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A note on total graph of commutative krasner hyperrings

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Abstract. The purpose of this paper is to introduce the notion of the total graph of Krasner hyperrings. In this regard, a connection between the graph theory and the theory of hyperrings is constructed, and some fundamental properties of the total graph of Krasner hyperrings are investigated. Finally, for a multiplicative-prime subset H of a hyperring R, the diameter and the girth of $\Gamma_H(R \setminus H)$ and $\Gamma_H(H)$ are computed precisely.

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1 Introduction

In the past three decades, graphs created from algebraic structures have been noticed by many researchers and have become a primary field of study (Please, see [1], [2], [3]). The total graph of a ring was first introduced by Badawi et al. in [2]. The total graph of R, denoted by $T(\Gamma(R))$, is the simple graph with all elements R as vertices, and for two distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R) = \{x \in R \,|\, xy = 0$, for some non-zero element $y \in R\}$. Recall that a non-empty subset H of R is said to be multiplicative-prime provided that: (1) $a \cdot b \in H$ for every $a \in H$ and $b \in R$ and (2) If $a \cdot b \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$. It is obvious that $Z(R)$ is a multiplicative-prime subset. Toward a generalization of the concept of the total graph, Badawi et al. in [3] defined the concept of the total graph with respect to a multiplicative-prime subset, where the set vertices of this graph are the elements of the ring R and two vertices x, y are adjacent if and only if $x + y$ is an element of H. Assigning graphs to algebraic structures has played a significant role in studying their structures. Studying these

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graphs means one may find significant results about the algebraic structures and vice versa. The study of derived graphs of algebraic structures helps to the interplay between the properties of the algebraic structures and the structures of the assigned graphs. The main goal of this paper is to study the total graph of Krasner hyperrings, which is a common generalization of the total graph introduced by Badawi and et al. The structure of the paper is as follows. First, in Section 2, we will provide some preliminary but essential concepts of hyperstructures that are crucial for us. In Section 3, the concept of a multiplicative-prime subset that plays the most important role in the structure of a total graph for hyperstructures will be formulated. According to [3], the multiplicative-prime subset can be explained in two segregated structures. Therefore, in Sections 4 and 5, the shape of the total graph, its connectedness and disjointness, and its related subgraphs regarding the girth and diameter of the subgraphs of the total graph will be investigated. Algebraic properties of the algebraic structures can be translated into the language of graph theory (Please see [4], [6], [8], [9]). Then, the geometric properties of the graphs can assist in finding some interesting facts about algebraic structures.

Let us first recall some preliminary concepts and definitions that are crucial in the sequel.

2 Preliminaries

Now, we recall some definitions and notations on graphs and hyperring theory. Let Γ be a simple graph. The vertex set of Γ is denoted by $V(\Gamma)$. We recall that a graph is connected if a path exists connecting any two distinct vertices. The distance $\mathrm{d}(a,b)$ is the length of the shortest path from a to b; if such a path does not exist, then $d(a,b)=\infty$. The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a,b) : a,b \in V(\Gamma)\}$. A graph is complete if it is connected with a diameter less than or equal to one. The girth of a graph Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma)=\infty$, in this case Γ is called an acyclic graph. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 (respectively, Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (respectively, Γ_2). We denote the complete bipartite graph on m and n vertices by $K^{m,n}$. A component (connected component) of graph Γ is a subgraph in which any two vertices are connected by paths and which is connected to no additional vertices in the graph Γ. We say that u is a universal vertex of Γ if u is adjacent to all other vertices of Γ. A vertex v in an undirected connected graph G is a cut-point (cut vertex) of G if removing it (and edges through it) disconnects the graph.

Now, we recall various notions from hyperring theory, which will be used in the sequel. Assume that G is a non-empty set and $P^*(G)$ is the set of all nonempty subsets of G. A hyperoperation on G is a map $\circ : H \times H \longrightarrow P^*(H)$ and the couple $(H,\circ)$ is called a hypergroupoid. For any two nonempty subsets A and B of G and $x \in G$, we define $A \circ B = \bigcup_{(a,b)\in A\times B} a \circ b$, where $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$. The more general structure that satisfies the ring-like axioms is the hyperring in the general sense: $(R,+,\cdot)$ is a hyperring if $+$ and $\cdot$ are two hyperoperations such that $(R,+)$ is a hypergroup and $\cdot$ is an associative hyperoperation, which is distributive with respect to $+$. There are different notions of hyperrings. If only the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a usual operation, then we say that R is an additive hyperring. A special case of this type is the hyperring introduced by Krasner. There are comprehensive references for hyperrings; for example see [5] and [12]. In fact, different kinds of hyperrings are defined; one of them is the Krasner hyperring described as follows (Please, see [5], [11]).

Definition 1. *A commutative Krasner hyperring is an algebraic structure* $(R, +, \cdot)$ *which satisfies the following axioms:*

(1) $(R, +)$ *is a canonical hypergroup, i.e.,*

  (i) *For every* $x, y, z \in R$, $x + (y + z) = (x + y) + z$,

  (ii) *For every* $x, y \in R$, $x + y = y + x$,

  (iii) *There exists* $0 \in R$ *such that* $0 + x = \{x\}$, *for all* $x \in R$,

  (iv) *For all* $x \in R$ *there exists a unique elements* $x' \in R$ *such that* $0 \in x + x'$ *(We write* $-x$ *for* x'*),*

  (v) $z \in x + y$ *implies* $y \in -x + z$ *and* $x \in z - y$;

(2) $(R, \cdot)$ *is a semigroup having zero as a bilaterally absorbing element, i.e.,* $x \cdot 0 = 0 \cdot x = 0$.

(3) *The multiplication is distributive with respect to the hyperoperation* $+$.

Throughout this paper, by a hyperring we mean a commutative Krasner hyperring. A non-empty subset A of a hyperring $(R, +, \cdot)$ is called a subhyperring of R if $(A, +, \cdot)$ is itself a hyperring. The subhyperring A of R is called normal, if $x + A - x \subseteq A$ for all $x \in R$. A non-empty subset I of a hyperring R is called a hyperideal if and only if and only if the following conditions hold.

(1) $a - b \subseteq I$, for all $a, b \in I$;

(2) $a \in I$ and $r \in R$ imply that $a \cdot r \in I$ and $r \cdot a \in I$.

A hyperideal P of a hyperring R is called a prime hyperideal of R if, for every pair of elements a and b of R, the fact that $ab \in P$, implies either $a \in P$ or $b \in P$. If A is a normal hyperideal of a hyperring R, then we define the relation $x \equiv y \,(\bmod A)$ if and only if $x - y \cap A \neq \emptyset$. This relation is denoted by xA^*y and is an equivalence relation. Let $A^*(x)$ be the equivalence class of the element $x \in R$.

Lemma 1. (cf. [5, Corollary 3.2.5]) *Let* A *be a normal hyperideal of* R. *Then*

(1) $(A + x) + (A + y) = A + x + y$ *for all* $x, y \in R$.

(2) $A + x = A + y$ *for all* $y \in A + x$.

(3) $A + x = A^*(x)$ *for all* $x \in R$.

Proposition 1. (cf. [5, Proposition 3.2.13]) *Let* R *be a Krasner hyperring and* A *be a normal hyperideal of* R. *We define the hyperoperation* $\oplus$ *and the multiplication* $\odot$ *on the set of all classes* $[R : A^*] = \{A^*(x) \mid x \in R\}$, *as follows:*

$$A^*(x) \oplus A^*(y) = \{A^*(z) \mid z \in A^*(x) + A^*(y)\} \text{ and}$$
$$A^*(x) \odot A^*(y) = A^*(x \cdot y).$$

Then $[R : A^*]$ *is a Krasner hyperring.*

3 Total Graph of a Krasner hyperring

In this section, we introduced the total graph of a Krasner hyperring. First, we define a multiplicative-prime subset of Krasner hyperrings as follows:

Definition 2. *A nonempty proper subset H of Krasner hyperring $(R, +, \cdot)$ is called a multiplicative-prime subset of R if the following two conditions hold.*

(1) *$a \cdot b \in H$ for every $a \in H$ and $b \in R$;*

(2) *If $a \cdot b \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$.*

The concept of total graphs of classical rings and modules was initially introduced and studied in ([1], [2], [3], [7], [9]). Now, we will extend these definitions and properties to the general case of Krasner hyperrings.

Definition 3. *Let R be a Krasner hyperring and H be a multiplicative-prime subset of R. The total graph $\Gamma_H(R)$ is a graph with all elements of the Krasner hyperring R as vertices and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \subseteq H$. For $A \subseteq R$, let $\Gamma_H(A)$ be the induced subhypergraph of $\Gamma_H(R)$. For example $\Gamma_H(R \setminus H)$ is subhypergraph of $\Gamma_H(R)$ with vertices $R \setminus H$.*

According to the definition, if $H = R$, then it is easily seen that $\Gamma_H(R)$ is a complete graph, and it is a disconnected graph when $H = 0$ and $|R| \geq 2$. So, in what follows, due to our approach, without loss of generality, one may assume that $H \neq 0$ and $H \neq R$. The study of $\Gamma_H(R)$ breaks naturally into two cases depending on whether or not H is a hyperideal of R. First, we consider the case when H is a hyperideal of R.

4 The Case at Which H is a Hyperideal of R

Let H be a multiplicative-prime subset of a commutative Krasner hyperring R. In this section, we study $\Gamma_H(R)$ in case that H is a (prime) hyperideal of R; i.e., when H is closed under the hyperaddition. We first begin with the following lemma.

Lemma 2. *Let H be a normal hyperideal of a commutative Krasner hyperring R. Then*

(1) *If $x - y \subseteq H$ for every $x, y \in R \setminus H$, then $x + H = y + H$.(Similarly, if $x + y \subseteq H$ for $x, y \in R \setminus H$ then $x + H = -y + H$.)*

(2) *If $z + y \subseteq H$ and $z \in H$, then y is an element of H.*

Proof. (1) Let $x, y \in R \setminus H$ be arbitrary elements and let $x - y \subseteq H$. If $z \in x - y$, then $z \in H$ and so $x \in z + y = y + z \subseteq \cup_{z \in H} y + z = y + H$. Therefore, by Lemma 1, we have $x + H = y + H$.

(2) Let $a \in z + y \subseteq H$. Since R is a Krasner hyperring, then $y \in a - z$. Considering that $y \in H$ and $a - z \subseteq H$, one may deduce that y is an element of H, as desired. □

Now, we are going to investigate the connectedness of the subgraphs $\Gamma_H(H)$ and $\Gamma_H(R \setminus H)$.

Theorem 1. *Let H be a prime normal hyperideal of a commutative Krasner hyperring R. Then $\Gamma_H(H)$ is a complete (induced) subgraph of $\Gamma_H(R)$ and $\Gamma_H(H)$ is disjoint from $\Gamma_H(R \setminus H)$. Consequently, while $\Gamma_H(H)$ is always connected, $\Gamma_H(R)$ is never connected.*

Proof. Suppose that there exist $x \in \Gamma_H(H)$ and $y \in \Gamma_H(R \setminus H)$ such that $x + y \subseteq H$. Therefore $y \in y + 0 \subseteq y + x - x = x + y - x \subseteq H$, and so, $y \in H$ which is a contradiction. Since H is a hyperideal of R, then for any $x, y \in H$ we have $x + y \subseteq H$, which implies that $\Gamma_H(H)$ is connected. □

The next theorem gives a complete description of $\Gamma_H(R)$ when H is a hyperideal of R. It also shows that non-isomorphic hyperrings may have isomorphic graphs. We allow α and β to be infinite cardinals; if β is infinite, then $\beta - 1 = (\beta - 1)/2 = \beta$.

Theorem 2. *Let H be a prime normal hyperideal of a commutative Krasner hyperring R and let $|H| = \alpha$ and $|[R : H^*]| = |\{x + H | x \in R\}| = \beta$. The following statements hold.*

(1) *If $1 + 1 \subseteq H$, then $\Gamma_H(R \setminus H)$ is the union of $\beta - 1$ disjoint K^α's.*

(2) *If $1 + 1 \nsubseteq H$, then $\Gamma_H(R \setminus H)$ is the union of $(\beta - 1)/2$ disjoint $K^{\alpha,\alpha}$'s.*

Proof. (1) Let $1+1 \subseteq H$, and suppose that $x \in R\setminus H$. First we show that the coset $H^*(x) = x+H$ is a complete subgraph of $\Gamma_H(R \setminus H)$. Let $a, b \in x + H$. Then $a \in x + h_1$ and $b \in x + h_2$ for some $h_1, h_2 \in H$. Since H is a hyperideal and $x(1+1) = x+x \subseteq H$, then $a+b \subseteq (x+h_1)+(x+h_2) = (x + x) + h_1 + h_2 \subseteq H$ and the claim is proved. Now, assume that $x, y \in R \setminus H$. We show that the subgraphs with vertices set of cosets $H^*(x)$ and $H^*(y)$ are disjoint in $\Gamma_H(R \setminus H)$. Let $a \in H^*(x) = H + x = \bigcup_{h\in H} h + x$ and $b \in H^*(y) = H + x = \bigcup_{h\in H} h + y$. Then $a \in h_1 + x$ and $b \in h_2 + x$ for some $h_1, h_2 \in H$ which means that $x \in a - h_1$ and $y \in b - h_2$. Hence $x + y \subseteq a - h_1 + b - h_2 = (a + b) - (h_1 + h_2) \subseteq H$. On the other hand, as $0 \in y - y$ and $y + y = y(1 + 1) \subseteq H$, then $x - y = x - y + 0 \subseteq x - y + y - y = x + y - (y + y) \subseteq H$. Therefore, by Lemma 1 and [5, Lemma 3.2.9], it is concluded that $H^*(x) = H^*(y)$, which in turn, implies that $\Gamma_H(R \setminus H)$ is the union of $\beta - 1$ disjoint (induced) subgraphs $H^*(x)$, each of which equals K^α, where $\alpha = |H| = |H^*(x)|$.

(2) Now, assume that $1 + 1 \nsubseteq H$, and let $x \in R \setminus H$. We will show that no two distinct elements in $x + H$ are adjacent. Assume the contrary and let $a + b \in H$ for some $a \in x + z_1$ and $b \in x + z_2$, where $z_1, z_2 \in H$. Then $x \in a - z_1$ and $x \in b - z_2$. Thinking of the fact that H is a hyperideal, then $x + x \subseteq (a - z_1) + (b - z_2) = (a + b) - (z_1 + z_2) \subseteq H$. Therefore, as H is a prime hyperideal, one may deduce that $x \in H$ is a contraction. Now, we will prove that the cosets $x + H$ and $-x + H$ are adjacent. Indeed, for $a \in x + z_1$ and $b \in -x + z_2$, where $z_1, z_2 \in H$, being H a normal hyperideal gives $a + b \subseteq (x + z_1) + (-x + z_2) = x + (z_1 + z_2) - x \subseteq H$. Therefore, by what we have proved, $(x + H) \cup (-x + H)$ is a complete bipartite (induced) subgraph of $\Gamma_H(R \setminus H)$. Furthermore, if $a \in x + z_1$ is adjacent to $\in y + z_2$ for some $y \in R \setminus H$ and $z_1, z_2 \in H$, then $a + b \subseteq H$ and $x + y \subseteq a - z_1 + b - z_2 = (a + b) - (z_1 + z_2) \subseteq H$ as H is a hyperideal. Thus $x + y \subseteq H$ and hence $y + H = -x + H$, by Lemma 2. Consequently, $\Gamma_H(R \setminus H)$ is the union of $(\beta - 1)/2$ disjoint (induced) subgraphs $(x + H) \cup (-x + H)$, each of which equals $K^{\alpha,\alpha}$, where $\alpha = |H| = |H^*(x)|$ and the result follows. □

It is easily seen that, by the proof of Theorem 2, that, when $\Gamma_H(R \setminus H)$ is complete or connected, and its diameter and girth could be computed, explicitly. So we have the following corollary.

Corollary 1. *Let H be a prime normal hyperideal of a commutative Krasner hyperring R. Then, the following statements hold.*

(1) $\mathrm{diam}(\Gamma_H(R \setminus H)) = 0, 1, 2,$ *or* ∞.

(2) $\mathrm{gr}(\Gamma_H(R \setminus H)) = 3, 4,$ *or* ∞.

We have already observed in Theorem 1 that, when H is a hyperideal of R, then $\Gamma_H(H)$ is always connected and $\Gamma_H(R)$ is never connected. The following propositions give some new criteria for $\Gamma_H(R \setminus H)$ to be connected.

Proposition 2. *Let H be a prime normal hyperideal of a commutative Krasner hyperring R. Then, the following hold.*

(1) *Let G be an induced subgraph of $\Gamma_H(R \setminus H)$, and let x and y be distinct vertices of G that are connected by a path in G. Then there is a path of length, at most two, between x and y in G. In particular, if $\Gamma_H(R \setminus H)$ is connected, then $\mathrm{diam}(\Gamma_H(R \setminus H) \leq 2$.*

(2) *Let x and y be distinct elements of $R \setminus H$ that are connected by a path in $\Gamma_H(R \setminus H)$. If $x + y \nsubseteq H$; i.e., if x and y are not adjacent, then $x - (-x) - y$ and $x - (-y) - y$ are paths of length two between x and y in $\Gamma_H(R \setminus H)$.*

Proof. (1) It suffices to show that if x_1, x_2, x_3 and x_4 are distinct vertices of G and there is a path $x_1 - x_2 - x_3 - x_4$ from x_1 to x_4, then x_1 and x_4 are adjacent. If $a \in x_1 + x_4$, then $0 \in x_2 - x_2$ and $0 \in x_3 - x_3$, and so, $a = a + 0 \in x_1 + x_4 + (x_2 - x_2) + (x_3 - x_3) = (x_1 + x_2) - (x_2 + x_3) + (x_3 + x_4) \subseteq H$. Thus $x_1 + x_4 \subseteq H$.

(2) Suppose that $x + y \nsubseteq H$. Then there exists $z \in R \setminus H$ such that $x - z - y$ is a path of length two by part (1). By Lemma 2, it is clear that $z \in R \setminus H$. Let $a \in x + z \subseteq H$ and $b \in z + y \subseteq H$. Then $x \in a - z$ and $-y \in z - b$ and, by view that H is a normal hyperideal of R, it is concluded that $x - y \subseteq a - z + z - b = z + (a - b) - z \subseteq H$. Similarly, $y - x \subseteq H$. Also, $x \neq -x$ and $y \neq -y$, otherwise, $y - x \subseteq H$ and $x - y \subseteq H$ imply that $x + y \subseteq H$, which is a contradiction. If $y = -x$ then $x + y = x + y + 0 \subseteq x + y + H = x - x + H = x + H - x \subseteq H$, where H being a normal hyperideal, results $x + y \subseteq H$ which is obviously a contradiction. Therefore $y \neq -x$ and $x - (-x) - y$ and $x - (-y) - y$ are paths of length two between x and y and the result follows. □

Now, we will characterize the connectedness of $\Gamma_H(R \setminus H)$ in terms of some algebraic properties.

Proposition 3. *Let H be a normal prime hyperideal of a commutative Krasner hyperring R. Then, the following statements are equivalent.*

(1) $\Gamma_H(R \setminus H)$ *is connected.*

(2) *Either $x + y \subseteq H$ or $x - y \subseteq H$ for every $x, y \in R \setminus H$.*

(3) *Either* $x + y \subseteq H$ *or* $x + y + y \subseteq H$ *for every* $x, y \in R \setminus H$.

In particular, for all $x \in R \backslash H$*, either* $x + x \subseteq H$ *or* $x + x + x \subseteq H$*, but not both.*

Proof. (1) $\Rightarrow$ (2) Assume that $\Gamma_H(R \setminus H)$ is connected, and let $x, y \in R \setminus H$. Let $x = y$ and $z \in x - y$. Then we have $z \in z + 0 \subseteq x - y + H = x - x + H = x + H - x \subseteq H$, as H is normal hyperideal. Hence $x - y \subseteq H$. Now assume that $x \neq y$ and let $x + y \nsubseteq H$. If $x + y \nsubseteq H$, then, by Proposition 2, it is conclude that $x - (-y) - y$ is a path from x to y, and so, $x - y \subseteq H$.

(2) $\Rightarrow$ (3) Let $x + y \nsubseteq H$ and let $z \in x + y$ be an arbitrary element. Then $x \in z - y$ which implies that $z - y \nsubseteq H$, as $x \in R \setminus H$, and hence, $y + z \subseteq H$ by assumption. Therefore $(x + y) + y = \bigcup_{t \in (x+y)} t + y \subseteq H$. Now assume that $x + x$ and $x + x + x + x$ both be in H and let $a \in x + x + x \subseteq H$. Then $a \in \bigcup_{t \in \{x+x\}} t + x$. This implies that $a \in t' + x$ for some $t' \in x + x \subseteq H$. Since H is a hyperideal, then $x \in a - t \subseteq H$ that contracts the assumption.

(3) $\Rightarrow$ (1) If $x, y \in R \setminus H$ and $x + y \nsubseteq H$, then, by assumption, $x + y + y \subseteq H$. First, suppose that $y + y \subseteq H$. If $a \in x + y + y$, then $a \in \bigcup_{t \in \{y+y\}} x + t$, and so, $a \in x + t''$ for some $t'' \in y + y$, which, in turn, implies that $x \in a - t''$. Considering that $a, t'' \in H$ and H is a hyperideal of R, we deduce that $x \in H$, which is a contradiction. Therefore, $y + y \nsubseteq H$. This means that $y + y + y \subseteq H$. Let $b \in H \setminus y + y$. If $x = b$, then $x + y = b + y \subseteq y + y + y \subseteq H$ is a contradiction, and so, $x \neq b$. Now, from $x + b \subseteq x + y + y \subseteq H$ and $b + y \subseteq y + y + y \subseteq H$, it is concluded that $x - b - y$ is a path from x to y in $\Gamma_H(R \setminus H)$, and so, $\Gamma_H(R \setminus H)$ is connected, as desired. $\square$

Now, we are going to provide an example for Theorem 2.

Example 1. Let $R = \{0, a, b, c\}$ be a set with hyperoperation $+$" and the operation $\cdot$" that are defined as follows:

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0, b\}$	$\{a, c\}$	$\{b\}$
b	$\{b\}$	$\{a, c\}$	$\{0, b\}$	$\{a\}$
c	$\{c\}$	$\{b\}$	$\{a\}$	$\{0\}$

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

By [4, Example 4.8] the algebraic structure $(R, +, \cdot)$ is a Krasner hyperring with $1_R = a$. Let $H = \{0, b\}$. It is easy to see that H is a normal prime hyperideal of R and both subgraphs $\Gamma_H(H)$ and $\Gamma_H(R \setminus H)$ are the complete graph K^2. Thinking of the facts that $\alpha = \beta = 1$ and $a + a = \{0, b\} \in H$, then, by Theorem 2, it is concluded that $\Gamma_H(R)$ is the disjoint union of its (complete) subgraphs $\Gamma_H(H)$ and $\Gamma_H(R \setminus H)$. Also, by Corollary 1, it is easily seen that $\mathrm{diam}(\Gamma_H(R \setminus H)) = 1$ and $\mathrm{gr}(\Gamma_H(R \setminus H)) = \infty$.

The following example shows the normal condition for the prime hyperideal H, in Theorem 2, is necessary.

Example 2. Let $(G = \{e, a, b, c\}, \cdot)$ be the Klein four-group. Set $R = G \cup \{0, u, v\}$ where 0 is a multiplicative absorbing element and u, v are distinct orthogonal idempotents, with

$$a \cdot 0 = 0 \cdot a = 0; \text{ for all } a \in R;$$
$$u \cdot v = v \cdot u = 0; \; u \cdot u = u; \; v \cdot v = v;$$
$$u \cdot g = g \cdot u = u \text{ and } v \cdot g = g \cdot v = v \text{ for all } g \in G.$$

Let the hyperoperation $+$ on R be as follows.

$$a+0=0+a=\{a\};\ a+a=\{0,a\} \text{ for all } g\in R;$$
$$a+b=b+a=R\setminus\{0,a,b\} \text{ for all } a,b\in R\setminus\{0\} \text{ and } a\neq b.$$

Then $(R,+,\cdot)$, by [5, Page 76], is a Krasner hyperring. Let $H=\{0,u\}$. Then H, by [10, Example 2.2], is a prime hyperideal of R. It is easy to see that H is not a normal hyperideal of R. Also $1_R+1_R=e_G+e_G=\{e_G,0\}\nsubseteq H$. So, by Theorem 2, $\Gamma_H(R\setminus H)$ should be a union of complete bipartite graphs. Note that $R\setminus H=\{e,a,b,c,v\}$. The following calculations show that a is not adjacent to any vertices in $\Gamma_H(R\setminus H)$.

$$e+a=R\setminus\{0,e,a\}=\{b,c,u,v\}\nsubseteq H;\ b+a=R\setminus\{0,b,a\}=\{e,c,u,v\}\nsubseteq H;$$
$$c+a=R\setminus\{0,c,a\}=\{e,b,u,v\}\nsubseteq H;\ v+a=R\setminus\{0,v,a\}=\{e,b,c,u\}\nsubseteq H.$$

Similarly, one may show that no vertices of $R\setminus H$ are adjacent in $\Gamma_H(R\setminus H)$, and so, $\Gamma_H(R\setminus H)$ is a totally disconnected graph.

5 The Case at Which H is not a Hyperideal of R

In this section, we consider the case that the multiplicative-prime subset H is not a hyperideal of R. Since H is always closed under multiplication by elements of R, then $0\in H$, and so, there are distinct elements $x,y\in H^*=H\setminus\{0\}$ such that $x+y\subseteq R\setminus H$. In this case, $\Gamma_H(H)$ is always connected, but never complete with $\mathrm{diam}(\Gamma_H(H))=2$.

Theorem 3. *Let R be a commutative Krasner hyperring and H be a multiplicative-prime subset of R that is not a hyperideal. Then $\Gamma_H(H)$ is connected with* $\mathrm{diam}(\Gamma_H(H))=2$.

Proof. Let $x,y\in H^*$. If $x+y\subseteq H$, then $d(x,y)=1$, otherwise, as every $x\in H^*$ is adjacent 0, then $x-0-y$ is a path, in $\Gamma_H(H)$, of length two between any two distinct $x,y\in H^*$, and so, $d(x,y)=2$. Therefore, $\Gamma_H(H)$ is connected with $\mathrm{diam}(\Gamma_H(H))=2$. □

Now, we will compute the girth of $\Gamma_H(H)$.

Theorem 4. *Let R be a commutative Krasner hyperring and H be a multiplicative-prime subset of R that is not a hyperideal. Then* $\mathrm{gr}(\Gamma_H(H))=3$ *or* ∞.

Proof. Since H is not a hyperideal of R, then there exist $x,y\in H^*$ such that $x+y\nsubseteq H$. This implies that $|H|\geq 3$. If $x+y\subseteq H$ for some distinct elements $x,y\in H^*$, then $0-x-y-0$ is a 3-cycle in $\Gamma_H(H)$ and so $\mathrm{gr}(\Gamma_H(H))=3$, otherwise, $x+y\nsubseteq H$, for all distinct elements $x,y\in H^*$, and so, every element $x\in H^*$ is adjacent to 0 and no two distinct elements of H are adjacent. Therefore, $\Gamma_H(H)$ is a star graph and so $\mathrm{gr}(\Gamma_H(H))=\infty$, as desired. □

Corollary 2. *Let R be a commutative Krasner hyperring and H be a multiplicative-prime subset of R that is not a hyperideal. If* $\mathrm{gr}(\Gamma_H(R))=4$, *then* $\mathrm{gr}(\Gamma_H(R\setminus H))=4$.

Proof. The proof is clear by Theorem 3. □

The following is an example of a Krasner hyperring R, at which, $\Gamma_H(R\setminus H)$ is a complete graph, but the subgraphs $\Gamma_H(H)$ and $\Gamma_H(R\setminus H)$ are not disjoint.

Example 3. Let $\mathbb{Z}_{n_\xi} := \mathbb{Z}_n \cup \{\xi\}$, where $\xi \notin \mathbb{Z}$. Then, by [1, Theorem 4.23], $(\mathbb{Z}_{n_\xi}, +_\xi, 0, \cdot_\xi, 1)$ is a Krasner hyperring where, for any $x, y \in \mathbb{Z}_n$, the hyperoperation $+_\xi$ and the operation $\cdot_\xi$ on $\mathbb{Z}_{n_\xi}$ are defined as follows.

$$x +_\xi y = \begin{cases} \{0, \xi\} & x = -y \\ \{x + y\} & x, y \in \mathbb{Z}_n, x \neq y \\ \{y\} & x = 0 \text{ or } (x = \xi \text{ and } y \notin \{\xi\}) \\ \{\xi\} & x = y = \xi \end{cases}$$

and

$$x \cdot_\xi y = \begin{cases} \{x \cdot y\} & x, y \in \mathbb{Z}_n \\ \{\xi\} & (y = \xi \text{ and } \gcd(x, n) \neq 1) \text{ or } (x = y = \xi) \\ \{0\} & x = \xi, \gcd(x, n) = 1. \end{cases}$$

Now, let $n = 10$. Then $I_1 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \xi\}$ and $I_2 = \{\bar{0}, \bar{5}, \xi\}$ are hyperideals of $\mathbb{Z}_{n_\xi}$, by [1, Theorem 4.24 (iii)]. It is easy to see that I_1 and I_2 are prime hyperideals of $\mathbb{Z}_{n_\xi}$. Set $H := I_1 \cup I_2 = \{\bar{0}, \bar{2}, \bar{4}, \bar{5}, \bar{6}, \bar{8}, \xi\}$. Then H is a multiplicative-prime subset of $\mathbb{Z}_{n_\xi}$. Since $\bar{5} +_\xi \bar{8} = \bar{5} + \bar{8} = \bar{3} \notin H$, then H is not a hyperideal. From the relations $\bar{0} +_\xi \bar{2} = \{\bar{2}\} \subseteq H$, $\bar{2} +_\xi \bar{8} = \{0, \xi\} \subseteq H$ and $\bar{8} +_\xi \bar{0} = \{\bar{8}\} \subseteq H$ it is evident that $\bar{0} - \bar{2} - \bar{8} - \bar{0}$ is a cycle in $\Gamma_H(H)$. Therefore, by Theorem 3, it is concluded that $\mathrm{gr}(\Gamma_H(H)) = 3$, which in turn, implies that $\mathrm{gr}(\Gamma_H(R)) = 3$. On the other hand, from $\bar{1} +_\xi \bar{4} = \{\bar{5}\} \subseteq H$ and $\bar{1} +_\xi \bar{5} = \{\bar{6}\} \subseteq H$, it is easily seen that, some vertices of two subgraphs $\Gamma_H(H)$ and $\Gamma_H(R \setminus H)$ are adjacent and these subgraphs are not disjoint. Also, the relations $\bar{1} +_\xi \bar{3} = \{\bar{4}\} \subseteq H$, $\bar{3} +_\xi \bar{7} = \{0, \xi\} \subseteq H$ and $\bar{7} +_\xi \bar{1} = \{\bar{8}\} \subseteq H$, imply that $\bar{1} - \bar{3} - \bar{7} - \bar{1}$ is a cycle of length 3 in $\Gamma_H(R \setminus H)$. Therefore, $\mathrm{gr}(\Gamma_H(R \setminus H)) = 3$. Finally, it is not hard to see that $\Gamma_H(R \setminus H)$ is a complete graph with vertex set $\{\bar{1}, \bar{3}, \bar{7}, \bar{9}\}$.

It is easy to see that the union of prime hyperideals of a hyperring is a multiplicative-prime subset that is not necessarily a hyperideal of R. So, we end this paper with the following results.

Lemma 3. *Assume that R is a commutative Krasner hyperring and let H be a multiplicative-prime subset of R that is not a hyperideal of R. Set $H := \bigcup_\alpha P_\alpha$, where the P_α's vary over a set of prime hyperideals of R. If the intersection $\bigcap_\alpha P_\alpha$ contains a non-zero element, h say, then the following statements hold.*

(1) *If $A \subseteq H$, then $A + h \subseteq H$.*

(2) *If $a \in R \setminus H$, then $(a + h) \cap H = \emptyset$.*

Proof. (1) Let $x \in A + h$. Then $x \in a + h$ for some $a \in A \subseteq H$, and so, $a \in P_\beta$ for some β. Therefore $a + h \subseteq P_\beta$, which means that $x \in P_\beta \subseteq H$.

(2) Suppose the contrary and let $y \in (a + h) \cap H$. Since $y \in H = \bigcup_\alpha P_\alpha$, then there exists a hyperideal P_β such that $y \in P_\beta$. On the other hand, $y \in a + h$ implies that $a \in y - h \in P_\beta$ as P_β is a hyperideal of R and $h \in \bigcap_\alpha P_\alpha \subseteq P_\beta$. Thus $a \in P_\beta \subseteq H$ which is a contradiction. □

Theorem 5. *Let R be a commutative Krasner hyperring and $H = \bigcup_\alpha P_\alpha$, where the P_α's vary over a set of prime hyperideals of R, be a multiplicative-prime subset of R that is not a hyperideal of R. Let $a - b - c$ be a path of length two in $\Gamma_H(R \setminus H)$ for distinct vertices $a, b, c \in R \setminus H$. If $k + k \subseteq H$ for some $k \in \{a, b, c\}$ and $\bigcap_\alpha P_\alpha \neq 0$, then $\mathrm{gr}(\Gamma_H(R \setminus H)) = 3$.*

Proof. Let $a - b - c$ be a path in $\Gamma_H(R \setminus H)$ and let $0 \neq h \in \bigcap_\alpha P_\alpha$. We proceed in three distinguished cases.

Case 1: Let $a + a \subseteq H$. If $t \in a + h$ then, by Lemma 3, we have $t \notin H$. We show that, in this case, $a - b - t - a$ is the desired cycle in $\Gamma_H(R \setminus H)$. Since $b + a \subseteq H$, then, by Lemma 5.5, it is concluded that $b + t \subseteq b + a + h \subseteq H$. Now, again by Lemma 3 and hypothesis, $t + a \subseteq a + h + a = a + a + h \subseteq H$.

Case 2: If $b + b \subseteq H$ then, by Lemma 3, we have $(b + h) \cap H = \emptyset$. Let $t' \in b + h$. Then, by 3 and the fact that $c + b \subseteq H$, one may deduce that $c + t' \subseteq c + b + h \subseteq H$. Similarly, it is easily seen that $t' + b \subseteq b + h + b = b + b + h \subseteq H$. Therefore, in this case, $b - c - t' - b$ is the desired cycle in $\Gamma_H(R \setminus H)$.

Case 3: If $c + c \subseteq H$, then the proof proceeds as in Case 1. □

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