subset Q_{\mathcal{A}}$ for all $x \in \mathcal{A}$, by Lemma 1.1, we conclude that $d(\mathcal{A}) \subset Rad_{\mathcal{A}}$. □

Remark 2.2. If $Q_{\mathcal{A}} = \{0\}$ or $\mathcal{A}$ is a semi-simple Banach algebra, in the hypotheses of Theorem 2.1, then $d = 0$.

We immediately get the following corollary from the above Theorem 2.1.

Corollary 2.3. *Let $\mathcal{A}$ be a local Banach algebra, $\mathcal{O}_1$ and $\mathcal{O}_2$ are two non-void open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous linear derivation d, satisfying :*

$$(\forall (x,y) \in \mathcal{O}_1 \times \mathcal{O}_2)(\exists (p,q) \in \mathbb{N}^{*2}) \ : \ [d(x^p), y^q] \in Q_{\mathcal{A}},$$

then $d(\mathcal{A}) \subset Rad_{\mathcal{A}}$.

Proof. Since $\mathcal{A}$ is local, then $\mathcal{A} = Q_{\mathcal{A}} \cup Inv(\mathcal{A})$. According to [[6], Lemma 3.1] , we have $Q_{\mathcal{A}} \cap Inv(\mathcal{A}) = \varnothing$, we conclude that $Inv(\mathcal{A})$ is the complement of $Q_{\mathcal{A}}$ in $\mathcal{A}$. Since $Inv(\mathcal{A})$ is open , $Q_{\mathcal{A}}$ is closed, if we take $M = Q_{\mathcal{A}}$ in the hypotheses of the Theorem 2.1, we obtain the result. □

Corollary 2.4. *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{O}_1$ and $\mathcal{O}_2$ are two non-void open subsets of $\mathcal{A}$ and M is a closed subspace of $\mathcal{A}$ included in $Q_{\mathcal{A}}$. If $\mathcal{A}$ admits a continuous derivation d, satisfying :*

$$(\forall (x,y) \in \mathcal{O}_1 \times \mathcal{O}_2)(\exists (p,q) \in \mathbb{N}^{*2}) \ : \ [d(x^p), y^q] \in M,$$

then d is necessarily not surjective.

Proof. According to Theorem 2.1, we have $d(\mathcal{A}) \subset Rad_{\mathcal{A}}$. If d is surjective, then $\mathcal{A} = Rad_{\mathcal{A}} = Q_{\mathcal{A}}$, but $1 \notin Q_{\mathcal{A}}$. We conclude that d is not surjective. □

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{O}$ is a non-void open subset of $\mathcal{A}$ and M a closed subspace of $\mathcal{A}$ included in $Q_{\mathcal{A}}$. If $\mathcal{A}$ admits a continuous linear derivation d, satisfying*

$$(\forall x \in \mathcal{G})(\exists (p,q) \in \mathbb{N}^{*2}) \ : \ [d(x^p), x^q] \in M,$$

then $d(\mathcal{A}) \subset Rad_{\mathcal{A}}$.

Proof. For all $(p,q) \in {\mathbb{N}^*}^2$, we define the following sets:

$$O_{p,q} = \{x \in \mathcal{A} \mid [d(x^p), x^q] \notin M\}$$

and

$$F_{p,q} = \{x \in \mathcal{A} \mid [d(x^p), x^q] \in M\}.$$

Proceeding in the same way as Theorem 2.1, we conclude that, there is $(n,m) \in \mathbb{N}^2$ and a non-void open subset $\mathcal{H}$ of $F_{n;m}$ such that :

$$[d(x^n), x^m] \in M \quad (\forall x \in \mathcal{H}).$$

Let $x \in \mathcal{H}$ and $y \in \mathcal{A}$, we have $x + ty \in \mathcal{H}$ for all sufficiently small real t, therefore

$$Q(t) = [d((x+ty)^n), (x+ty)^m] \in M.$$

Since d is continuous, can be written as

$$Q(t)=\sum_{k=0}^{k=n}\sum_{l=0}^{l=m}t^{k+l}[d(B_k(x,y)),A_l(x,y)]=\sum_{p=0}^{n+m}t^pC_p,$$

where $C_p=\sum_{k+l=p}[d(B_k(x,y)),A_l(x,y)]$. Therefore, the first coefficient in this polynomial is $C_0=[d(B_0(x,y));A_0(x,y)]=[d(x^n);x^m]\in M$ and last coefficient is $C_{n+m}=[d(y^n);y^m]$.
For a $t\neq 0$, we write

$$Q(t)=C_0+tC_1+t^2C_2+\cdots+t^{n+m}C_{n+m}.$$

Since M is a subspace of $\mathcal{A}$, we have

$$tC_1+t^2C_2+\cdots+t^{n+m}C_{n+m}\in M.$$

Multiplying by t^{-1} (M is a subspace of $\mathcal{A}$), we obtain

$$R(t)=C_1+tC_2+\cdots+t^{n+m-1}C_{n+m}\in M,$$

and take t to zero (R is continuous at 0 and M is closed), we get $C_1\in M$, we conclude that

$$tC_2+\cdots+t^{n+m-1}C_{n+n}\in M.$$

Multiplying by t^{-1} and take t to zero, we have $C_2\in M$. And so on, we have $C_p\in M$ for all $0\leq p\leq n+m$. In particular for $p=n+m$, we get $C_{n+m}=[d(y^n),y^m]\in M$. And from him $[d(y^n),y^m]\in M$ for all $y\in\mathcal{A}$. Let $x\in\mathcal{A}$ and a be a non-zero element of $Z(\mathcal{A})$, for all $t\in\mathbb{R}$, we have $S(t)=[d((a+tx)^n),(a+tx)^m]\in M$.
We can write $S(t)=\sum_{k=0}^{n}\sum_{p=0}^{m}\binom{k}{n}\binom{l}{m}t^{k+l}[d(a^{n-k}x^k),a^{m-l}x^l]\in M.$
By following the same steps of the proof above, we conclude that $\binom{k}{n}\binom{l}{m}t^{k+l}[d(a^{n-k}x^k),a^{m-l}x^l]\in M$, for all
$0\leq k\leq n$ and for all $0\leq l\leq m$. The coefficient of t^2 in this polynomial is

$$\sum_{k+l=2}\binom{k}{n}\binom{l}{m}[d(a^{n-k}x^k),a^{m-l}x^l]\in M.$$

We can write

$$\binom{0}{n}\binom{2}{m}[d(a^nx^0),a^{m-2}x^2]+\binom{1}{n}\binom{1}{m}[d(a^{n-1}x^1),a^{m-1}x^1]+$$

$$\binom{2}{n}\binom{0}{m}[d(a^{n-2}x^2),a^mx^0]\in M.$$

Since a is arbitrary in $Z(\mathcal{A})$, we take $a = 1$ and we get

$$\binom{0}{n}\binom{2}{m}[d(1^n x^0), 1^{m-2}x^2] + \binom{1}{n}\binom{1}{m}[d(1^{n-1}x^1), 1^{m-1}x^1] +$$

$$\binom{2}{n}\binom{0}{m}[d(1^{n-2}x^2), 1^m x^0] \in M.$$

As $d(1) = 0$ and $1 \in Z(\mathcal{A})$, consequently

$$\binom{1}{n}\binom{1}{m}[d(x), x] \in M.$$

Since M is a subspace of $\mathcal{A}$, we obtain

$$[d(x), x] \in M.$$

We conclude that

$$[d(x), x] \in Q(\mathcal{A}) \ (\forall x \in \mathcal{A}).$$

According to Lemma 1.1, we have $d(\mathcal{A}) \subset Rad_{\mathcal{A}}$. □

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{O}$ a non-void open subset of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous derivation d satisfying for all $x \in \mathcal{H}$, there is $n \in \mathbb{N}^*$ such that $d(x^n) \in Z(\mathcal{A})$, then $d(\mathcal{A}) \subseteq Rad_{\mathcal{A}}$.*

Proof. It resembles to the proof of Theorem 2.1, we conclude that there is $m \in \mathbb{N}^*$ and a non-void open subset $\mathcal{O}$ in $\mathcal{A}$ such that $d(x^m) \in Z(\mathcal{A})$ for all $x \in \mathcal{O}$.
Let $x \in \mathcal{O}$ and $y \in \mathcal{A}$, then $x + ty \in \mathcal{O}$ for all sufficiently small real t, then $P(t) = d((x+ty)^n) = \sum_{k=0}^{m} t^k A_k = A_0 + tA_1 + ... + t^m A_m \in Z(\mathcal{A})$ (because d is continuous), while the first term in this polynomial $A_0 = d(x^m)$ who belongs to $Z(\mathcal{A})$, we conclude that:

$$tA_1 + t^2 A_2 + \cdots + t^m A_m \in Z(\mathcal{A}).$$

We can simplify by t (*because $Z(\mathcal{A})$ is a subspace of $\mathcal{A}$*), we obtain

$$Q(t) = A_1 + tA_2 \cdots + t^{m-1} A_m \in Z(\mathcal{A}),$$

and take limit to zero (Q is continuous at 0 and $Z(\mathcal{A})$ is closed), we get $A_1 \in Z(\mathcal{A})$, and we obtain

$$t^2 A_2 + \cdots + t^n A_n \in Z(\mathcal{A}).$$

We can further simplify by t^2 and take limit to zero, we get $A_2 \in Z(\mathcal{A})$. And so on we get $A_k \in Z(\mathcal{A})$ for all $k \leq n$. we conclude that $A_m(x, y) = d(y^m) \in Z(\mathcal{A})$. Consequently $d(y^m) \in Z(\mathcal{A})$ for all $y \in \mathcal{A}$.

Let $x \in \mathcal{A}$, we have $d((x+t.1)^m) \in Z(\mathcal{A})$ for all $t \in \mathbb{R}$. Since $1^k \in Z(\mathcal{A})$ $(\forall k \in \mathbb{N}^*)$, we can write

$$d((x+t.1)^m) = \sum_{k=0}^{m} \binom{n}{k} t^k d(x^k) \in Z(\mathcal{A}).$$

The first term in this polynomial is $d(1^n) = 0$ who belongs to $Z(\mathcal{A})$, as preceding we conclude that $\binom{m}{k} d(x^k) \in Z(\mathcal{X})$ for all $0 \leq k \leq m$, in particular for $k = 1$, we have $m.d(x) \in Z(\mathcal{A})$, we can simplify by m (because $Z(\mathcal{A})$ is a subspace of $\mathcal{A}$), so $d(x) \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$ and d is a centralizing derivation. By theorem of Mathieu and Runde [7] we conclude that $d(\mathcal{A}) \subseteq \mathrm{Rad}_{\mathcal{A}}$. □

Theorem 2.7. *Let d and g be continuous derivations of a Banach algebra $\mathcal{A}$ and $\mathcal{O}$ a non-void open subset of $\mathcal{A}$. If for all $x \in \mathcal{O}$, there is $n \in \mathbb{N}^*$ such that $[d^2(x)+g(x), x^n] \in M$ where M is a closed subspace of $\mathcal{A}$ included in $Q_{\mathcal{A}}$, then d maps $\mathcal{A}$ into $Rad_{\mathcal{A}}$.*

Proof. For all $p \in \mathbb{N}^*$, we define the following sets:

$$O_p = \{x \in \mathcal{A} \mid [d^2(x) + g(x), x^p] \notin M\}$$

and

$$F_p = \{x \in \mathcal{A} \mid [d^2(x) + g(x), x^p] \in M\}.$$

We will repeatedly use the same method as in the proof of Theorem 2.1, we conclude that, there is $n \in \mathbb{N}^*$ and a non-void open subset $\mathcal{H}$ such that

$$[d^2(x) + g(x), x^n] \in M \quad (\forall x \in \mathcal{H}).$$

Let $x \in \mathcal{H}$ and $y \in \mathcal{A}$, we have $x + ty \in \mathcal{H}$ for all sufficiently small real t, thus

$$Q(t) = [d^2(x+ty) + g(x+ty), (x+ty)^n] \in M.$$

If we write $(x+ty)^n = A_0 + tA_1 + t^2A_2 + \cdots + t^nA_n = \sum_{k=0}^{n} t^k A_k$, we obtain

$$Q(t) = [d^2(x+ty) + g(x+ty), A_0] + t[d^2(x+ty) + g(x+ty), A_1] + \cdots + t^n[d^2(x+ty) + g(x+ty), A_n].$$

Since $d^2(x+ty) = d^2(x) + td^2(y)$ and $g(x+ty) = g(x) + tg(y)$. We have

$$[d^2(x+ty)+g(x+ty), A_k]t^k = t^k[d^2(x)+g(x), A_k]+t^{k+1}[d^2(y)+g(y), A_k].$$

Consequently

$$Q(t) = B_0 + tB_1 + t^2B_2 + \cdots + t^nB_n + t^{n+1}B_{n+1} \in M,$$

where $B_0 = [d^2(x)+g(x), A_0] \in M$ and $B_{n+1} = [d^2(y)+g(y), A_n]$. As the proof of the precedent theorem, we arrive at $B_{n+1} = [d^2(y)+g(y), y^n] \in M$ for all $y \in \mathcal{A}$.
Now, let $t \in \mathbb{R}$ and $y \in \mathcal{A}$, we have

$$[d^2(1+ty) + g(1+ty), (1+ty)^n] \in M.$$

That is $[d^2(y) + g(y), (1+ty)^n] \in M$, which implies $\sum_{k=0}^{n} \binom{k}{n} t^k [d^2(y) + g(y), y^k] \in M$. Repeating the same thing as precisely, we arrive at $[d^2(y)+g(y), y] \in M$. According to Lemma 1.2, we conclude that d and g map $\mathcal{A}$ into $\mathrm{Rad}_{\mathcal{A}}$. □

Theorem 2.8. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{O}$ a non-void open subset of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous derivation d satisfying for all $x \in \mathcal{O}$, there is $(n,m) \in \mathbb{N}^{*2}$ such that $([d(x^n), x^m])^2 \in M$ where M is a closed subspace of $\mathcal{A}$ included in $Q_{\mathcal{A}}$, then d maps $\mathcal{A}$ into $Rad_{\mathcal{A}}$.*

Proof. For all $(p,q) \in \mathbb{N}^{*2}$, we define the following sets:

$$O_{p,q} = \{x \in \mathcal{A} / [d(x^p), x^q]^2 \notin M\}$$

and

$$F_{p,q} = \{x \in \mathcal{A} / [d(x^p), x^q]^2 \in M\}.$$

Proceeding in the same way as Theorem 2.1, we deduce that, there is $(n,m) \in \mathbb{N}^{*2}$ and a non-void open subset $\mathcal{H}$ of $F_{n,m}$ such that :

$$[d(x^n), x^m]^2 \in M \quad (\forall x \in \mathcal{H}).$$

Let $x \in \mathcal{H}$ and $y \in \mathcal{A}$, we have $x + ty \in \mathcal{H}$ for all sufficiently small real t, therefore

$$Q(t) = [d((x+ty)^n), (x+ty)^m]^2 \in M.$$

Since d is continuous, can be written as

$$Q(t) = (\sum_{k=0}^{k=n} \sum_{l=0}^{l=m} [d(B_k(x,y)), A_l(x,y)] t^{k+l})^2 = (\sum_{p=0}^{n+m} C_p t^p)^2,$$

where $C_p = \sum_{k+l=p} [d(B_k(x,y)), A_l(x,y)]$. The first coefficient in this polynomial is
$C_0^2 = [d(B_0(x,y)), A_0(x,y)]^2 = [d(x^n), x^m]^2 \in M$ and last coefficient is $C_{n+m}^2 = [d(y^n), y^m]^2$.
For a $t \neq 0$, we write

$$Q(t) = (C_0 + tC_1 + t^2C_2 + \cdots + t^{n+m}C_{n+m})^2 \in M.$$

We write $Q(t)$ in the form

$$Q(t) = K_0 + tK_1 + t^2K_2 + \cdots + K_{(n+m)^2}t^{(n+m)^2} \in M,$$

where $K_0 = (C_0)^2 \in M$ and $K_{(n+m)^2} = [d(y^n), y^m]^2$. Since M is a closed subspace of $\mathcal{A}$, we have

$$tK_1 + K_2t^2 + ... + t^{(n+m)^2}K_{(n+m)^2} \in M.$$

Multiplying by t^{-1} (M is a subspace of $\mathcal{A}$), we obtain

$$R(t) = K_1 + tK_2 + \cdots + t^{(n+m)^2-1}K_{(n+m)^2} \in M,$$

and take t to zero (R is continuous at 0 and M is closed), we get $K_1 \in M$, we conclude that

$$tK_2 + \cdots + t^{(n+m)^2-1}K_{(n+m)^2} \in M.$$

Multiplying by t^{-1} and put t to zero, we have $K_2 \in M$. And so on, we have $K_p \in M$ for all $0 \leq p \leq (n+m)^2$. In particular for $p = (n+m)^2$, we get $K_{(n+m)^2} = [d(y^n), y^m]^2 \in M$. And from him $[d(y^n), y^m]^2 \in M$ for all $y \in \mathcal{A}$. Let $x \in \mathcal{A}$ and a be a non-zero element of $Z(\mathcal{A})$, for all $t \in \mathbb{R}$, we have $S(t) = [d((a+tx)^n), (a+tx)^m]^2 \in M$. We can write $S(t) = (\sum_{k=0}^{n}\sum_{l=0}^{m} \binom{k}{n}\binom{l}{m} t^{k+l}[d(a^{n-k}x^k), a^{m-l}x^l])^2 \in M$. By following the same steps of the proof above, we conclude that,

$$(\sum_{k+l=2} \binom{k}{n}\binom{l}{m}[d(a^{n-k}x^k), a^{m-l}x^l])^2 \in M,$$

We can write

$$\Bigg(\binom{0}{n}\binom{2}{m}[d(a^n), a^{m-2}x^2] + \binom{1}{n}\binom{1}{m}[d(a^{n-1}x), a^{m-1}x]$$
$$+\binom{2}{n}\binom{0}{m}[d(a^{n-2}x^2), a^m]^2 \in M.$$

Since a is arbitrary in $Z(\mathcal{A})$, we take $a = 1$ and we get

$$(\binom{0}{n}\binom{2}{m}[d(1^nx^0), 1^{m-2}x^2] + \binom{1}{n}\binom{1}{m}[d(x), x] + \binom{2}{n}\binom{0}{m}[d(x^2), 1])^2 \in M.$$

Therefore

$$(\binom{1}{n}\binom{1}{m}[d(x), x])^2 \in M.$$

Since M is a subspace of $\mathcal{A}$, we obtain

$$[d(x), x]^2 \in M.$$

Finally,

$$[d(x), x]^2 \in Q(\mathcal{A}) \ (\forall x \in \mathcal{A}).$$

By Lemma 1.3, we have $d(\mathcal{A}) \subset Rad_{\mathcal{A}}$. □

Application 1. Let $\mathcal{A}$ be a commutative Banach algebra and $d : \mathcal{A} \to \mathcal{A}$ is a continuous linear derivation, then d is not-surjective. Indeed:
Since $\mathcal{A}$ is commutative, then $y^q \in Z(\mathcal{A})$ for all $(q, y) \in \mathbb{N}^* \times \mathcal{A}$.
Therefore, for all $(p, q) \in \mathbb{N}^{*2}$ we have:

$$[d(x^p), y^q] = 0 \quad \forall (x, y) \in \mathcal{A}^2.$$

According to Corollary 2.4; we conclude that, d is not surjective.

Application 2. Let $\mathcal{A}$ be a Banach algebra and $d : \mathcal{A} \to \mathcal{A}$ a continuous linear derivation, then the restriction of d to $Z(\mathcal{A})$ is not-surjective.
Indeed:
Let d_1 be the restriction of d to $Z(\mathcal{A})$. Since $d_1(Z(\mathcal{A})) \subset Z(\mathcal{A})$, and $Z(\mathcal{A})$ is a closed subspace of the Banach algebra $\mathcal{A}$, then $Z(\mathcal{A})$ is a Banach algebra and $d_1 : Z(\mathcal{A}) \to Z(\mathcal{A})$ is a continuous linear derivation.
According to Application 1 we conclude that, d_1 is not surjective.

The following example shows that the Theorem 2.1 is false, if we replace derivation by a Banach algebra homomorphism.

Example 2.9. Let $\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}$ endowed with usual matrix addition and multiplication and of norm defined by $\| \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \| = \max(| a | + | b |, | c |)$, is an unital Banach algebra with neutral element $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let G be the linear mapping defined by:

$$G\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Observe that $G\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^n\right) = \begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix} \in Z(\mathcal{A})$ for all $n \in \mathbb{N}^*$ and G is a continuous Banach algebra homomorphism.
It is easy to verify that for all $A, B \in \mathcal{A}$ and for all $n, m \in \mathbb{N}$: $[G(A^n), B^m] = 0 \in \{0\} \subset Q_{\mathcal{A}}$.
But $G(\mathcal{A}) \nsubseteq Rad_{\mathcal{A}}$, because $G(I) = I \notin Rad_{\mathcal{A}}$.

Acknowledgments

The author wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

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