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Solving a class of auto-convolution Volterra integral equations via differential transform method

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Abstract. The aim of this paper is to solve a class of auto-convolution Volterra integral equations by the well-known differential transform method. The analytic property of solution and convergence of the method under some assumptions are discussed and some illustrative examples are given to clarify the theoretical results, accuracy and performance of the proposed method.

Keywords: Volterra integral equations, Auto-convolution, Differential transform method, Analytic function.
AMS Subject Classification 2010: 45D05.

1 Introduction

The Differential Transform Method (DTM) is a semi-analytical numerical method used in numerical analysis for solving differential, integral, and integro-differential equations. It was first introduced by Zhou to solve linear and nonlinear initial value problems in electric circuits [27]. Since then, it has been developed by many authors for addressing various types of ordinary, partial, integral, and integro-differential equations. For example, it was extended in [3, 4] for solving systems of differential and differential-algebraic equations. Its applications have been further expanded to address various types of partial differential equations and one-dimensional Volterra integral and integro-differential equations in [2, 6, 10] and [1, 15]. The DTM has also been generalized to solve single-order and multi-order fractional differential equations, systems of conformable fractional differential equations, and conformable fractional partial differential equations in various studies [7–9, 16].

For solving a system of non-linear Volterra integro-differential equations with variable coefficients and two-dimensional integral equations it is investigated in [22–24]. A modification of this method is

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suggested for solving delay differential, integro-differential equations, and partial differential equations in [5, 14, 20]. Application of this method for stiff differential equations may be found in [19]. It is also extended for some applied problems [12, 17, 18, 21, 25].
The subject of this article is to investigate the application of the DTM to a class of generalized auto-convolution Volterra integral equations (AVIEs), i.e.

$$u(t) = g(t) + \int_0^t K(t,s)u(t-s)u(s)\,\mathrm{d}s, \quad t \in \mathbb{I} := [0,T], \tag{1}$$

where $T > 0$ is a real number, K and g are given smooth functions, and u represents the unknown function for which we aim to determine its power series.

Application of Eq. (1) is, e.g., in the identification of memory kernels in the theory of viscoelasticity or in the computation of certain special functions [26]. The existence and uniqueness of solution for this kind of equations may be found in [26]. Equations of the form (1) have been solved by the piecewise polynomial collocation method in [26] and by the barycentric rational quadrature method in [13]. More details about these equations and other solution methods may be found in literature.

The next sections of this paper are organized as follows. In Section 2, we recall DTM and its important properties. In Section 3, we present sufficient conditions for the existence and uniqueness of an analytic solution for Eq. (1) around the point $t = 0$. Formulation of the DTM on Eq. (1) will be described in the last part of Section 3. In Section 4, some illustrative examples are given to clarify the accuracy and convergence of the method. Finally, in Section 5, we provide a brief conclusion.

2 Preliminaries

In this section, we present the preliminary results concerning the DTM that will be used throughout the paper. Throughout this paper, the differential transform of any function is denoted by the corresponding capital letter of the function's name.

Definition 1. *Let the function $f : \mathbb{I} \to \mathbb{R}$ be differentiable of any order on the open interval $\mathbb{I} \subset \mathbb{R}$. Then for any positive integer n the differential transform of f at the point $t_0 \in \mathbb{I}$ is defined by*

$$F_n = \frac{1}{n!}\left[\frac{d^n f(t)}{dt^n}\right]_{t=t_0}, \tag{2}$$

and the inverse transform is defined by

$$f(t) = \sum_{n=0}^{\infty} F_n (t-t_0)^n. \tag{3}$$

From (2) and (3), we obtain

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{d^n f(t)}{dt^n}\right]_{t=t_0} (t-t_0)^n, \tag{4}$$

which is the Taylor series of $f(t)$ around the point t_0.

We summarize the main properties of the differential transform in Proposition 1.

Proposition 1 ([15]). *Let F_n, G_n, U_n, and V_n denote the differential transforms of the functions f, g, u and v around the point $t_0 \in \mathbb{I}$, respectively. Then*

a. *If $f(t) = cu(t) \pm dv(t)$ for some arbitrary constants c and d, then*

$$F_n = cU_n \pm dV_n.$$

b. *If $f(t) = u(t)v(t)$, then $F_n = \sum_{k=0}^{n} U_k V_{n-k}$.*

c. *If $f(t) = (t-t_0)^k$, then $F_n = \delta_{n,k}$, where $\delta_{n,k}$ denotes the Kronecker delta function.*

d. *If $f(t) = t^k$, then $F_n = \begin{cases} \binom{k}{n} t_0^{k-n}, & n = 0,1,\ldots,k, \\ 0, & n > k. \end{cases}$*

e. *If $f(t) = \sin(ct+d)$, then $F_n = \frac{c^n}{n!}\sin(ct_0 + d + \frac{n\pi}{2})$.*

f. *If $f(t) = \cos(ct+d)$, then $F_n = \frac{c^n}{n!}\cos(ct_0 + d + \frac{n\pi}{2})$.*

g. *If $f(t) = e^{ct}$, then $F_n = \frac{c^n}{n!}e^{ct_0}$.*

h *If $f(t) = \int_{t_0}^{t} u(s)\,\mathrm{d}s$, then $F_n = \frac{U_{n-1}}{n}, \ n \geq 1, \ F_0 = 0.$*

i. *If $f(t) = g(t)\int_{t_0}^{t} u(s)\,\mathrm{d}s$, then $F_n = \sum_{k=0}^{n-1} \frac{G_k U_{n-k-1}}{n-k}, \ n \geq 1, \ F_0 = 0.$*

j. *If $f(t) = \int_{t_0}^{t} g(s)u(s)\,\mathrm{d}s$, then $F_n = \sum_{k=0}^{n-1} \frac{G_k U_{n-k-1}}{n}, \ n \geq 1, \ F_0 = 0.$*

Theorem 1 ([21]). *If $h(t) = \int_{t_0}^{t} f(t_0 + t - s)g(s)\,\mathrm{d}s$, then*

$$H_0 = 0, \quad H_n = \sum_{k=0}^{n-1} \frac{k!(n-k-1)!}{n!} F_k G_{n-k-1}, \ n \geq 1. \tag{5}$$

We recall the following definition which will be used in the next section.

Definition 2. *[11] Assume that $f(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n$ and $g(t) = \sum_{n=0}^{\infty} b_n (t-t_0)^n$ be two power series with positive radius of convergence and assume that $b_n \geq 0$ for all n. We say that $g(x)$ is a majorant of $f(x)$ if for all $n \geq 0$ we have $|a_n| \leq b_n$.*

Remark 1. *From the definition above, we can conclude that, if R_1 and R_2 are convergence radius of $f(x)$ and $g(x)$ respectively, then $R_2 \leq R_1$.*

3 Main results

In this section, first we recall Lemma 1 and Theorems 2 and 3 below from [26], then in Theorem 4, we show under appropriate assumptions, the integral equation (1) has a unique continuous solution.

Lemma 1 ([26]). Assume that $g \in C(\mathbb{I})$ and $K \in C(D)$ in which $D := \{(t,s)\colon 0 \le s \le t \le T\}$. Then, (1) has a unique local solution $u \in C[0,\sigma]$, where

$$\sigma := \min\left\{\frac{1}{4\bar{K}(\bar{G}+1)^2}, T\right\},\ \bar{G} := \max_{t\in I}|g(t)|,\ \bar{K} := \max_{(t,s)\in D}|K(t,s)|.$$

This local solution can be extended to cover the interval $[0,T]$, so we have the following theorem.

Theorem 2 ([26]). Let in the AVIE (1) we have $g \in C(\mathbb{I})$, $K \in C(D)$. Then (1) has a unique solution $u \in C(I)$.

The next theorem discusses the regularity property of the solution of (1).

Theorem 3 ([26]). Assume that in the Eq. (1) we have $g \in C^m(\mathbb{I})$ and $K \in C^m(D)$ for some $m \ge 1$. Then, the solution u of (1) possess the regularity $u \in C^m(\mathbb{I})$.

In this part, we will prove that under appropriate conditions, the solution of (1) is analytic at $t = 0$ with a convergence interval that includes $[0,\varepsilon]$.

Theorem 4. Suppose that the given functions g and K are analytic at $t = 0$ with their convergence intervals and domains contained in $\mathbb{I}$ and D, respectively. Then, the solution of the AVIE (1) is also analytic at $t = 0$, with its convergence interval containing $[0,\varepsilon]$, where ε will be specified during the proof.

Proof. Since g and K are analytic at $t = 0$, then we have

$$g(t) = \sum_{i=0}^{\infty} G_i t^i, \quad K(t,s) = \sum_{\rho,\sigma=0}^{\infty} \bar{K}_{\rho,\sigma} t^{\rho} s^{\sigma}. \tag{6}$$

We now assume that on the interval $[0,\varepsilon]$, the series $u(t) = \sum_{k=0}^{\infty} U_k t^k$ is convergent, then from Eq. (1) we get

$$\begin{cases} U_0 = G_0, \\ U_k = G_k + \displaystyle\sum_{\rho+\sigma+i+j+1=k} \bar{K}_{\rho,\sigma} U_i U_j B(j+1,\sigma+i+1), \qquad k \ge 1, \end{cases}$$

where B denotes the beta function. Assume the equation

$$y(t) = \tilde{g}(t) + \int_0^t \widetilde{K}(t,s) y(t-s) y(s)\,\mathrm{d}s,$$

in which $\tilde{g}(t)$ and $\widetilde{K}(t,s)$ are majorant functions of $g(t)$ and $K(t,s)$, respectively. Following the same approach as previously mentioned for $u(t)$, we can determine that $y(t)$ is a majorant of $u(t)$, this means that $y(t) = \sum_{k=0}^{\infty} Y_k t^k$ with $Y_k \geq |U_k|$. We demonstrate that the latter series converges on $[0, \varepsilon]$. To achieve this, let $N_l(t) := \sum_{k=0}^{l} Y_k t^k$. It is evident that the sequence $\{N_l(t)\}$ is increasing, and since

$$\begin{aligned}\tilde{g}(t) + \int_0^t \widetilde{K}(t,s) N_l(t-s) N_l(s)\,\mathrm{d}s &= \sum_{k=0}^{\infty} \tilde{G}_k t^k + \sum_{\rho,\sigma=0}^{\infty} \sum_{i,j=0}^{l} \widetilde{K}_{\rho,\sigma} Y_i Y_j B(j+1, \sigma+i+1) t^{\sigma+\rho+i+j+1} \\ &= N_{l+1}(t) + \sum_{k=l+2}^{\infty} \tilde{G}_k t^k \\ &\quad + \sum_{\rho+\sigma+i+j+1 \geq l+2} \widetilde{K}_{\rho,\sigma} Y_i Y_j B(j+1, \sigma+i+1) t^{\sigma+\rho+i+j+1},\end{aligned}$$

where $\sum_{k=l+2}^{\infty} \tilde{G}_k t^k$ and $\sum_{\rho+\sigma+i+j+1 \geq l+2} \widetilde{K}_{\rho,\sigma} Y_i Y_j B(j+1, \sigma+i+1) t^{\sigma+\rho+i+j+1}$ are non-negative, we conclude the following inequality:

$$N_{l+1}(t) \leq \tilde{g}(t) + \int_0^t \widetilde{K}(t,s) N_l(t-s) N_l(s)\,\mathrm{d}s.$$

Let us define $D_1 := \max_{t \in I} \tilde{g}(t)$ and $D_2 := D_1^2 \max_{(t,s) \in D} \widetilde{K}(t,s)$. Then, for $\varepsilon := \min\{T, \frac{D_1}{4D_2}\}$ we derive $N_0(t) = Y_0 \leq D_1 \leq 2D_1$. We get by induction

$$\begin{aligned}N_{l+1}(t) &\leq \tilde{g}(t) + \int_0^t \widetilde{K}(t,s) N_l(t-s) N_l(s)\,\mathrm{d}s \\ &\leq D_1 + \max_{s \in [0,t]} \{\widetilde{K}(t,s) N_l(t-s) N_l(s)\} t \\ &\leq D_1 + 4D_1^2 \max_{(t,s) \in D} \{\widetilde{K}(t,s)\} \varepsilon \\ &\leq D_1 + 4D_2 \varepsilon \\ &\leq 2D_1.\end{aligned}$$

This implies that the sequence $\{N_l(t)\}$ is uniformly bounded, and thus, the series $\sum_{k=0}^{\infty} Y_k t^k$ is uniformly convergent on $[0, \varepsilon]$. Consequently, we conclude that the series $\sum_{k=0}^{\infty} U_k t^k$ is also convergent, at least on $[0, \varepsilon]$. □

Let us assume that the functions g and K are analytic at $t = 0$ with convergence interval and domain included in $[0, T]$ and D, respectively. Then Eq. (1) has a unique analytic (at $t = 0$) solution u, which can be expressed as

$$u(t) = \sum_{n=0}^{\infty} U_n t^n. \tag{7}$$

To find an approximation to $u(t)$, we can apply the DTM to construct a recurrence relation for the unknown coefficients U_n, where $n = 0, 1, 2, \ldots$. To achieve this, we expand the functions $k(t,s)$ and $g(t)$ as follows:

$$K(t,s) = \sum_{n=0}^{\infty} k_n(s) t^n, \qquad g(t) = \sum_{n=0}^{\infty} G_n t^n, \tag{8}$$

where

$$k_n(s) := \frac{1}{n!}\left[\frac{\partial^n K(t,s)}{\partial t^n}\right]_{t=0}, \qquad G_n := \frac{1}{n!}\left[\frac{d^n g(t)}{dt^n}\right]_{t=0}. \tag{9}$$

Then, using the expansions (7) and (8) in Eq. (1) and setting

$$h_n(t) = \int_0^t k_n(s)u(t-s)u(s)\,\mathrm{d}s,$$

it becomes

$$\sum_{n=0}^{\infty} U_n t^n = \sum_{n=0}^{\infty} G_n t^n + \sum_{n=0}^{\infty} t^n h_n(t). \tag{10}$$

Comparing now coefficients of t^n on both sides of (10), implies

$$U_n = G_n + \sum_{j=0}^{n} H_{n-j,j}, \;\; n = 0,1,2,\ldots, \tag{11}$$

where $H_{n,k}$ stands for the differential transform of $h_n(t)$.

On the other hand, if we set $z_n(s) = k_n(s)u(s)$, then

$$h_n(t) = \int_0^t u(t-s)z_n(s)\,\mathrm{d}s, \tag{12}$$

and it follows from Theorem 1 that

$$H_{n,j} = \begin{cases} 0, & j = 0 \\ \sum_{l=0}^{j-1} \frac{l!(j-l-1)!}{j!} U_l Z_{n,j-l-1}, & j \geq 1. \end{cases} \tag{13}$$

We also apply Proposition 1, part (b)., to the function $z_n(s) = k_n(s)u(s)$, to get

$$Z_{n,j} = \sum_{r=0}^{j} A_{n,r} U_{j-r}, \;\; j = 0,1,2,\ldots. \tag{14}$$

Combining the relations (11), (13) and (14), yields the recurrence relation

$$U_n = \begin{cases} G_0, & n = 0, \\ G_n + \sum_{j=1}^{n} \sum_{l=0}^{j-1} \sum_{r=0}^{j-l-1} \frac{l!(j-l-1)!}{j!} A_{n-j,r} U_{j-l-r-1} U_l, & n = 1,2,\ldots, \end{cases} \tag{15}$$

for the coefficients $\{U_n\}_{n=0}^{\infty}$.

4 Examples

To illustrate the structure of the recurrence relation (15) and the accuracy of the method, we present the following examples. A comparison of our results with those provided in [13, 26] demonstrates the effectiveness of our method. All computations are performed using Maple software.

Example 1 ([13, 26])**.** As the first example, we consider the AVIE

$$u(t) = \beta t e^{-\gamma t} + \beta^2 e^{-\gamma t}(2\sin t - t - t\cos t) - \int_0^t \cos(t-s)u(t-s)u(s)\,\mathrm{d}s, \;\; t \in [0,1], \tag{16}$$

where the exact solution is $u(t) = \beta t e^{-\gamma t}$.

We set $\beta = \gamma = 1$ as in [26], then we have

$$\begin{aligned} u(t) = e^{-t}(2\sin t - t\cos t) - \cos t \int_0^t \cos s \cdot u(t-s)u(s)\,\mathrm{d}s \\ - \sin t \int_0^t \sin s \cdot u(t-s)u(s)\,\mathrm{d}s. \end{aligned} \tag{17}$$

By using Proposition 1, the recurrence relation (15) is written in the form of

$$\begin{cases} U_0 = 0 \\ U_n = 2\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \sin\left(\frac{(n-k)\pi}{2}\right) \\ \quad - \frac{(-1)^k}{l!(n-k-1)!} \cos\left(\frac{(n-k-1)\pi}{2}\right) \\ \quad - \sum_{k=1}^{n}\sum_{l=0}^{k-1}\sum_{r=0}^{k-l-1} \frac{l!(k-l-1)!}{k!r!(n-k)!} \cos\left(\frac{(n-k)\pi}{2}\right) \cos\left(\frac{r\pi}{2}\right) U_l U_{k-l-r-1} \\ \quad - \sum_{k=1}^{n}\sum_{l=0}^{k-1}\sum_{r=0}^{k-l-1} \frac{l!(k-l-1)!}{k!r!(n-k)!} \sin\left(\frac{(n-k)\pi}{2}\right) \sin\left(\frac{r\pi}{2}\right) U_l U_{k-l-r-1}, \\ \quad n = 1, 2, \ldots. \end{cases}$$

Then as an example, for $n = 10$ we obtain the approximate solution as

$$u_{10}(t) = t - t^2 + \frac{1}{2}t^3 - \frac{1}{6}t^4 + \frac{1}{24}t^5 - \frac{1}{120}t^6 + \frac{1}{720}t^7 - \frac{1}{5040}t^8 + \frac{1}{40320}t^9 - \frac{1}{362880}t^{10},$$

which is exactly the truncated Taylor expansion of degree 8.

Example 2 ([26])**.** Consider the equation

$$u(t) = \frac{1}{2}\sin t + \frac{1}{2}\int_0^t u(t-s)u(s)\,\mathrm{d}s, \;\; t \in [0,1], \tag{18}$$

where the exact solution is $u(t) = J_1(t)$, in which $J_1(t)$ denotes the Bessel function of order one that is defined by

$$J_1(t) = t\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{2^{2k+1}k!(k+1)!}. \tag{19}$$

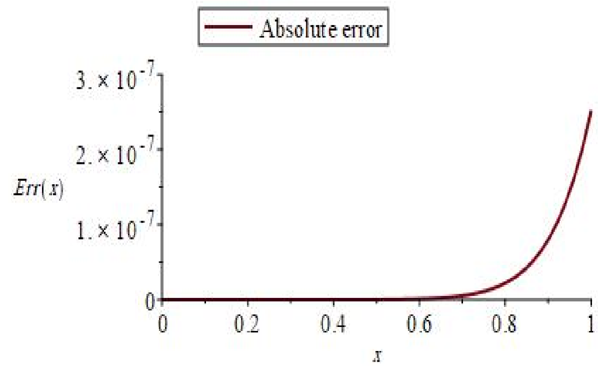

Figure 1: Plot of absolute error for Example 1.

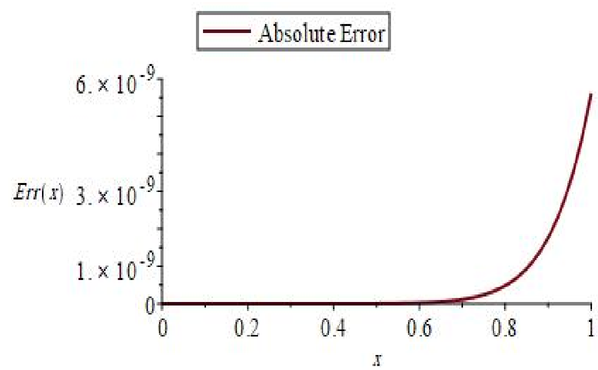

Figure 2: Plot of absolute error for Example 2.

The recurrence relation (15) for this equation is obtained as follows

$$\begin{cases} U_n = \frac{1}{2n!}\sin(\frac{n\pi}{2}) + \frac{1}{2}\sum_{k=0}^{n-1}\dfrac{k!(n-k-1)!}{n!}U_k U_{n-k-1}, & n = 1,2,\ldots, \\ U_0 = 0. \end{cases}$$

As an example, the approximate solution for $n = 10$ is then given by

$$u_{10}(t) = \frac{1}{2}t - \frac{1}{16}t^3 + \frac{1}{384}t^5 - \frac{1}{18432}t^7 + \frac{1}{1474560}t^9,$$

which coincides with partial sum of the series (19).

Example 3 ([26])**.** Consider the equation

$$u(t) = 50te^{-50t} + \frac{1250}{3}t^3e^{-50t} - \int_0^t u(t-s)u(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{20}$$

with the exact solution $u(t) = 50te^{-50t}$.

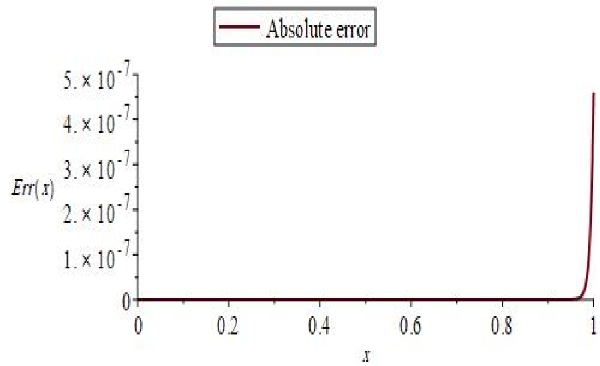

Figure 3: Plot of absolute error for Example 3.

By Theorem 1 and the recurrence relation (15), we have

$$\begin{aligned} U_n =& 50\sum_{k=0}^{n}\delta_{k,1}\frac{(-50)^{n-k}}{(n-k)!}-\frac{1250}{3}\sum_{k=0}^{n}\delta_{k,3}\frac{(-50)^{n-k}}{(n-k)!} \\ &-\sum_{k=0}^{n-1}\frac{k!(n-k-1)!}{n!}U_kU_{n-k-1}, \end{aligned}$$

for $n = 1, 2, ...,$ with $U_0 = 0$. For $n = 6$, we obtain the approximate solution as

$$u_6(t) = 50t - 2500t^2 + 62500t^3 - \frac{3125000}{3}t^4 + \frac{39062500}{3}t^5 - \frac{390625000}{3}t^6,$$

which exactly coincides with the truncated Taylor expansion of the solution.

Example 4. Consider the equation

$$u(t) = \sin t + \frac{1}{4}t^2(t\cos t - \sin t) + t\int_0^t su(t-s)u(s)\,ds, \quad t \in [0,2], \tag{21}$$

with the exact solution $u(t) = \sin(t)$. The approximate solution for $n = 10$ is obtained as follows:

$$u_{10}(t) = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \frac{1}{362880}t^9.$$

The plot of absolute error is shown in Fig. 4

5 Conclusion

In this article, we discussed the application of the Differential Transform Method for a class of non-standard Volterra integral equations. We demonstrated that, under suitable conditions, the solutions of these types of equations are analytic at the origin ($t = 0$). Subsequently, we derived a recurrence relation to obtain the Taylor expansion of the exact solution around the origin. Our findings contribute to the development of effective methods for solving non-standard Volterra integral equations, which can be applied in various fields of science and engineering.

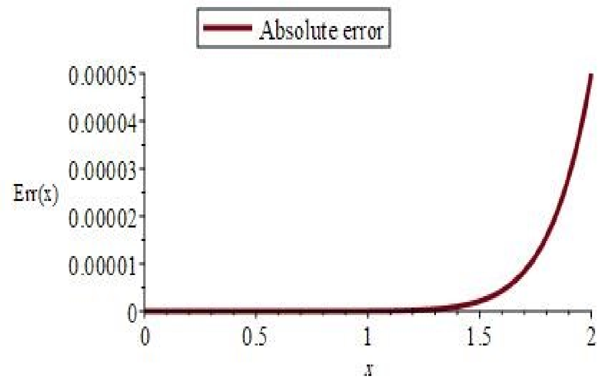

Figure 4: Plot of absolute error for Example 4.

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