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Stability of the Kawahara equation with time-varying delay

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Abstract. In this work, we consider the nonlinear Kawahara equation with internal time-dependent delay in a bounded domain. We prove that this equation has a unique solution. Moreover, we use a Lyapunov functional approach to prove the exponential stability of the nonlinear system, under some assumptions on the weights of the feedbacks and on the time-dependent delay. We present some numerical simulations to illustrate the obtained results.

Keywords: Exponential stability, internal feedback with delay, energy of the system, Lyapunov functional.
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1 Introduction

This article concerns the stabilization of the nonlinear Kawahara equation with a time-varying delay on the internal feedback. This equation is given by

$$y_t + y_x + y_{xxx} + \eta y_{xxxxx} + yy_x = 0.$$

It is a fifth-order nonlinear one-dimensional equation that describes water waves with surface tension. It is a model for small amplitude long waves, as plasma wave, water waves and other physical phenomena arising in fluid dynamics. In control systems, sensors act with a certain delay, which motivates the interest in studying the equations in the presence of a constant or time-dependent delay. Time delay phenomena appear in many problems of engineering and biology. We know that even a small delay can destabilize a system. In [4] the authors analyzed the stability for one-dimensional heat and wave equations involving time varying delay and in [5] the authors addressed the same problem for wave equations in domains in $\mathbb{R}^n$. The stability of the Kawahara equation has been studied by many authors without delay (see [9] and [1]) and with constant delay (see [2]). The problem of stabilization of the Korteweg-de Vries equation has recently been studied in the case of time-dependent delay (see [6]) and

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we propose to extend these results for the Kawahara equation using the Lyapunov approach. In our best knowledge, there is no work dealing with this problem for the Kawahara equation.

In this work, we consider the following system

$$\begin{cases} y_t(x,t)+y_x(x,t)+y_{xxx}(x,t)+\eta y_{xxxxx}(x,t)+y(x,t)y_x(x,t) & \\ \qquad +a(x)y(x,t)+b(x)y(x,t-\sigma(t))=0, & t>0,\ x\in(0,L), \\ y(0,t)=y(L,t)=y_x(0,t)=y_x(L,t)=y_{xx}(L,t)=0, & t>0, \\ y(x,0)=y_0(x), & x\in(0,L), \\ y(x,t-\sigma(0))=z_0(x,t-\sigma(0)), & 0<t<\sigma(0),\ x\in(0,L), \end{cases} \tag{1}$$

where $L>0$ is the length of the spatial domain, $y(x,t)$ is the amplitude of the water wave at position x at time t and η is a negative real number. The initial data y_0 is supposed to belong to $L^2(0,L)$ and the delayed data z_0 belongs to $L^2(0,L)\times L^2(-\sigma(0),0)$. We assume that the delay σ is a function of time t, which satisfies the following conditions

$$0<\sigma_0\leq\sigma(t)\leq M, \qquad \forall t\geq 0, \tag{2}$$

$$\dot{\sigma}(t)\leq\delta<1, \qquad \forall t\geq 0, \tag{3}$$

where M is a positive constant, $0\leq\delta<1$, and

$$\sigma\in W^{2,\infty}([0,T]), \qquad \forall T>0. \tag{4}$$

The functions a and b are nonnegative and belong to $L^\infty(0,L)$. Let ω = supp b be an open nonempty subset of $(0,L)$ and assume that

$$b(x)\geq b_0>0, \quad \text{in} \quad \omega. \tag{5}$$

We assume that a and b satisfy the following assumption

$$\exists q>0, \quad \frac{2-\delta}{2-2\delta}b(x)+q\leqslant a(x), \quad \text{in} \quad \omega. \tag{6}$$

Then ω = supp $b\subset$ supp a and $a(x)\geq b_0+q>0$ in ω.

We recall the definitions of spaces used in this work for an open subset $\Omega\subset\mathbb{R}$, $1\leq p,m\leq+\infty$ and a constant $c>0$:

$$L^p(\Omega)=\left\{f:\Omega\longrightarrow\mathbb{R},\ f \text{ measurable}/\int_\Omega |f(x)|^p dx<+\infty\right\},$$

$$L^\infty(\Omega)=\left\{f:\Omega\longrightarrow\mathbb{R},\ f \text{ measurable}/|f(x)|\leq c\right\},$$

$$W^{p,m}(\Omega)=\left\{f\in L^m(\Omega)/D^n f\in L^m(\Omega),\ \text{for all } n\in\mathbb{N}, n\leq p\right\}.$$

For a Banach space X, we define the space $L^p((0,T);X)$ as the usual Lebesgue space with X valued functions defined on $(0,T)$:

$$L^p((0,T);X)=\left\{f:(0,T)\longrightarrow X\ f\text{measurable}/\int_0^T \|f(t)\|_X^p dt<+\infty\right\}.$$

The plan of this paper is as follows. In Section 2, we prove the well-posedness results for the linear system using semigroups theory and then for the nonlinear system thanks to the fixed-point argument. The exponential stability result is proven in Section 3 using a Lyapunov functional approach and an estimation of the decay rate is given. The last section is devoted to some numerical simulations to illustrate our results.

2 Well-posedness results

The aim of this section is to prove that the system (1) has a unique solution. We start by proving that the linear system has a unique solution using semigroups theory and finally, we prove the well-posedness of the nonlinear system using the fixed-point argument.

2.1 Study of the linear system

We consider the following linearization around 0 of the system (1)

$$\begin{cases} y_t(x,t)+y_x(x,t)+y_{xxx}(x,t)+\eta y_{xxxxx}(x,t) & \\ \qquad +a(x)y(x,t)+b(x)y(x,t-\sigma(t))=0, & t>0,\ x\in(0,L),\\ y(0,t)=y(L,t)=y_x(0,t)=y_x(L,t)=y_{xx}(L,t)=0, & t>0,\\ y(x,0)=y_0(x), & x\in(0,L),\\ y(x,t-\sigma(0))=z_0(x,t-\sigma(0)), & 0<t<\sigma(0),\ x\in(0,L). \end{cases} \tag{7}$$

Now, following [6], we introduce a new variable, $z(x,\rho,t)=y_{|\omega}(x,t-\sigma(t)\rho)$ for any $x\in\omega$, $\rho\in(0,1)$ and $t>0$. We can see that z satisfies the following transport equation

$$\begin{cases} \sigma(t)z_t(x,\rho,t)+(1-\dot\sigma(t)\rho)z_\rho(x,\rho,t)=0, & x\in\omega,\ \rho\in(0,1), t>0,\\ z(x,0,t)=y_{|\omega}(x,t), & x\in\omega, t>0,\\ z(x,\rho,0)=z_0(x,-\sigma(0)\rho), & x\in\omega,\rho\in(0,1). \end{cases} \tag{8}$$

We set $\psi=\begin{pmatrix} y\\ z\end{pmatrix}$. Then we obtain

$$\psi_t=\begin{pmatrix} y_t\\ z_t\end{pmatrix}=\begin{pmatrix} -y_x-y_{xxx}-\eta y_{xxxxx}-ay-b\tilde z(.,1)\\ \dfrac{\dot\sigma(t)\rho-1}{\sigma(t)}z_\rho \end{pmatrix},$$

where $\tilde z(.,1)\in L^2(0,L)$ is the extension of $z(.,1)$ by zero outside ω. We can rewrite this problem as the following first-order evolution equation

$$\begin{cases} \psi_t(t)=\mathscr{A}(t)\psi(t), & t>0,\\ \psi(0)=\psi_0=(y_0,z_0(\cdot,-\sigma(0)\cdot))^T, \end{cases} \tag{9}$$

where the operator $\mathscr{A}(t)$ is defined by

$$\mathscr{A}(t)\begin{pmatrix} y\\ z\end{pmatrix}=\begin{pmatrix} -y_x-y_{xxx}-\eta y_{xxxxx}-ay-b\tilde z(.,1)\\ \dfrac{\dot\sigma(t)\rho-1}{\sigma(t)}z_\rho \end{pmatrix},$$

with domain

$$D(\mathscr{A}(t))=\big\{(y,z)\in H^5(0,L)\times L^2(\omega,H^1(0,1)),\ y(0)=y(L)=y_x(0)=y_x(L)=0,\\ y_{xx}(L)=0,\ z(x,0)=y_{|\omega}(x)\big\}.$$

We note that $D(\mathscr{A}(t)) = D(\mathscr{A}(0))$, for all $t > 0$, that means that the domain of the operator $\mathscr{A}(t)$ is independent of the time. The Hilbert space $H = L^2(0,L) \times L^2(\omega \times (0,1))$, is equipped with the usual inner product

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} \right\rangle_H = \int_0^L y\tilde{y}dx + \int_\omega \int_0^1 z\tilde{z}d\rho dx.$$

Now, due to (6), we can choose a positive function ξ in $L^\infty(0,L)$ such that supp ξ = supp $b = \omega$ and

$$\frac{1}{1-\delta} b(x) + q \leqslant \xi(x) \leqslant 2a(x) - b(x) - q \quad \text{in} \quad \omega. \tag{10}$$

We introduce the following time-dependent inner product on H

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_0^L y\tilde{y}dx + \sigma(t) \int_\omega \int_0^1 \xi(x) z\tilde{z}d\rho dx.$$

Obviously, the two norms $\|\cdot\|_t$ and $\|\cdot\|_H$ are equivalent on H. Indeed,

$$\forall t \geq 0,\, \forall (y,z) \in H,\, (1+\sigma_0 b_0)\|(y,z)\|_H^2 \leq \|(y,z)\|_t^2 \leq (1+2M\|a\|_\infty)\|(y,z)\|_H^2, \tag{11}$$

using (2), (5) and (10).

To prove the well-posedness of (9), we follow [4,6]. We will use the following theorem which gives the existence and uniqueness results of the solution. The reader can find the proof of this theorem in [3].

Theorem 1. *Assume that*

1. $\mathscr{Y} = D(\mathscr{A}(0))$ *is a dense subset of* H,
2. $D(\mathscr{A}(t)) = D(\mathscr{A}(0))$, *for all* $t > 0$,
3. *for all* $t \in [0,T]$, $\mathscr{A}(t)$ *generates a strongly continuous semigroup on* H *and the family* $\mathscr{A} = \{\mathscr{A}(t) : t \in [0,T]\}$ *is stable with stability constants* C *and* m *independent of* t *(i.e. the semigroup* $(S_t(s))_{s\geq 0}$ *generated by* $\mathscr{A}(t)$ *satisfies* $\|S_t(s)\psi\|_H \leq Ce^{ms}\|\psi\|_H$, *for all* $\psi \in H$ *and* $s \geq 0$*),*
4. $\partial_t \mathscr{A}(t)$ *belongs to* $L_*^\infty([0,T], B(\mathscr{Y}, H))$, *the space of equivalent classes of essentially bounded, strongly measure functions from* $[0,T]$ *into the set* $B(\mathscr{Y}, H)$ *of bounded operators from* $\mathscr{Y}$ *into* H.

Then, problem (9) *has a unique solution* $\psi \in C([0,T], \mathscr{Y}) \cap C^1([0,T], H)$ *for any initial datum in* $\mathscr{Y}$.

Now we are able to prove the following result.

Theorem 2. *Assume that the conditions* (2)-(6) *hold and that* $\psi_0 \in H$. *Then there exists a unique mild solution* $\psi \in C([0,+\infty), H)$ *to* (9). *If* $\psi_0 \in D(\mathscr{A}(0))$ *then* $\psi \in C([0,+\infty), D(\mathscr{A}(0))) \cap C^1([0,+\infty), H)$.

Proof. We are going to prove the four assumptions of the Theorem 1.
1. $\mathscr{Y} = D(\mathscr{A}(0))$ is a dense subset of H.
2. We have $D(\mathscr{A}(t)) = D(\mathscr{A}(0))$, for all $t > 0$.

3. The proof of the third point of Theorem 1: Let $t \in [0,T]$ be fixed. In order to prove that the operator is dissipative, we compute $\langle \mathscr{A}(t)\psi, \psi\rangle_t$ for $\psi = (y,z) \in D(\mathscr{A}(t))$. Then we have

$$\begin{aligned}\langle \mathscr{A}(t)\psi, \psi\rangle_t =& -\int_0^L y_{xxx} y dx - \int_0^L y_x y dx - \int_0^L a(x) y^2 dx - \int_\omega b(x) z(x,1) y(x) dx \\ & - \eta \int_0^L y_{xxxxx} y dx + \int_\omega \int_0^1 \xi(x)(\dot{\sigma}(t)\rho - 1) z_\rho z d\rho dx.\end{aligned}$$

After some integration by parts in space and in ρ we get

$$\begin{aligned}\langle \mathscr{A}(t)\psi, \psi\rangle_t =& \int_0^L y_{xx} y_x dx - [y_{xx} y]_0^L - \frac{1}{2}[y^2]_0^L - \int_0^L a(x) y^2 dx - \int_\omega b(x) z(x,1) y(x) dx \\ & - \eta \int_0^L y_{xx} y_{xxx} dx + \frac{1}{2}\int_\omega \xi(x)[(\dot{\sigma}(t)\rho - 1) z^2]_0^1 dx \\ & - \frac{1}{2}\dot{\sigma}(t) \int_\omega \int_0^1 \xi(x) z^2(x,\rho) d\rho dx,\end{aligned}$$

then

$$\begin{aligned}\langle \mathscr{A}(t)\psi, \psi\rangle_t =& -\int_0^L a(x) y^2(x,t) dx - \int_\omega b(x) z(x,1) y(x,t) dx - \frac{\eta}{2}[y_{xx}^2(x,t)]_0^L \\ & + \frac{1}{2}\int_\omega \xi(x)(\dot{\sigma}(t) - 1) z^2(x,1) dx + \frac{1}{2}\int_\omega \xi(x) z^2(x,0) dx \\ & - \frac{1}{2}\dot{\sigma}(t) \int_\omega \int_0^1 \xi(x) z^2(x,\rho) d\rho dx.\end{aligned}$$

We have

$$-2z(x,1)y(x,t) \leqslant z^2(x,1) + y^2(x,t),$$

then

$$\begin{aligned}\langle \mathscr{A}(t)\psi, \psi\rangle_t \leq& \frac{\eta}{2} y_{xx}^2(0,t) - \frac{1}{2}\int_\omega (2a(x) - b(x) - \xi(x))\, y^2(x,t) dx \\ & - \frac{1}{2}\int_\omega (\xi(x)(1-\delta) - b(x))\, z^2(x,1) dx - \frac{1}{2}\dot{\sigma}(t) \int_\omega \int_0^1 \xi(x) z^2(x,\rho) d\rho dx.\end{aligned}$$

From (10) and (3), we have $2a(x) - \xi(x) - b(x) > 0$ and $\xi(x)(1-\delta) - b(x) > 0$. Therefore, we obtain

$$\langle \mathscr{A}(t)\psi, \psi\rangle_t \leq -\frac{1}{2}\dot{\sigma}(t) \int_\omega \int_0^1 \xi(x) z^2(x,\rho) d\rho dx.$$

We set

$$\nu(t) = \frac{(\dot{\sigma}(t)^2 + 1)^{1/2}}{2\sigma(t)}.$$

Hence

$$\langle \mathscr{A}(t)\psi, \psi\rangle_t - \nu(t)\langle \psi, \psi\rangle_t \leq -\frac{1}{2}((\dot{\sigma}(t)^2 + 1)^{1/2} + \dot{\sigma}(t)) \int_\omega \int_0^1 \xi(x) z^2(x,\rho) d\rho dx \leqslant 0,$$

then the operator $\widetilde{\mathscr{A}}(t) := \mathscr{A}(t) - \nu(t) I$ is dissipative.

The adjoint $\mathscr{A}(t)^*$ of $\mathscr{A}(t)$ is defined by

$$\mathscr{A}(t)^*\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y_x + y_{xxx} + \eta y_{xxxxx} - ay + \xi(x)\tilde{z}(.,0) \\ \dfrac{1-\dot{\sigma}(t)\rho}{\sigma(t)} z_\rho - \dfrac{\dot{\sigma}(t)}{\sigma(t)} z \end{pmatrix},$$

with domain

$$D(\mathscr{A}(t)^*) = \{(y,z) \in H^5(0,L) \times L^2(\omega, H^1(0,1)),\ y(0) = y(L) = y_x(0) = y_x(L) = 0,$$
$$y_{xx}(0) = 0,\ z(x,1) = \frac{b(x)}{\xi(x)(\dot{\sigma}(t)-1)} y_{|\omega}(x)\}.$$

Now, we will prove that the operator $\widetilde{\mathscr{A}}(t)^* = \mathscr{A}(t)^* - \nu(t)I$ is dissipative. Let $\psi = (y,z) \in D(\mathscr{A}(t)^*)$, we have

$$\langle \mathscr{A}(t)^*\psi, \psi\rangle_t = \int_0^L y_{xxx} y dx + \eta \int_0^L y_{xxxxx} y dx - \int_0^L a(x) y^2 dx + \int_\omega \xi(x) z(x,0) y(x) dx$$
$$+ \int_0^L y_x y dx + \int_\omega \int_0^1 \xi(x)(1-\dot{\sigma}(t)\rho) z_\rho z d\rho dx - \int_\omega \int_0^1 \xi(x)\dot{\sigma}(t) z^2 d\rho dx.$$

Using integration by parts in space and in ρ, we obtain

$$\langle \mathscr{A}(t)^*\psi, \psi\rangle_t = -\int_0^L a(x) y^2(x,t) dx + \int_\omega \xi(x) y(x) z(x,0) dx + \frac{\eta}{2}[y_{xx}^2(x,t)]_0^L$$
$$+ \frac{1}{2}\int_\omega \int_0^1 \xi(x)\dot{\sigma}(t) z^2(x,\rho) d\rho dx + \frac{1}{2}\int_\omega \xi(x)[(1-\dot{\sigma}(t)\rho) z^2(x,\rho)]_0^1 dx$$
$$- \int_\omega \int_0^1 \xi(x)\dot{\sigma}(t) z^2(x,\rho) d\rho dx.$$

From the boundary conditions, we get

$$\langle \mathscr{A}(t)^*\psi, \psi\rangle_t = -\int_0^L a(x) y^2(x,t) dx + \int_\omega \xi(x) y(x) z(x,0) dx + \frac{\eta}{2} y_{xx}^2(L,t)$$
$$- \frac{\dot{\sigma}(t)}{2}\int_\omega \int_0^1 \xi(x) z^2(x,\rho) d\rho dx + \frac{1}{2}\int_\omega \frac{b^2(x)}{\xi(x)(1-\dot{\sigma}(t))} y^2(x) dx$$
$$- \frac{1}{2}\int_\omega \xi(x) z^2(x,0) dx,$$

then

$$\langle \mathscr{A}(t)^*\psi, \psi\rangle_t \le \frac{\eta}{2} y_{xx}^2(L,t) - \frac{1}{2}\int_\omega \left(2a(x) - \xi(x) - \frac{b^2(x)}{\xi(x)(1-\dot{\sigma}(t))}\right) y^2(x,t) dx$$
$$- \frac{\dot{\sigma}(t)}{2}\int_\omega \int_0^1 \xi(x) z^2(x,\rho) d\rho dx.$$

From (10) and (3), we have $2a(x) - \xi(x) - \dfrac{b^2(x)}{\xi(x)(1-\dot{\sigma}(t))} \ge 0$. Consequently

$$\langle \mathscr{A}(t)^*\psi, \psi\rangle_t \le -\frac{\dot{\sigma}(t)}{2}\int_\omega \int_0^1 \xi(x) z^2(x,\rho) d\rho dx,$$

hence

$$\langle \mathscr{A}(t)^*\psi,\psi\rangle_t - \nu(t)\langle \psi,\psi\rangle_t \leqslant 0.$$

Therefore, the operator $\widetilde{\mathscr{A}}(t)^* = \mathscr{A}(t)^* - \nu(t)I$ is dissipative. $\widetilde{\mathscr{A}}(t)$ is a densely defined closed linear operator and in addition $\widetilde{\mathscr{A}}(t)$ and $\widetilde{\mathscr{A}}(t)^*$ are dissipative, then $\widetilde{\mathscr{A}}(t)$ is the infinitesimal generator of a C_0 semigroup of contraction on H for any $t\in[0,T]$ (see [7]). We can easily prove that

$$\frac{\|\psi\|_t}{\|\psi\|_s} \leq e^{\frac{c}{2\sigma_0}|t-s|}, \quad \forall t,s\in[0,T], \tag{12}$$

where $\psi=(y,z)\in H$ and c is a positive constant (see [6]).

Then, $\widetilde{\mathscr{A}}(t)$ generates a strongly continuous semigroup on H for all $t\in[0,T]$ and the family $\widetilde{\mathscr{A}} = \{\widetilde{\mathscr{A}}(t) : t\in[0,T]\}$ is stable with stability constants C and m independent of t (see Proposition 3.4 of [3]).
4. From (4), we can prove that

$$\frac{d}{dt}\widetilde{\mathscr{A}}(t) \in L^\infty_*([0,T],B(D(\mathscr{A}(0)),H)),$$

since

$$\dot{\nu}(t) = \frac{\ddot{\sigma}(t)\dot{\sigma}(t)}{2\sigma(t)(\dot{\sigma}(t)^2+1)^{1/2}} - \frac{\dot{\sigma}(t)(\dot{\sigma}(t)^2+1)^{1/2}}{2\sigma(t)^2},$$

is bounded on $[0,T]$ for all $T>0$ and

$$\frac{d}{dt}\mathscr{A}(t)\psi = \begin{pmatrix} 0 \\ \dfrac{\ddot{\sigma}(t)\sigma(t)\rho - \dot{\sigma}(t)(\dot{\sigma}(t)\rho-1)}{\sigma(t)^2} z_\rho \end{pmatrix}.$$

Finally, all assumptions of Theorem 1 are verified, then the problem

$$\begin{cases} \tilde{\psi}_t(t) = \tilde{\mathscr{A}}(t)\tilde{\psi}, \\ \tilde{\psi}(0) = \psi_0, \end{cases}$$

has a unique solution $\tilde{\psi}\in C([0,+\infty),D(\mathscr{A}(0)))\cap C^1([0,+\infty),H)$ for $\psi_0\in D(\mathscr{A}(0))$. We can check that the solution of (9) is then given by $\psi(t) = e^{\int_0^t \nu(s)ds}\tilde{\psi}(t)$. Indeed,

$$\begin{aligned}\mathscr{A}(t)\psi(t) =& e^{\int_0^t \nu(s)ds}\mathscr{A}(t)\tilde{\psi}(t) = e^{\int_0^t \nu(s)ds}(\nu(t)\tilde{\psi}(t) + \tilde{\mathscr{A}}(t)\tilde{\psi}(t)) \\ =& \nu(t)e^{\int_0^t \nu(s)ds}\tilde{\psi}(t) + e^{\int_0^t \nu(s)ds}\tilde{\psi}_t(t) = \psi_t(t).\end{aligned}$$

□

2.2 Linear Kawahara equation with a source term

Now, to consider the nonlinear term of the equation, we study the well-posedness of the linear Kawahara equation with a source term f, that is

$$\begin{cases} y_t(x,t)+y_x(x,t)+y_{xxx}(x,t)+\eta y_{xxxxx}(x,t) & \\ \qquad +a(x)y(x,t)+b(x)y(x,t-\sigma(t))=f(x,t), & t>0,\ x\in(0,L), \\ y(0,t)=y(L,t)=y_x(0,t)=y_x(L,t)=y_{xx}(L,t)=0, & t>0, \\ y(x,0)=y_0(x), & x\in(0,L), \\ y(x,t-\sigma(0))=z_0(x,t-\sigma(0)), & 0<t<\sigma(0),\ x\in(0,L). \end{cases} \tag{13}$$

Let $T>0$ and introduce the space $B=C([0,T],L^2(0,L))\cap L^2((0,T),H_0^2(0,L))$ endowed with the norm

$$\|y\|_B=\|y\|_{C([0,T],L^2(0,L))}+\|y\|_{L^2((0,T),H_0^2(0,L))}.$$

Proposition 1. *Assume that the conditions (2)-(6) are fulfilled. Then for $f\in L^1(0,T,H_0^1(0,L))$ and $(y_0,z_0(.,-\sigma(0).))\in H$, there exists a unique mild solution $(y,y(.,t-\sigma(t).))\in B\times C([0,T],L^2(\omega\times(0,1)))$ to (13). Moreover, there exists $C>0$ independent of T and $C_T>0$ such that*

$$\|(y,z)\|_{C([0,T],H)}\leq C\left(\|(y_0,z_0(.,-\sigma(0).))\|_H+\|f\|_{L^1(0,T,H_0^1(0,L))}\right), \tag{14}$$

$$\|y_{xx}\|_{L^2(0,T;L^2(0,L))}+\|y_x\|_{L^2(0,T;L^2(0,L))}\leq C_T\left(\|(y_0,z_0(.,-\sigma(0).))\|_H+\|f\|_{L^1(0,T,H_0^1(0,L))}\right). \tag{15}$$

Proof. The proof is similar to the proof of [2, Proposition 2] (see also [8, 9]). We can write the system (13) as

$$\psi_t=\mathscr{A}(t)\psi+\begin{pmatrix} f \\ 0 \end{pmatrix}.$$

□

2.3 Well-posedness result of the nonlinear system

To complete the study of the existence of the solution, we prove the well-posedness result of the nonlinear system (1).

Theorem 3. *Assume that the conditions (2)-(6) are satisfied and let $L>0$. Then for any $(y_0,z_0(.,-\sigma(0).))\in H$, there exists a unique solution $y\in C(0,\infty;L^2(0,L))\cap L^2_{loc}((0,\infty),H_0^2(0,L))$ of the system (1).*

Proof. Following [8] (see also [2], we can prove the local (in time) existence and uniqueness of the solution of the nonlinear system (1). Let $\tilde{y}\in B$, we consider the map $\Phi: B\longrightarrow B$ defined by $\Phi(\tilde{y})=y$, where y is the solution of the following system

$$\begin{cases} y_t(x,t)+y_x(x,t)+y_{xxx}(x,t)+\eta y_{xxxxx}(x,t) & \\ \qquad +a(x)y(x,t)+b(x)y(x,t-\sigma(t))=-\tilde{y}(x,t)\tilde{y}_x(x,t), & t>0,\ x\in(0,L), \\ y(0,t)=y(L,t)=y_x(0,t)=y_x(L,t)=y_{xx}(L,t)=0, & t>0, \\ y(x,0)=y_0(x), & x\in(0,L), \\ y(x,t-\sigma(0))=z_0(x,t-\sigma(0)), & 0<t<\sigma(0),\ x\in(0,L). \end{cases} \tag{16}$$

Let $r > 0$ and $B_r = \{y \in B / \|y\|_B \leq r\}$ a closed ball. Using Proposition 1, we can prove that Φ is a contraction on B_r for r small enough similarly to the proof of [8, Proposition 4] (see also [6, Theorem 2.6]). Finally, from the Banach fixed point theorem, the map Φ has a unique fixed point $y \in B$ which is the requested solution of the nonlinear Kawahara equation (1). Using the decay of the energy, we can get the global existence of the solution. □

3 Exponential stability result

In this section, we prove the exponential stability of the nonlinear Kawahara equation using a new Lyapunov functional. We consider the following definition of the energy of the nonlinear system (1)

$$E(t) = \int_0^L y^2(x,t)dx + \sigma(t)\int_\omega \int_0^1 \xi(x) y^2(x,t-\sigma(t)\rho)d\rho dx, \tag{17}$$

where ξ is defined by (10).

3.1 The decay of the energy

We start by proving the decay of the energy of the nonlinear system (1) in the following proposition.

Proposition 2. *Assume that the conditions* (2)-(6) *hold. Then for any regular solution of* (1)*, the energy* E *defined by* (17) *is decreasing and satisfies*

$$\begin{aligned}\frac{d}{dt}E(t) \leq & \eta y_{xx}^2(0,t) + \int_\omega (-2a(x) + b(x) + \xi(x)) y^2(x,t)dx \\ & + \int_\omega (\xi(x)(\delta - 1) + b(x)) y^2(x,t-\sigma(t))dx \leq 0.\end{aligned} \tag{18}$$

Proof. We differentiate E:

$$\frac{d}{dt}E(t) = 2\int_0^L yy_t dx + \dot{\sigma}(t)\int_\omega \int_0^1 \xi(x) y^2(x,t-\sigma(t)\rho)d\rho dx + 2\sigma(t)\int_\omega \int_0^1 \xi(x) zz_t d\rho dx,$$

then

$$\begin{aligned}\frac{d}{dt}E(t) = & 2\int_0^L y(y_x + y_{xxx} + \eta y_{xxxxx} + yy_x + a(x)y(x,t) + b(x)y(x,t-\sigma(t)))dx \\ & + \dot{\sigma}(t)\int_\omega \int_0^1 \xi(x) y^2(x,t-\sigma(t)\rho)d\rho dx + 2\int_\omega \int_0^1 \xi(x)(\dot{\sigma}(t)\rho - 1) zz_\rho d\rho dx.\end{aligned} \tag{19}$$

Integrating by parts and using boundary conditions, we obtain

$$\begin{aligned}\frac{d}{dt}E(t) \leq & \eta y_{xx}^2(0,t) + \int_\omega (-2a(x) + b(x) + \xi(x)) y^2(x,t)dx \\ & + \int_\omega (\xi(x)(\delta - 1) + b(x)) y^2(x,t-\sigma(t))dx \leq 0.\end{aligned}$$

□

3.2 Exponential stability by Lyapunov functional

To prove the exponential stability, we consider the following Lyapunov functional

$$V(t) = E(t) + \alpha_1 V_1(t) + \alpha_2 V_2(t), \tag{20}$$

where α_1, $\alpha_2 > 0$ are fixed constants taken small enough to obtain the decrease of the energy E defined by (17). The functions V_1 and V_2 are defined by

$$V_1(t) = \int_0^L e^{\lambda x} y^2(x,t)dx, \tag{21}$$

$$V_2(t) = \sigma(t) \int_\omega \int_0^1 (1-\rho) y^2(x, t-\sigma(t)\rho) d\rho dx, \tag{22}$$

where $\lambda > 0$ will be chosen small enough to obtain the decay of the energy. V_1 is classical for the Kawahara equation and V_2 comes from the delay term depending on time. We can prove from the definition of $V(t)$ and $E(t)$ that, for any $t > 0$,

$$E(t) \leq V(t) \leq \left(1 + \max\{e^{\lambda L}\alpha_1, \frac{\alpha_2}{b_0}\}\right) E(t). \tag{23}$$

Indeed, from (5) and (10), we have

$$\begin{aligned} E(t) \leq V(t) &= E(t) + \alpha_1 \int_0^L e^{\lambda x} y^2(x,t)dx + \alpha_2 \sigma(t) \int_\omega \int_0^1 (1-\rho) y^2(x, t-\sigma(t)\rho) d\rho dx \\ &\leq E(t) + \alpha_1 e^{\lambda L} \int_0^L y^2(x,t)dx + \alpha_2 \sigma(t) \int_\omega \int_0^1 \frac{\xi(x)}{b_0} y^2(x, t-\sigma(t)\rho) d\rho dx \\ &\leq \left(1 + \max\{e^{\lambda L}\alpha_1, \frac{\alpha_2}{b_0}\}\right) E(t). \end{aligned}$$

The main result of this paper is given in the following theorem. We will prove that the energy of the nonlinear system (1) decays exponentially.

Theorem 4. *Assume that the conditions (2)-(6) are fulfilled. Then, there exists $r > 0$ small enough, such that, for every $(y_0, z_0) \in H$ satisfying $\|(y_0, z_0)\|_0 \leq r$, the energy of the nonlinear system (1) decays exponentially. More precisely, there exist two positive constants γ and K such that*

$$E(t) \leq Ke^{-2\gamma t} E(0), \ \forall t > 0,$$

where, for α_1, $\alpha_2 > 0$ and $\lambda > 0$ small enough,

$$\gamma \leq \min\left\{ \frac{(-15\eta\lambda\pi^2 - \sqrt{2\lambda} L^3 e^{\lambda L} r)\alpha_1 \pi^2}{6L^4(1+\alpha_1 e^{\lambda L})}, \frac{(1-\delta)\alpha_2}{2M(\alpha_2 + \|\xi\|_{L^\infty(0,L)})} \right\}, \tag{24}$$

$$K \leq 1 + \max\left\{ e^{\lambda L}\alpha_1, \frac{\alpha_2}{b_0} \right\}.$$

Proof. We start by proving that V decays exponentially, so we have to prove that

$$\forall t > 0,\ \frac{d}{dt}V(t) + 2\gamma V(t) \leq 0.$$

Consider y as the solution of (1) with $(y_0, z_0(., -\sigma(0).)) \in D(\mathscr{A}(0))$ and satisfying the condition

$$\|(y_0, z_0(., -\sigma(0).))\|_H \leq r.$$

Differentiating V_1 and using integration by parts, we obtain

$$\begin{aligned}
\frac{d}{dt}V_1(t) &= 2\int_0^L e^{\lambda x} y(x,t) y_t(x,t)dx \\
&= -2\int_0^L e^{\lambda x} y(x,t)(y_x(x,t) + y_{xxx}(x,t) + \eta y_{xxxxx}(x,t) + y(x,t)y_x(x,t))dx \\
&\quad - 2\int_0^L e^{\lambda x} a(x) y^2(x,t)dx - 2\int_0^L e^{\lambda x} b(x) y(x,t) y(x,t-\sigma(t))dx \\
&= (\lambda + \lambda^3 + \eta\lambda^5)\int_0^L e^{\lambda x} y^2(x,t)dx - (3\lambda + 5\eta\lambda^3)\int_0^L e^{\lambda x} y_x^2(x,t)dx \\
&\quad + 5\eta\lambda\int_0^L e^{\lambda x} y_{xx}^2(x,t)dx + \frac{2}{3}\lambda\int_0^L e^{\lambda x} y^3(x,t)dx + \eta y_{xx}^2(0,t) \\
&\quad - 2\int_0^L e^{\lambda x} a(x) y^2(x,t)dx + \int_0^L e^{\lambda x} b(x) y^2(x,t)dx + \int_0^L e^{\lambda x} b(x) y^2(x,t-\sigma(t))dx.
\end{aligned}$$

In the same way, we differentiate V_2. Using integration by parts and the relation

$$\sigma(t)\partial_t y(x,t-\sigma(t)\rho) = (\dot{\sigma}(t)\rho - 1)\partial_\rho y(x,t-\sigma(t)\rho),$$

we get

$$\begin{aligned}
\frac{d}{dt}V_2(t) =& \dot{\sigma}(t)\int_\omega\int_0^1 (1-\rho)y^2(x,t-\sigma(t)\rho)d\rho dx \\
& + 2\sigma(t)\int_\omega\int_0^1 (1-\rho)y(x,t-\sigma(t)\rho)\partial_t(y(x,t-\sigma(t)\rho))d\rho dx \\
=& \dot{\sigma}(t)\int_\omega\int_0^1 (1-\rho)y^2(x,t-\sigma(t)\rho)d\rho dx \\
& + 2\int_\omega\int_0^1 (\dot{\sigma}(t)\rho - 1)(1-\rho)y(x,t-\sigma(t)\rho)\partial_\rho y(x,t-\sigma(t)\rho)d\rho dx \\
=& \dot{\sigma}(t)\int_\omega\int_0^1 (1-\rho)y^2(x,t-\sigma(t)\rho)d\rho dx + \int_\omega [(\dot{\sigma}(t)\rho - 1)(1-\rho)y^2(x,t-\sigma(t)\rho)]_0^1 dx \\
& - \int_\omega\int_0^1 (1+\dot{\sigma}(t) - 2\dot{\sigma}(t)\rho)y^2(x,t-\sigma(t)\rho)d\rho dx \\
=& \int_\omega y^2(x,t)dx - \int_\omega\int_0^1 (1-\dot{\sigma}(t)\rho)y^2(x,t-\sigma(t)\rho)d\rho dx.
\end{aligned}$$

Now, we are ready to calculate $\frac{d}{dt}V(t)+2\gamma V(t)$. We obtain

$$\begin{aligned}\frac{d}{dt}V(t)+2\gamma V(t)\leq&\int_{\omega}(-2a(x)+b(x)+\xi(x)+\alpha_1 e^{\lambda L}b(x)+\alpha_2)y^2(x,t)dx\\&+\int_{\omega}(b(x)+(\delta-1)\xi(x)+\alpha_1 e^{\lambda L}b(x))y^2(x,t-\sigma(t))dx\\&+5\eta\lambda\alpha_1\int_0^L e^{\lambda x}y_{xx}^2(x,t)dx+2\gamma(1+\alpha_1 e^{\lambda L})\int_0^L y^2(x,t)dx\\&+\frac{2}{3}\alpha_1\lambda\int_0^L e^{\lambda x}y^3(x,t)dx+\alpha_1\int_0^L(\lambda+\lambda^3+\eta\lambda^5-2a(x))e^{\lambda x}y^2(x,t)dx\\&-\alpha_1\lambda(3+5\eta\lambda^2)\int_0^L e^{\lambda x}y_x^2(x,t)dx+\eta(1+\alpha_1)y_{xx}^2(0,t)\\&+\int_{\omega}\int_0^1(2\gamma\xi(x)\sigma(t)+2\gamma\alpha_2\sigma(t)-\alpha_2(1-\delta))y^2(x,t-\sigma(t)\rho)d\rho dx.\end{aligned}$$

Note that from Cauchy-Schwarz's inequality, we get

$$\int_0^L e^{\lambda x}y^3dx\leq\|y\|^2_{L^\infty(0,L)}\int_0^L e^{\lambda x}|y|dx\leq\|y\|^2_{L^\infty(0,L)}\ \|y\|_{L^2(0,L)}\sqrt{\frac{e^{2\lambda L}}{2\lambda}}.$$

By the injection of $H_0^1(0,L)$ into $L^\infty(0,L)$, we have

$$\|y\|^2_{L^\infty(0,L)}\leq L\|y_x\|^2_{L^2(0,L)},$$

and from Poincare's inequality, we obtain

$$\int_0^L e^{\lambda x}y^3dx\leq\frac{L^3}{\pi^2}\|y_{xx}\|^2_{L^2(0,L)}\frac{e^{\lambda L}}{\sqrt{2\lambda}}\|y\|_{L^2(0,L)}.$$

Since we have $q\leq a(x)$, we get

$$\begin{aligned}\frac{d}{dt}V(t)+2\gamma V(t)\leq&\int_{\omega}(-2a(x)+b(x)+\xi(x)+\alpha_1 e^{\lambda L}b(x)+\alpha_2)y^2(x,t)dx\\&+\int_{\omega}(b(x)+(\delta-1)\xi(x)+\alpha_1 e^{\lambda L}b(x))y^2(x,t-\sigma(t))dx\\&+\left(2\gamma\frac{L^4}{\pi^4}(1+\alpha_1 e^{\lambda L})+5\eta\lambda\alpha_1+\frac{L^3\alpha_1\sqrt{2\lambda}e^{\lambda L}r}{3\pi^2}\right)\int_0^L y_{xx}^2(x,t)dx\\&+\alpha_1(\lambda+\lambda^3+\eta\lambda^5-2q)\int_0^L e^{\lambda x}y^2(x,t)dx\\&-\alpha_1\lambda(3+5\eta\lambda^2)\int_0^L e^{\lambda x}y_x^2(x,t)dx+\eta(1+\alpha_1)y_{xx}^2(0,t)\\&+\int_{\omega}\int_0^1(2\gamma\xi(x)\sigma(t)+2\gamma\alpha_2\sigma(t)-\alpha_2(1-\delta))y^2(x,t-\sigma(t)\rho)d\rho dx.\end{aligned}$$

To obtain $\frac{d}{dt}V(t)+2\gamma V(t)\leq 0$, from (10), we can choose α_1, α_2, λ, γ and r such that

$$\alpha_1 \leq \inf_{x\in\omega}\left\{\frac{2a(x)-b(x)-\xi(x)}{e^{\lambda L}b(x)}, \frac{(1-\delta)\xi(x)-b(x)}{e^{\lambda L}b(x)}\right\},$$

$$\alpha_2 \leq \inf_{x\in\omega}\{2a(x)-b(x)-\xi(x)-\alpha_1 e^{\lambda L}b(x)\},$$

$$\lambda \leq \min\left\{\lambda_0, \sqrt{\frac{3}{-5\eta}}\right\},$$

where λ_0 can be chosen such that $\lambda_0+\lambda_0^3+\eta\lambda_0^5-2q\leq 0$, and

$$\gamma \leq \min\left\{\frac{(-15\eta\lambda\pi^2-\sqrt{2\lambda}L^3e^{\lambda L}r)\alpha_1\pi^2}{6L^4(1+\alpha_1 e^{\lambda L})}, \frac{(1-\delta)\alpha_2}{2M(\alpha_2+\|\xi\|_{L^\infty(0,L)})}\right\}, \tag{25}$$

where r fulfilles the inequality $-15\eta\lambda\pi^2-\sqrt{2\lambda}L^3e^{\lambda L}r>0$ which implies that

$$0<r<\frac{-15\eta\sqrt{\lambda}\pi^2}{\sqrt{2}L^3e^{\lambda L}}.$$

By integrating $\frac{d}{dt}V(t)+2\gamma V(t)\leq 0$ over $(0,t)$, we get for all $t>0$, $V(t)\leq V(0)e^{-2\gamma t}$. Since E and V are equivalent from (23), we finally obtain

$$E(t)\leq\left(1+\max\{e^{\lambda L}\alpha_1, \frac{\alpha_2}{b_0}\}\right)E(0)e^{-2\gamma t},\ \forall t>0.$$

Lastly, we note that $D(\mathscr{A}(0))$ is dense in H, then we can take $(y_0, z_0(., -\sigma(0).))\in H$. □

We can notice that the decay rate γ decreases when the upper bound M of the time-delay function increases. We have the same remark when δ tends to 1.

4 Numerical simulations and conclusion

This section is devoted to some numerical simulations that adapt the schemes used in [6,8]. We illustrate the stability result obtained in this study. We set a final time T and consider a uniform spatial and time discretization. Let N_x, N_ρ and N_t be three positive integers. Let $dx = L/N_x$ be the spatial step, $dt = L/N_t$ the time step and $d\rho = 1/N_\rho$ be the delay step. We introduce the notation $y(idx, ndt) = y_i^n$ and $z(idx, kd\rho, ndt) = z_{i,k}^n$ for $i = 0,\dots,N_x$, $k = 0,\dots,N_\rho$ and $n = 0,\dots,N_t$. For the first derivative with respect to x, we will use the approximations

$$D_x^+y_i = \frac{y_{i+1}-y_i}{dx}, \quad D_x^-y_i = \frac{y_i-y_{i-1}}{dx}, \quad D_xy_i = \frac{y_{i+1}-y_{i-1}}{2dx}. \tag{26}$$

The same definitions hold for the first derivative with respect to ρ. The third derivative will be given as $D_{xxx} = D_x^+D_x^+D_x^-$ and the fifth by $D_{xxxxx} = D_x^+D_x^-D_x^+D_x^+D_x^-$. Let $s = E(L/2dx)$, where E is the floor

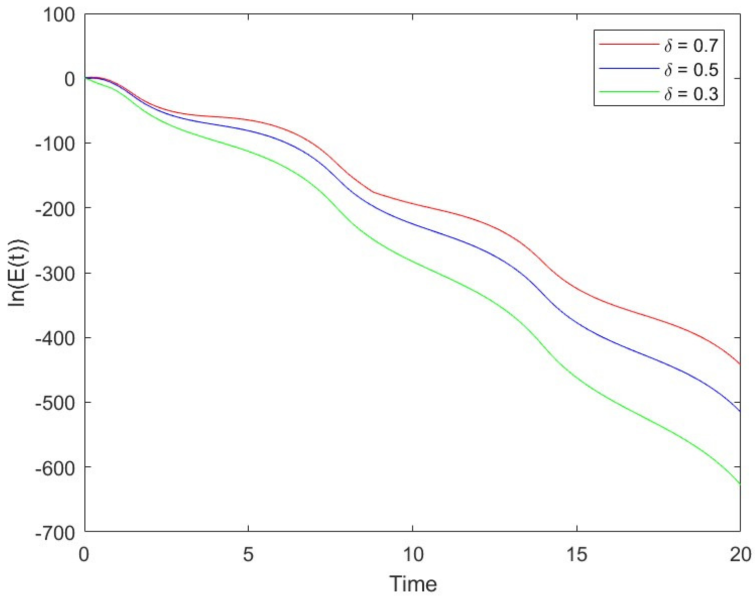

Figure 1: Representation of $t \mapsto \ln(E(t))$ for different values of δ.

function. The system is discretized as follows

$$
\begin{cases}
\dfrac{y_i^n - y_i^{n-1}}{dt} + (D_x + D_{xxx} - D_{xxxxx}) y_i^n & \\
\qquad = -y_i^{n-1} D_x^+ y_i^{n-1} - a_i y_i^{n-1} - b_i z_{i,N_\rho}^{n-1} & n = 1 \dots, N_t,\ i = 2, \dots, N_x - 1, \\
y_0^n = y_1^n = y_{Nx-2}^n = y_{Nx-1}^n = y_{Nx}^n = 0 & n = 0, \dots, N_t, \\
y_i^0 = y_0(idx), & i = 0, \dots, N_x, \\
\dfrac{z_{i,k}^n - z_{i,k}^{n-1}}{dt} + \dfrac{1}{\sigma^n}(1 - \dot{\sigma}^n k d\rho) D_\rho^+ z_{i,k}^n = 0 & n = 1, \dots, N_t,\ i = 0, \dots, s, \\
& k = 1, \dots, N_\rho - 1, \\
z_{i,N_\rho}^n = \dfrac{b_i y_i^n}{\xi(\dot{\sigma}^n - 1)} & n = 1, \dots, N_t,\ i = 0, \dots, s, \\
z_{i,k}^0 = z_0(idx, kd\rho) & i = 0, \dots, s,\ k = 1, \dots, N_\rho - 1, \\
z_{i,0}^n = y_i^n & n = 0, \dots, N_t,\ i = 0, \dots, s, \\
z_{i,k}^n = 0 & n = 0, \dots, N_t,\ i = s+1, \dots, N_x,\ k = 0, \dots, N_\rho.
\end{cases}
\tag{27}
$$

We take the following parameters $L = 1$, $T = 20$, $\eta = -1$ and the feedback terms are constant in their support supp a = supp $b = (0, L/2)$, $a(x) = 2{,}2$, $b(x) = 1{,}5$ and $\xi(x) = 2.5$. The initial conditions are $y_0(x) = 1 - \cos(2\pi x)$, $z_0(x, \rho) = \cos(2\pi\rho)(1 - \cos(2\pi x))$ and the delay is $\sigma(t) = \delta(1, 1 - \sin(t))$. We solve the discretized system using the backward Euler method. We observe that the decay rate γ decreases when δ tends to 1 as shown in the estimation (25). Figure 1 represents $t \mapsto \ln(E(t))$ for different values of δ.

In this work, we present an internal stability result for the nonlinear Kawahara equation with time-varying delay. We prove the existence and the uniqueness of the solution of the system and we study the exponential stability using an appropriate Lyapunov functional. Finally, we present some numerical simulations to illustrate the theoretical result obtained.

An interesting question to investigate is to consider the study of the stability of the Kawahara equation with a delay in the nonlinear term.

References

[1] F.D. Araruna, R.A. Capistrano-Filho, G.G Doronin, *Energy decay for the modified Kawahara equation posed in a bounded domain*, J. Math. Anal. Appl. **385** (2012) 743–756.

[2] B. Chentouf, *Well-posedness and exponential stability of the Kawahara equation with a time-delayed localized damping*, Math. Methods Appl. Sci. **45** (2022) 10312–10330.

[3] T. Kato, *Linear evolution equations of "hyperbolic" type*, J. Fac. Sci. Univ. Tokyo Sect. I **17** (1970) 241–258.

[4] S. Nicaise, J. Valein, E. Fridman, *Stability of the heat and of the wave equations with boundary time-varying delays*, Discrete Contin. Dyn. Syst. S **2** (2009) 559–581.

[5] S. Nicaise, C. Pignotti, J. Valein, *Exponential stability of the wave equation with boundary time-varying delay*, Discrete Contin. Dyn. Syst. S **4** (2011) 693–722.

[6] H. Parada, C. Timimoun and J. Valein, *Stability results for the KdV equation with time-varying delay*, Syst. Control Lett. **177** (2023) 105547.

[7] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations (Applied Mathematical Sciences)*, Springer-Verlag, New York, 1983.

[8] J. Valein, *On the asymptotic stability of the Korteweg-de Vries equation with time-delayed internal feedback*, Math. Control Relat. Fields **12** (2022) 667–694.

[9] C.F. Vasconcellos, P.N. Da Silva, *Stabilisation of the Kawahara equation with localized damping*, ESAIM Control Optim. Ca. **17** (2011) 102–116.