Journal of Algebra and Related Topics
Vol. 13, No. 01, 2025, pp. 135-157.

Ordered maps and kernels of OBCI-algebras

Eunsuk Yang[†∗], **Eun Hwan Roh**[‡], **Young Bae Jun**[§]

[†] *Department of Philosophy, Institute of Critical Thinking and Writing, Jeonbuk National University, Jeonju, Korea*
[‡] *Department of Mathematics Education, Chinju National University of Education, Jinju, Korea*
[§] *Department of Mathematics Education, Gyeongsang National University, Jinju, Korea*
Emails: eunsyang@jbnu.ac.kr, ehroh9988@gmail.com, skywine@gmail.com

Abstract. Yang-Roh-Jun recently introduced the notion of ordered BCI-algebras as a generalization of BCI-algebras. They further introduced the notions of homomorphisms of ordered BCI-algebras and studied associated properties. Here we generalize homomorphisms into ordered maps, i.e., order-preserving maps. More precisely, the notions of ordered maps and kernels of ordered BCI-algebras are first defined. Next, properties related to (ordered) subalgebras, (ordered) filters and direct products of ordered BCI-algebras are addressed.

Keywords: (ordered) BCI-algebra, Ordered map, Kernel, (ordered) Subalgebra, (ordered) Filter.
AMS Subject Classification 2010: 03B05, 03G25, 06F35.

1 Introduction: Background and Motivation

In universal algebra and logic, one of the important research plans is to introduce the classes of algebras and logics with more general structures. One of its good examples is BCK-algebras and their generalizations. Note that Imai and Iséki [11] first introduced the notion of BCK-algebras as logic algebras and then many algebras have been introduced as generalizations of those algebras. Its representative case is BCI-algebras and their generalizations.

In 1966 Imai and Iséki [11] first introduced BCK-algebras, Iséki [12] introduced the notion of BCI-algebras as a generalization of BCK-algebras. Since then, more general algebras such as B-algebras [16, 17], BH-algebras [13] and BCH-algebras [8, 9] have been introduced as generalizations of BCI-algebras. In particular, homomorphisms of such algebras have been studied (see e.g. [1–4, 6, 7, 10, 13–15, 17–19]). Note that homomorphisms are a useful tool for studying the relationship between two algebras.

∗Corresponding author
Received: 01 June 2023/ Revised: 15 January 2024/ Accepted: 27 January 2024
DOI: 10.22124/JART.2024.24864.1546

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All the generalizations of BCI-algebras above are defined by equations. One interesting fact is that Yang-Roh-Jun [20], very recently introduced a generalization of BCI-algebras defined by inequations. They, to be more precise, introduced the notion of ordered BCI-algebras (briefly OBCI-algebras). (Note that an OBCI-algebra is a structure with an underlying partial order.) They [21], moreover investigated homomorphisms of OBCI-algebras. One important difference between homomorphisms of BCI-algebras and homomorphisms of OBCI-algebras is that the former homomorphisms are order-preserving in themselves (see [6, 10]), whereas the latter homomorphisms are not necessary. This means that order-preserving maps, i.e., monotone maps, of OBCI-algebras can be studied independently of the homomorphisms of OBCI-algebras.

Davey and Priestley [5], studied ordered sets and lattices. In it, they defined the notion of order-preserving maps between two ordered sets as the *basic* notion of maps between ordered sets. However, they just dealt with these maps in a general context of ordered sets and lattices. In [20,21], Yang-Roh-Jun did not investigate such maps. This situation motivates us to introduce the notion of order-preserving maps between OBCI-algebras. To this end, first we define order-preserving maps and kernels of OBCI-algebras. Next we study properties related to (ordered) subalgebras, (ordered) filters and direct products of OBCI-algebras to emphasize the similarities and differences with the research on homomorphisms of OBCI-algebras in [20].

2 Preliminaries

Definition 1 ([20]). *Let L be a set with a binary operation " $\to$ ", a constant "e" and a binary relation " $\leq_e$ ". Then $\boldsymbol{L} := (L, \to, e, \leq_e)$ is called an ordered BCI-algebra (briefly, OBCI-algebra) if it satisfies the following conditions:*

$$(\forall w, c, v \in L)(e \leq_e (w \to c) \to ((c \to v) \to (w \to v))), \tag{1}$$

$$(\forall w, c \in L)(e \leq_e w \to ((w \to c) \to c)), \tag{2}$$

$$(\forall w \in L)(e \leq_e w \to w), \tag{3}$$

$$(\forall w, c \in L)(e \leq_e w \to c,\ e \leq_e c \to w \ \Rightarrow\ w = c), \tag{4}$$

$$(\forall w, c \in L)(w \leq_e c \ \Leftrightarrow\ e \leq_e w \to c), \tag{5}$$

$$(\forall w, c \in L)(e \leq_e w,\ w \leq_e c \ \Rightarrow\ e \leq_e c). \tag{6}$$

Proposition 1 ([20]). *Every OBCI-algebra $\boldsymbol{L} := (L, \to, e, \leq_e)$ satisfies:*

$$(\forall w \in L)(e \to w = w). \tag{7}$$

$$(\forall w, c, v \in L)(v \to (c \to w) = c \to (v \to w)). \tag{8}$$

$$(\forall w, c, v \in L)(e \leq_e w \to c \ \Rightarrow\ e \leq_e (c \to v) \to (w \to v)). \tag{9}$$

$$(\forall w, c, v \in L)(e \leq_e w \to c,\ e \leq_e c \to v \ \Rightarrow\ e \leq_e w \to v). \tag{10}$$

$$(\forall w, c, v \in L)(e \leq_e (c \to v) \to ((w \to c) \to (w \to v))). \tag{11}$$

$$(\forall w, c, v \in L)(e \leq_e w \to c \ \Rightarrow\ e \leq_e (v \to w) \to (v \to c)). \tag{12}$$

For future convenience, $\mathbf{L} := (L, \to, e, \leq_e)$ represents the OBCI-algebra unless otherwise specified.

Definition 2 ([20]). *A subset A of L is called*

- *a subalgebra of $\boldsymbol{L} := (L, \to, e, \leq_e)$ if it satisfies:*

$$(\forall w, c \in L)(w, c \in A \ \Rightarrow \ w \to c \in A). \tag{13}$$

- *an ordered subalgebra of $\boldsymbol{L} := (L, \to, e, \leq_e)$ if it satisfies:*

$$(\forall w, c \in L)(w, c \in A, \, e \leq_e w, \, e \leq_e c \ \Rightarrow \ w \to c \in A). \tag{14}$$

Definition 3 ([20]). *A subset F of L is called*

- *a filter of $\boldsymbol{L} := (L, \to, e, \leq_e)$ if it satisfies:*

$$e \in F, \tag{15}$$
$$(\forall w, c \in L)(w \to c \in F, \, w \in F \ \Rightarrow \ c \in F). \tag{16}$$

- *an ordered filter of $\boldsymbol{L} := (L, \to, e, \leq_e)$ if it satisfies (15) and*

$$(\forall w, c \in L)(w \in F, \, e \leq_e w \to c \ \Rightarrow \ c \in F). \tag{17}$$

Proposition 2 ([20]). *If an ordered filter F of $\boldsymbol{L} := (L, \to, e, \leq_e)$ satisfies*

$$(\forall w \in L)(w \in F \ \Rightarrow \ e \leq_e w), \tag{18}$$

then it is a filter of $\boldsymbol{L} := (L, \to, e, \leq_e)$.

Definition 4 ([20]). *An (ordered) filter F of $\boldsymbol{L} := (L, \to, e, \leq_e)$ is said to be closed if it is a subalgebra of $\boldsymbol{L} := (L, \to, e, \leq_e)$; ordered closed if it is an ordered subalgebra of $\boldsymbol{L} := (L, \to, e, \leq_e)$.*

Definition 5 ([20]). *Let $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ and $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ be OBCI-algebras. Consider a binary operation " $\Rightarrow$ ", a constant "$\mathbf{e}$" and a binary relation " $\ll$ " in the Cartesian product $L \times Y$ defined as follows:*

$$(w, c) \Rightarrow (c, u) = (w \to_L c, c \to_Y u),$$
$$\mathbf{e} = (e_L, e_Y),$$
$$(w, c) \ll (c, u) \ \Leftrightarrow \ w \leq_L c, \, c \leq_Y u)$$

for all $(w, c), (c, u) \in L \times Y$. $\boldsymbol{L} \times \boldsymbol{Y} := (L \times Y, \Rightarrow, \mathbf{e}, \ll)$ is said to be a direct product OBCI-algebra if it is an OBCI-algebra.

3 Ordered maps

In this section, we first introduce ordered maps and kernels and then investigate several properties related to (ordered) subalgebras, (ordered) filters and direct products.

3.1 Ordered maps and kernels

Definition 6. *Let* $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ *and* $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$ *be OBCI-algebras. A mapping* $\zeta : L \rightarrow Y$ *is called an ordered map (briefly, O-mapping) if it satisfies:*

$$(\forall w, c \in L)(e_L \leq_L w \rightarrow_L c \;\Rightarrow\; e_Y \leq_Y \zeta(w) \rightarrow_Y \zeta(c)). \tag{19}$$

Given OBCI-algebras $\mathbf{L} := (L, \rightarrow_L, e_L, \leq_L)$ and $\mathbf{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$, the mapping $\zeta : L \rightarrow Y,\ w \mapsto e_Y$, is an O-map. In fact, let $w, c \in L$ be such that $e_L \leq_L w \rightarrow_L c$. Then $e_Y \leq_Y e_Y \rightarrow_Y e_Y = \zeta(w) \rightarrow_Y \zeta(c)$ by (3).

Example 1. Let $L := \{e, w, c\}$ and $Y = \{e, c\}$ be sets with binary operations " $\rightarrow_L$ " and " $\rightarrow_Y$ " given by Table 1 and Table 2, respectively.

Table 1: Cayley table for the binary operation "$\rightarrow$"

$\rightarrow_L$	e	w	c
e	e	w	c
w	e	e	c
c	c	c	e

Table 2: Cayley table for the binary operation "$\rightarrow$"

$\rightarrow_Y$	e	c
e	e	c
c	c	e

Let $\leq_L := \{(e, e), (w, w), (c, c), (w, e)\}$ and $\leq_Y := \{(e, e), (c, c)\}$. Then $\mathbf{L} := (L, \rightarrow_L, e, \leq_L)$ and $\mathbf{Y} := (Y, \rightarrow_Y, e, \leq_Y)$ are OBCI-algebras. Define a mapping

$$\zeta : L \rightarrow Y,\ v \mapsto \begin{cases} e & \text{if } v \in \{e, w\}, \\ c & \text{if } v = c. \end{cases} \tag{20}$$

It is routine to verify that ζ is an O-map.

Lemma 1. *Let* ζ *be a mapping from an OBCI-algebra* $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ *to an OBCI-algebra* $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$*. Let* $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ *and* $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$ *be OBCI-algebras. If* $\zeta : L \rightarrow Y$ *is an O-map, then*

$$e_Y \leq_Y \zeta(e_L) \rightarrow_Y \zeta(e_L). \tag{21}$$

Proof. It is straightforward by the combination of (3) and (19). □

Proposition 3. *Let* $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ *and* $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$ *be OBCI-algebras.*

(i) *Every O-mapping* $\zeta : L \to Y$ *satisfies* (21) *and*

$$(\forall w, c \in L)(w \leq_L c \ \Rightarrow \zeta(w) \leq_Y \zeta(c)). \tag{22}$$

(ii) *Suppose that* $e_Y = \zeta(e_L)$. *Every O-mapping* $\zeta : L \to Y$ *further satisfies:*

$$(\forall w, c \in L)(e_L \leq_L w \to_L c \ \Rightarrow \ e_Y \leq_Y \zeta(w \to_L c)). \tag{23}$$

$$(\forall w, c \in L)(w \leq_L c \ \Rightarrow \ e_Y \leq_Y \zeta(w \to_L c)). \tag{24}$$

Proof. (i) Clearly, (21) is true by (3). Let $w, c \in L$ be such that $w \leq_L c$. Then $e_L \leq_L w \to_L c$ by (5), and so $e_Y \leq_Y \zeta(w) \to_Y \zeta(c)$, that is, $\zeta(w) \leq_Y \zeta(c)$.

(ii) For (23), let $e_Y = \zeta(e_L)$ and $w, c \in L$ be such that $e_L \leq_L w \to_L c$. Then $e_L \leq_L e_L \to_L (w \to_L c)$ by (7), and so

$$e_Y \leq_Y \zeta(e_L) \to_Y \zeta(w \to_L c)$$

by the O-mapping of ζ. Hence $e_Y \leq_Y \zeta(w \to_L c)$ by (5) and $e_Y = \zeta(e_L)$.

The proof for (24) is almost the same as (23). We just note that (5) assures that $e_L \leq_L w \to_L c$ if and only if $w \leq_L c$. □

Let ζ be a mapping from an OBCI-algebra $\mathbf{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\mathbf{Y} := (Y, \to_Y, e_Y, \leq_Y)$. The map ζ satisfying (23) may not be an O-map, as seen in the following example.

Example 2. Let $L := \{1, \frac{2}{3}, \frac{1}{3}, 0\}$ and $Y := \{1, e, \partial, 0\}$ be sets with binary operation " $\to_L$ " and " $\to_Y$ " given by Table 3 and Table 4, respectively.

Table 3: Cayley table for the binary operation "$\to_Y$"

$\to_L$	1	$\frac{2}{3}$	$\frac{1}{3}$	0
1	1	0	0	0
$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0
$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	0
0	1	1	1	1

Table 4: Cayley table for the binary operation "$\to_L$"

$\to_Y$	1	e	∂	0
1	1	0	0	0
e	1	e	∂	0
∂	1	∂	e	0
0	1	1	1	1

Let

$$\leq_L := \{(1,1), (\tfrac{2}{3}, \tfrac{2}{3}), (\tfrac{1}{3}, \tfrac{1}{3}), (0,0), (\tfrac{2}{3}, 1), (\tfrac{1}{3}, \tfrac{2}{3}), (0, \tfrac{1}{3})\},$$

and

$$\leq_Y := \{(0,0), (e,e), (\partial,\partial), (1,1), (0,e), (0,\partial), (e,1), (\partial,1)\}.$$

Then $\mathbf{L} := (L, \rightarrow_L, \frac{2}{3}, \leq_L)$ and $\mathbf{Y} := (Y, \rightarrow_Y, e, \leq_Y)$ are OBCI-algebras. Define a mapping ζ from L to Y as follows:

$$\zeta : L \to Y,\ w \mapsto \begin{cases} 1 & \text{if } w = 1, \\ e & \text{if } w = \frac{2}{3}, \\ \partial & \text{if } w = \frac{1}{3}, \\ 0 & \text{if } w = 0. \end{cases} \tag{25}$$

For $\frac{2}{3}, \frac{1}{3} \in L$, we have $\frac{2}{3} \leq_L \frac{1}{3} \rightarrow_L \frac{2}{3} = \frac{2}{3}$ and $e \leq_Y \zeta(\frac{1}{3} \rightarrow_L \frac{2}{3}) = e$. However, $e \not\leq_Y \zeta(\frac{1}{3}) \rightarrow_Y \zeta(\frac{2}{3}) = \partial \rightarrow_Y e = \partial$. Therefore, the map ζ does not form an O-map.

We provide a condition for a map ζ satisfying (23) to be an O-map.

Theorem 1. *Let ζ be a mapping from an OBCI-algebra $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$. If ζ satisfies (23) and*

$$(\forall w, c \in L)(e_Y \leq_Y \zeta(w \rightarrow_L c) \ \Rightarrow \zeta(w) \leq_Y \zeta(c)), \tag{26}$$

then it is an O-map.

Proof. Let ζ be a mapping from an OBCI-algebra $\mathbf{L}$ to an OBCI-algebra $\mathbf{Y}$ and satisfy (23) and (26). Let $w, c \in L$ be such that $e_L \leq_L w \rightarrow_L c$. Then $e_Y \leq_Y \zeta(w \rightarrow_L c)$ by (23), and so $\zeta(w) \leq_Y \zeta(c)$ by (26). Hence $e_Y \leq_Y \zeta(w) \rightarrow_Y \zeta(c)$ by (5). Therefore, ζ is an O-map. □

Definition 7. *Let ζ be a mapping from an OBCI-algebra $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$. The kernel of ζ is defined to be a subset, denoted by* $\ker(\zeta)$*, of L that satisfies:*

$$(\forall w \in L)(w \in \ker(\zeta) \ \Leftrightarrow \ e_Y \leq_Y \zeta(w)). \tag{27}$$

It is clear that $\ker(\zeta) := \{w \in L \mid e_Y \leq_Y \zeta(w)\}$, and it is unique.

Example 3. Let $L := \{1, e_L, w, c\}$ and $Y := \{1, e_Y, c, u\}$ be sets with binary operation "$\rightarrow_L$" and "$\rightarrow_Y$" given by Table 5 and Table 6, respectively. Let

$$\leq_L := \{(1,1), (e_L, e_L), (w,w), (c,c), (e_L, 1), (w, e_L), (c, w)\},$$

and

$$\leq_Y := \{(1,1), (e_Y, e_Y), (c,c), (u,u), (e_Y, 1), (c, e_Y)\}.$$

Then $\mathbf{L} := (L, \rightarrow_L, e_L, \leq_L)$ and $\mathbf{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$ are OBCI-algebras. Define a mapping ζ from L to Y as follows:

$$\zeta : L \to Y,\ w \mapsto \begin{cases} 1 & \text{if } w = 1, \\ e_Y & \text{if } w = e_L, \\ c & \text{if } w = w, \\ u & \text{if } w = c. \end{cases} \tag{28}$$

Table 5: Cayley table for the binary operation "$\rightarrow_Y$"

$\rightarrow_L$	1	e_L	w	c
1	1	w	w	c
e_L	1	e_L	w	c
w	1	1	1	c
c	1	1	1	1

Table 6: Cayley table for the binary operation "$\rightarrow_L$"

$\rightarrow_Y$	1	e_Y	c	u
1	1	c	c	u
e_Y	1	e_Y	c	u
c	1	1	1	u
u	u	u	u	1

Then ζ is not an O-mapping since $e_L \leq_L c \rightarrow_L 1 = 1$ but $e_Y \not\leq_Y \zeta(c) \rightarrow_Y \zeta(1) = u \rightarrow_Y 1 = u$. It is routine to calculate that $\ker(\zeta) = \{1, e_L\}$.

Proposition 4 ([21])**.** *If ζ is a mapping from an OBCI-algebra $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$, then its kernel* $\ker(\zeta)$ *is given by the following set.*

$$\ker(\zeta) = \{c \in L \mid (\exists w \in L)(e_Y \leq_Y \zeta(w), e_Y \leq_Y \zeta(w) \rightarrow_Y \zeta(c))\}. \tag{29}$$

Example 4. Let $L = \{1, e, \partial, 0\}$ be a set with the binary operation " $\rightarrow$ " given by Table 4. If

$$\leq_e := \{(0,0), (e,e), (\partial,\partial), (1,1), (0,e), (0,\partial), (e,1), (\partial,1)\},$$

then $\mathbf{L} := (L, \rightarrow, e, \leq_e)$ is an OBCI-algebra (see [20]). Consider the identity map $\zeta : L \rightarrow L$. Clearly, ζ is an O-map, and we have $\ker(\zeta) = \{1, e\}$.

3.2 Kernels and (ordered) subalgebras

In the following example, we know that kernels may not be (ordered) subalgebras.

Example 5. Let $L = \{1, e, \partial, 0\}$ be a set with a binary operation " $\rightarrow$ " given by Table 7. Let

$$\leq_e := \{(0,0), (e,e), (\partial,\partial), (1,1), (0,e), (0,\partial), (e,1), (\partial,1)\}.$$

Then it is routine to verify that $\mathbf{L} := (L, \rightarrow, e, \leq_e)$ is an OBCI-algebra. Let $\zeta : L \rightarrow L$ be an automorphism as the identity map. Then $\ker(\zeta) = \{e, 1\}$ and it is neither a subalgebra of $\mathbf{L} := (L, \rightarrow, e, \leq_e)$ nor an ordered subalgebra of $\mathbf{L} := (L, \rightarrow, e, \leq_e)$ since $1 \rightarrow e = 0 \notin \ker(\zeta)$.

Definition 8. *Let $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ and $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$ be OBCI-algebras, and $\zeta : L \rightarrow Y$ be an O-map. The kernel of ζ is said to be closed if it is a subalgebra of $\boldsymbol{L}$; and ordered closed (briefly O-closed) if it is an ordered subalgebra of $\boldsymbol{L}$.*

Table 7: Cayley table for the binary operation "→"

$\rightarrow$	1	e	∂	0
1	1	0	0	0
e	1	e	∂	0
∂	1	0	e	0
0	1	1	1	1

For simplicity, by the expression '(O-)closed,' we ambiguously denote both 'closed' and 'O-closed' together if these are not be distinguished.

Example 6 ([20]). Let $L = \{e, c, u, z, w\}$ be a set with a binary operation " → " given by Table 8.

Table 8: Cayley table for the binary operation "→"

$\rightarrow$	e	c	u	z	w
e	e	c	u	z	w
c	e	e	u	z	z
u	e	c	e	z	w
z	z	w	z	e	c
w	z	z	z	e	e

Let $\leq_e := \{(e, e), (c, c), (u, u), (z, z), (w, w), (c, e), (u, e), (w, z)\}$. Then $\mathbf{L} := (L, \rightarrow, e, \leq_e)$ is an OBCI-algebra. Let $\zeta : L \rightarrow L$ be an automorphism as the identity map. Then $\ker(\zeta) = \{e\}$ and it is the (O-)closed kernel of ζ.

Proposition 5. *Let* $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ *and* $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$ *be OBCI-algebras, and* $\zeta : L \rightarrow Y$ *be an* O*-map.*

(i) *If* $\ker(\zeta)$ *is the closed kernel of* ζ*, then it satisfies:*

$$(\forall w \in L)(e_Y = \zeta(e_L) \leq_Y \zeta(w) \;\Rightarrow\; w \rightarrow_L e_L \in \ker(\zeta)). \tag{30}$$

(ii) *If* $\ker(\zeta)$ *is the* O*-closed kernel of* ζ *and satisfies* (18)*, then it further satisfies* (30)*.*

Proof. (i) Let $\ker(\zeta)$ be the closed kernel of ζ and $w \in L$ be such that $e_Y = \zeta(e_L) \leq_Y \zeta(w)$. Then $w \in \ker(\zeta)$ by (27), and so $w \rightarrow_L e_L \in \ker(\zeta)$ by (13) because $e_L \in \ker(\zeta)$.

(ii) Let $\ker(\zeta)$ is the O-closed kernel of ζ and $w \in L$ be such that $e_Y = \zeta(e_L) \leq_Y \zeta(w)$. Then as above $e_Y \leq_Y \zeta(w)$ and so $w \in \ker(\zeta)$. Hence $e_L \leq_L w$ by (18), and so $w \rightarrow_L e_L \in \ker(\zeta)$ by (14) because $e_L \leq_L e_L$ and $e_L \in \ker(\zeta)$. □

Let the kernel $\ker(\zeta)$ of ζ be not (O-)closed in Proposition 5. Then the condition (30) does not hold as we can see in the following example.

Example 7. The kernel $\ker(\zeta) = \{e, 1\}$ in Example 5 does not satisfy (30) since $\zeta(e) \leq_e \zeta(1)$ but $1 \to e = 0 \notin \ker(\zeta)$.

Given an OBCI-algebra $\mathbf{L} := (L, \to_L, e_L, \leq_L)$ and a set $A \subseteq L$, consider the following assertion.

$$(\forall w, c \in L)(w \to c \in A \ \Rightarrow \ w \leq_L c). \tag{31}$$

The following example verifies that $\ker(\zeta)$ may not satisfy the condition (31).

Example 8. Let $L := \{e_L, c\}$ and $Y := \{e_Y\}$ be sets with binary operation " $\to_L$ " and " $\to_Y$ " given by Table 2 and $e_Y \to_Y e_Y = e_Y$, respectively. Let $\leq_L := \{(e_L, e_L), (c, c)\}$ and $\leq_Y := \{(e_Y, e_Y)\}$. Then $\mathbf{L} := (L, \to_L, e_L, \leq_L)$ and $\mathbf{Y} := (Y, \to_Y, e_Y, \leq_Y)$ are OBCI-algebras. Define a mapping ζ from L to Y as follows:

$$\zeta : L \to Y,\ w, \in L, \mapsto e_Y. \tag{32}$$

It is clear that ζ is an O-mapping and L is the kernel $\ker(\zeta)$ of ζ. Then $e_L \to_L c \in \ker(\zeta)$ but $e_L \not\leq_L c$. Hence it does not satisfy the condition (31).

We provide conditions for kernels to be (O-)closed.

Theorem 2. *Let* $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ *and* $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ *be OBCI-algebras, and* $\zeta : L \to Y$ *be an* O*-map.*

(i) *If the kernel* $\ker(\zeta)$ *of* ζ *satisfies* (30) *and* (31)*, then it is closed.*

(ii) *If the kernel* $\ker(\zeta)$ *of* ζ *satisfies* (30)*, then it is* O*-closed.*

Proof. (i) Suppose that $\ker(\zeta)$ satisfies (30) and (31). It suffices to show that $\ker(\zeta)$ is a subalgebra of $\mathbf{L}$. Let $w, c \in \ker(\zeta)$. Then $e_Y \leq_Y \zeta(w)$ and $e_Y \leq_Y \zeta(c)$ by (27). Thus $w \to_L e_L \in \ker(\zeta)$ and $c \to_L e_L \in \ker(\zeta)$ by (30), and so $e_Y \leq_Y \zeta(w \to_L e_L)$ and $e_Y \leq_Y \zeta(c \to_L e_L)$ by (27). Since

$$e_L \leq_L (w \to_L e_L) \to_L ((e_L \to_L c) \to_L (w \to_L c))$$

by (1), we have

$$e_Y \leq_Y \zeta(w \to_L e_L) \to_Y \zeta((e_L \to_L c) \to_L (w \to_L c))$$

by the O-mapping of ζ, and so

$$e_Y \leq_Y \zeta((e_L \to_L c) \to_L (w \to_L c))$$

by (5) and (6). Then $(e_L \to_L c) \to_L (w \to_L c) \in \ker(\zeta)$, and so $c \to_L (w \to_L c) \in \ker(\zeta)$ by (7). Using (31), we further obtain $c \leq_L w \to_L c$, and so $e_L \leq_L c \to_L (w \to_L c)$ by (5). Hence

$$e_Y \leq_Y \zeta(c) \to_L \zeta(w \to_L c)$$

by the O-mapping of ζ, and so similarly $e_Y \leq_Y \zeta(w \to_L c)$, i.e., $w \to_L c \in \ker(\zeta)$. Therefore $\ker(\zeta)$ is a subalgebra of $\mathbf{L}$, and the proof is completed.

(ii) Suppose that $\ker(\zeta)$ satisfies (30). It suffices to show that $\ker(\zeta)$ is an ordered subalgebra of **L**. Let $w, c \in \ker(\zeta)$ such that $e_L \leq_L w$ and $e_L \leq_L c$. Then, as the proof in (i), we can show that $c \to_L (w \to_L c) \in \ker(\zeta)$ and so $e_Y \leq_Y \zeta(c \to_L (w \to_L c))$. Note that

$$e_L \leq_L (e_L \to_L c) \to_L ((c \to_L (w \to_L c)) \to_L (e_L \to_L (w \to_L c))$$

by (1), and so

$$e_L \leq_L c \to_L ((c \to_L (w \to_L c)) \to_L (w \to_L c))$$

by (7). Then

$$e_L \leq_L (c \to_L (w \to_L c)) \to_L (w \to_L c)$$

by (5) and (6), and so

$$e_Y \leq_Y \zeta(c \to_L (w \to_L c)) \to_Y \zeta(w \to_L c)$$

by the O-mapping of ζ. Hence

$$e_Y \leq_Y \zeta(w \to_L c), i.e., w \to_L c \in \ker(\zeta)$$

by (5) and (6). Therefore $\ker(\zeta)$ is an ordered subalgebra of **L**, and the proof is completed. □

Let $\zeta : L \to Y$ be a mapping from an OBCI-algebra $\mathbf{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\mathbf{Y} := (Y, \to_Y, e_Y, \leq_Y)$. For $A \subseteq L$ and $B \subseteq Y$, consider the following three assertions.

$$(\forall w, c \in L)(w \leq_e c,\, w \in A \;\Rightarrow\; c \in A). \tag{33}$$

$$(\forall w, c \in L)(w \to_L c \in A \;\Rightarrow\; \zeta(w) \leq_Y \zeta(c)). \tag{34}$$

$$(\forall w, c \in L)(\zeta(w) \to_Y \zeta(c) \in B \;\Rightarrow\; w \leq_L c). \tag{35}$$

By adding these assertions to subalgebras of **L** and **Y**, we can show that ζ is a mapping between such subalgebras.

Theorem 3. *Let $\zeta : L \to Y$ be a mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$.*

- (i) *If H is a subalgebra of $\boldsymbol{Y}$ and satisfies (35), then $\zeta^{-1}(H)$ containing e_L and satisfying (33) is a subalgebra of $\boldsymbol{L}$.*
- (ii) *Suppose that ζ is subjective. If F is a subalgebra of $\boldsymbol{L}$ and satisfies (34), then $\zeta(F)$ containing e_Y and satisfying (33) is a subalgebra of $\boldsymbol{Y}$.*

Proof. (i) Let H be a subalgebra of **Y** and satisfy (35), $\zeta^{-1}(H)$ contain e_L and satisfy (33), and $w, c \in \zeta^{-1}(H)$. Then $\zeta(w) \in H$ and $\zeta(c) \in H$. Hence $\zeta(w) \to_Y \zeta(c) \in H$ by (13), and so $w \leq_L c$ by (35). Then $e_L \leq_L w \to_L c$ by (5). Hence, by (33), $w \to_L c \in \zeta^{-1}(H)$ since $e_L \in \zeta^{-1}(H)$. Therefore the $\zeta^{-1}(H)$ is a subalgebra of **L**.

(ii) Let ζ be surjective, F be a subalgebra of **L** and satisfy (34), $\zeta(F)$ contain e_Y and satisfy (33), and $c, u \in \zeta(F)$. Then there are $w, c \in F$ such that $c = \zeta(w)$ and $u = \zeta(c)$, and so $w \to_L c \in F$ by (13). Hence $\zeta(w) \leq_Y \zeta(c)$ by (34), and so $e_Y \leq_L \zeta(w) \to_L \zeta(c)$ by (5). Then, by (33), we have $c \to_Y u \in \zeta(F)$ since $c \to_Y u = \zeta(w) \to_L \zeta(c)$ and $e_Y \in \zeta(F)$. Therefore the $\zeta(F)$ is a subalgebra of **Y**. □

Let $\zeta : L \to Y$ be an O-mapping from an OBCI-algebra $\mathbf{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\mathbf{Y} := (Y, \to_Y, e_Y, \leq_Y)$. Then similarly we can show that ζ is an O-mapping between some (ordered) subalgebras.

Lemma 2. *Let $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ be an OBCI-algebra. Given a set $A \subseteq L$, the assertion (31) is equivalent to the assertion (18).*

Proof. (31) $\Longrightarrow$ (18) : Let $w \in L$ be such that $w \in A$. Then $e_L \to_L w \in A$ by (7), and so $e_L \leq_L w$ by (31). (18) $\Longrightarrow$ (31) : Let $w, c \in L$ be such that $w \to_L c \in A$. Then $e_L \leq_L w \to_L c$ by (18), and so $w \leq_L c$ by (5). □

Theorem 4. *Let $\zeta : L \to Y$ be an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a subalgebra of $\boldsymbol{Y}$, contains e_Y and satisfies (35) and (33), then $\zeta^{-1}(H)$ is a subalgebra of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective. If F is a subalgebra of $\boldsymbol{L}$ and satisfies (31), then $\zeta(F)$ containing e_Y and satisfying (33) is a subalgebra of $\boldsymbol{Y}$.*

Proof. (i) Let H be a subalgebra of $\mathbf{Y}$, contain e_Y and satisfy (35) and (33), and $w, c \in \zeta^{-1}(H)$. Then, as in the proof of (i) in Theorem 3, we can obtain $w \leq_L c$ using (35), and so

$$e_Y \leq_Y \zeta(w \to_L c)$$

by (24) in Proposition 3. Hence $\zeta(w \to_L c) \in H$ by (33), and so $w \to_L c \in \zeta^{-1}(H)$. Therefore the $\zeta^{-1}(H)$ is a subalgebra of $\mathbf{L}$.

(ii) Let ζ be surjective, F be a subalgebra of $\mathbf{L}$ and satisfy (31), $\zeta(F)$ contain e_Y and satisfy (33), and $c, u \in \zeta(F)$. Then, as in the proof of (ii) in Theorem 3, we can construct $w, c \in F$ such that $c = \zeta(w)$ and $u = \zeta(c)$, and so $w \to_L c \in F$ by (13). Hence $e_L \leq_L w \to_L c$ by Lemma 2, and so

$$e_Y \leq_Y \zeta(w) \to_Y \zeta(c) = c \to_Y u$$

by the O-mapping of ζ. Then, by (33), we have $c \to_Y u \in \zeta(F)$ since $e_Y \in \zeta(F)$. Therefore the $\zeta(F)$ is a subalgebra of $\mathbf{Y}$. □

Theorem 5. *Let $\zeta : L \to Y$ be an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is an ordered subalgebra of $\boldsymbol{Y}$, contains e_Y and satisfies (35) and (33), then $\zeta^{-1}(H)$ is an ordered subalgebra of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective. If F is an ordered subalgebra of $\boldsymbol{L}$ and satisfies (18), then $\zeta(F)$ containing e_Y and satisfying (33) is an ordered subalgebra of $\boldsymbol{Y}$.*

Proof. (i) Let H be an ordered subalgebra of $\mathbf{Y}$, contain e_Y and satisfy (35) and (33), and $w, c \in L$ be such that $w, c \in \zeta^{-1}(H)$, $e_L \leq_L w$ and $e_L \leq_L c$. Then $e_L \leq_L e_L \to_L w$ and $e_L \leq_L e_L \to_L c$ by (7), and so

$$e_Y \leq_Y \zeta(e_L) \to_Y \zeta(w) = \zeta(w)$$

and

$$e_Y \leq_Y \zeta(e_L) \rightarrow_Y \zeta(c) = \zeta(c)$$

by the O-mapping of ζ, $e_Y = \zeta(e_L)$ and (7). Then, as in the proof of (i) in Theorem 4, we can show $w \rightarrow_L c \in \zeta^{-1}(H)$. Therefore the $\zeta^{-1}(H)$ is an ordered subalgebra of $\mathbf{L}$.

(ii) Let ζ be surjective, F be an ordered subalgebra of $\mathbf{L}$ and satisfy (31), $\zeta(F)$ contain e_Y and satisfy (33), and $c, u \in Y$ be such that $c, u \in \zeta(F)$, $e_Y \leq_Y c$ and $e_Y \leq_Y u$. Then as above we can construct $w, c \in F$ such that $c = \zeta(w)$ and $u = \zeta(c)$, and so $e_L \leq_L w$ and $e_L \leq_L c$ by (18). Hence $w \rightarrow_L c \in F$ by (14), and so $e_L \leq_L w \rightarrow_L c$ by (18). Then

$$e_Y \leq_Y \zeta(w) \rightarrow_Y \zeta(c) = c \rightarrow_Y u$$

by the O-mapping of ζ. Then, as in the proof of (ii) in Theorem 4, we can show $c \rightarrow_Y u \in \zeta(F)$. Therefore the $\zeta(F)$ is an ordered subalgebra of $\mathbf{Y}$. □

3.3 Kernels and (ordered) filters

Let $\zeta : L \rightarrow Y$ be an O-mapping from an OBCI-algebra $\mathbf{L} := (L, \rightarrow_L, e_L, \leq_L)$ to an OBCI-algebra $\mathbf{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$. The following theorem shows that (31) is the condition for a kernel of ζ to be a filter of $\mathbf{L}$.

Theorem 6. *Let $\zeta : L \rightarrow Y$ be an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$. If* $\ker(\zeta)$ *satisfies* (31)*, then it is a filter of $\boldsymbol{L}$.*

Proof. Let $\ker(\zeta)$ satisfy (31) and $w, c \in L$ be such that $w \rightarrow_L c \in \ker(\zeta)$ and $w \in \ker(\zeta)$. Then $w \leq_L c$ by (31), and so $e_L \leq_L w \rightarrow_L c$ by (5). Hence $e_Y \leq_Y \zeta(w)$ by (27) and

$$e_Y \leq_Y \zeta(w) \rightarrow_Y \zeta(c)$$

by the O-mapping of ζ. Then $e_Y \leq_Y \zeta(c)$ by (5) and (6), and so $c \in \ker(\zeta)$ by (27). Therefore $\ker(\zeta)$ is a filter of $\mathbf{L}$. □

Theorem 7. *If $\zeta : L \rightarrow Y$ is an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$, then* $\ker(\zeta)$ *is an ordered filter of $\boldsymbol{L}$.*

Proof. It is clear that $e_L \in \ker(\zeta)$. Let $w, c \in L$ be such that $w \in \ker(\zeta)$ and $e_L \leq_L w \rightarrow_L c$. Then $e_Y \leq_Y \zeta(w)$ by (27) and $e_Y \leq_Y \zeta(w) \rightarrow_Y \zeta(c)$ by the O-mapping of ζ. It follows from (5) and (6) that $e_Y \leq_Y \zeta(c)$, that is, $c \in \ker(\zeta)$. Hence $\ker(\zeta)$ is an ordered filter of $\mathbf{L}$. □

Here we address several mappings between (ordered) filters of OBCI-algebras. The order of dealing with mappings between filters of OBCI-algebras is as follows: 1. mappings between subsets/filters of OBCI-algebras; 2. O-maps between subsets/filters of OBCI-algebras; 3. mappings between subsets/filters of OBCI-algebras containing kernels; 4. O-maps between subsets/filters of OBCI-algebras containing kernels.

Theorem 8. *Let $\zeta : L \rightarrow Y$ be a mapping from an OBCI-algebra $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a subset of Y, then $\zeta^{-1}(H)$ satisfying* (33) *and* (18) *is a filter of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective. If F is a subset of L and satisfies (33) and (18), then $\zeta(F)$ satisfying (35) is a filter of $\boldsymbol{Y}$.*

Proof. (i) Let $H \subseteq Y$ and $\zeta^{-1}(H)$ satisfy (33) and (18). Since $e_Y = \zeta(e_L)$ and so $e_L = \zeta^{-1}(e_Y)$, we have $e_L \in \zeta^{-1}(G)$. Let $w, c \in L$ be such that $w \in \zeta^{-1}(H)$ and $w \to_L c \in \zeta^{-1}(H)$. Then $e_L \leq_L w$ and $e_L \leq_L w \to_L c$ by (18), and so $e_L \leq_L c$ by (5) and (6). Hence $c \in \zeta^{-1}(H)$ by (33), and therefore the $\zeta^{-1}(H)$ is a filter of $\mathbf{L}$.

(ii) Let ζ be surjective, F $(\subseteq L)$ satisfy (33) and (18) and $\zeta(F)$ satisfy (35). Since $e_L \in F$, we have $e_Y = \zeta(e_L) \in \zeta(F)$. Let $c, u \in Y$ be such that $c \in \zeta(F)$ and $c \to_Y u \in \zeta(F)$. Then there are $w \in F$ and $c \in L$ such that $\zeta(w) = c$ and $\zeta(c) = u$, and so $\zeta(w) \in \zeta(F)$ and $\zeta(w) \to_Y \zeta(c) \in \zeta(F)$. Thus $e_L \leq_L w$ by (18) and $w \leq_L c$ by (35), and so $e_L \leq_L c$ by (6). Hence $c \in F$ by (33) and so $u = \zeta(c) \in \zeta(F)$. Therefore the $\zeta(F)$ is a filter of $\mathbf{Y}$. □

Corollary 1. *Let $\zeta : L \to Y$ be a mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a filter of $\mathbf{Y}$, then $\zeta^{-1}(H)$ satisfying (33) and (18) is a filter of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective. If F is a filter of $\boldsymbol{L}$ and satisfies (33) and (18), then $\zeta(F)$ satisfying (35) is a filter of $\boldsymbol{Y}$.*

Theorem 9. *Let $\zeta : L \to Y$ be an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a subset of Y and satisfies (33) and (18), then $\zeta^{-1}(H)$ is a filter of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective. If F is a subset of L and satisfies (18), then $\zeta(F)$ satisfying (33) and (35) is a filter of $\boldsymbol{Y}$.*

Proof. (i) Let $H \subseteq Y$ and $\zeta^{-1}(H)$ satisfy (33) and (18). As above, $e_L \in \zeta^{-1}(G)$. Let $w, c \in L$ be such that $w \in \zeta^{-1}(H)$ and $w \to_L c \in \zeta^{-1}(H)$. Then $e_L \leq_L w = e_L \to_L w$ by (18) and (7) and $e_L \leq_L w \to_L c$ by (18), and so

$$e_Y \leq_Y \zeta(e_L) \to_Y \zeta(w) = \zeta(w)$$

and

$$e_Y \leq_Y \zeta(w) \to_Y \zeta(c)$$

by the O-mapping of ζ and (7). Hence $e_Y \leq_Y \zeta(c)$ by (5) and (6), and so $\zeta(c) \in H$, that is, $c \in \zeta^{-1}(H)$, by (33). Therefore the $\zeta^{-1}(H)$ is a filter of $\mathbf{L}$.

(ii) Let ζ be surjective, F $(\subseteq L)$ satisfy (18) and $\zeta(F)$ satisfy (33) and (35). As above, $e_Y = \zeta(e_L) \in \zeta(F)$. Let $c, u \in Y$ be such that $c \in \zeta(F)$ and $c \to_Y u \in \zeta(F)$. There are $w \in F$ and $c \in L$ such that $\zeta(w) = c$ and $\zeta(c) = u$, and so $\zeta(w) \to_Y \zeta(c) \in \zeta(F)$. Then $e_L \leq_L w$ by (18) and $w \leq_L c$ by (35), and so $e_L \leq_L c = e_L \to_L c$ by (6) and (7). Hence

$$e_Y \leq_Y \zeta(e_L) \to_Y \zeta(c) = \zeta(c)$$

by the O-mapping of ζ, $e_Y = \zeta(e_L)$ and (7), and so $u = \zeta(c) \in \zeta(F)$ by (33). Therefore the $\zeta(F)$ is a filter of $\mathbf{Y}$. □

Corollary 2. *Let $\zeta : L \to Y$ be an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a filter of $\boldsymbol{Y}$ and satisfies (33) and (18), then $\zeta^{-1}(H)$ is a filter of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective. If F is a filter of $\boldsymbol{L}$ and satisfies (18), then $\zeta(F)$ satisfying (33) and (35) is a filter of $\boldsymbol{Y}$.*

If we take $\zeta^{-1}(H)$ and F as filters containing kernels of ζ in Theorems 8 and 9, we can drop the condition (33) as follows.

Theorem 10. *Let $\zeta : L \to Y$ be a mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a subset of Y, then $\zeta^{-1}(H)$ containing $\ker(\zeta)$ and satisfying (34) is a filter of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective. If F is a subset of L containing $\ker(\zeta)$, then $\zeta(F)$ satisfying (18) is a filter of $\boldsymbol{Y}$.*

Proof. (i) Let $H \subseteq Y$, and $\zeta^{-1}(H)$ contain $\ker(\zeta)$ and satisfy (34). As above, $e_L \in \zeta^{-1}(G)$. Let $w, c \in L$ be such that $w \in \zeta^{-1}(H)$ and $w \to_L c \in \zeta^{-1}(H)$. Then $e_Y \leq_Y \zeta(w)$ and $\zeta(w) \leq_Y \zeta(c)$ by (34), (7) and $e_Y = \zeta(e_L)$, and so $e_Y \leq_Y \zeta(c)$ by (6). Hence $c \in \ker(\zeta) \subseteq \zeta^{-1}(H)$, and therefore the $\zeta^{-1}(H)$ is a filter of $\mathbf{L}$.

(ii) Let ζ be surjective, F ($\subseteq L$) contain $\ker(\zeta)$ and $\zeta(F)$ satisfy (18). As above $e_Y = \zeta(e_L) \in \zeta(F)$. Let $c, u \in Y$ be such that $c \in \zeta(F)$ and $c \to_Y u \in \zeta(F)$. There are $w \in F$ and $c \in L$ such that $\zeta(w) = c$ and $\zeta(c) = u$, and so $\zeta(w) \in \zeta(F)$ and $\zeta(w) \to_Y \zeta(c) \in \zeta(F)$. Then $e_Y \leq_Y \zeta(w)$ and $\zeta(w) \leq_Y \zeta(c)$ by (18). Then as in (i) we have $e_Y \leq_Y \zeta(c)$, and so $c \in \ker(\zeta) \subseteq F$. Hence $u = \zeta(c) \in \zeta(F)$, and therefore the $\zeta(F)$ is a filter of $\mathbf{Y}$. □

Corollary 3. *Let $\zeta : L \to Y$ be a mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a filter of $\boldsymbol{Y}$, then $\zeta^{-1}(H)$ containing $\ker(\zeta)$ and satisfying (34) is a filter of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective. If F is a filter of $\boldsymbol{L}$ containing $\ker(\zeta)$, then $\zeta(F)$ satisfying (18) is a filter of $\boldsymbol{Y}$.*

Theorem 11. *Let $\zeta : L \to Y$ be an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a subset of Y, then $\zeta^{-1}(H)$ containing $\ker(\zeta)$ and satisfying (18) is a filter of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective. If F is a subset of L containing $\ker(\zeta)$ and satisfying (18), then $\zeta(F)$ satisfying (35) is a filter of $\boldsymbol{Y}$.*

Proof. (i) Let $H \subseteq Y$, and $\zeta^{-1}(H)$ contain $\ker(\zeta)$ and satisfy (18). As above, $e_L \in \zeta^{-1}(G)$. Let $w, c \in L$ be such that $w \in \zeta^{-1}(H)$ and $w \to_L c \in \zeta^{-1}(H)$. Then $e_L \leq_L w$ and $e_L \leq_L w \to_L c$ by (18), and so $e_L \leq_L c$ by (5) and (6). Hence, using the O-mapping of ζ, (7) and $e_Y = \zeta(e_L)$, we obtain $e_Y \leq_Y \zeta(c)$, and so $c \in \ker(\zeta) \subseteq \zeta^{-1}(H)$. Therefore the $\zeta^{-1}(H)$ is a filter of $\mathbf{L}$.

(ii) Let ζ be surjective, F $(\subseteq L)$ contain $\ker(\zeta)$ satisfy (18) and $\zeta(F)$ satisfy (35). As above $e_Y = \zeta(e_L) \in \zeta(F)$. Let $c, u \in Y$ be such that $c \in \zeta(F)$ and $c \to_Y u \in \zeta(F)$. There are $w \in F$ and $c \in L$ such that $\zeta(w) = c$ and $\zeta(c) = u$, and so $\zeta(w) \to_Y \zeta(c) \in \zeta(F)$. Then $e_L \leq_L w$ by (18) and $w \leq_L c$ by (35). Hence $e_L \leq_L c$ by (6). Then as in (i) we have $e_Y \leq_Y \zeta(c)$, and so $c \in \ker(\zeta) \subseteq F$. Hence $u = \zeta(c) \in \zeta(F)$, and therefore the $\zeta(F)$ is a filter of $\mathbf{Y}$. □

Corollary 4. *Let $\zeta : L \to Y$ be an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a filter of $\mathbf{Y}$, then $\zeta^{-1}(H)$ containing* $\ker(\zeta)$ *and satisfying* (18) *is a filter of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective. If F is a filter of $\boldsymbol{L}$ containing* $\ker(\zeta)$ *and satisfying* (18)*, then $\zeta(F)$ satisfying* (35) *is a filter of $\boldsymbol{Y}$.*

Similarly we deal with several mappings between ordered filters of OBCI-algebras.

Theorem 12. *Let $\zeta : L \to Y$ be a mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a subset of Y satisfying* (33)*, then $\zeta^{-1}(H)$ is an ordered filter of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective and satisfies*

$$(\forall w, c \in L)(\zeta(w) \leq_Y \zeta(c) \;\Rightarrow\; w \leq_L c). \tag{36}$$

If F is a subset of L satisfying (33)*, then $\zeta(F)$ is an ordered filter of $\boldsymbol{Y}$.*

Proof. (i) Let $H \subseteq Y$ satisfy (33). Clearly $e_L \in \zeta^{-1}(G)$. Let $w, c \in L$ be such that $w \in \zeta^{-1}(H)$ and $e_L \leq_L w \to_L c$. Then $w \leq_L c$ by (5), and so $c \in \zeta^{-1}(H)$ by (33). Therefore $\zeta^{-1}(H)$ is an ordered filter of $\mathbf{L}$.

(ii) Let ζ be surjective and satisfy (36), and $F \subseteq L$ satisfy (33). Clearly $e_Y \in \zeta(F)$. Let $c, u \in Y$ be such that $c \in \zeta(F)$ and $e_Y \leq_Y c \to_Y u$. As above, there exist $w \in F$ and $c \in L$ such that $\zeta(w) = c \in \zeta(F)$ and $\zeta(c) = u$, and so $e_Y \leq_Y \zeta(w) \to_Y \zeta(c)$. Then $w \leq_L c$ by (36) and (5), and so $c \in F$ by (33). Hence $u = \zeta(c) \in \zeta(F)$, and therefore $\zeta(F)$ is an ordered filter of $\mathbf{Y}$. □

Note that the assertion (33) is equivalent to (17) by (5). Hence we obtain the following as a corollary of Theorem 12.

Corollary 5. *Let $\zeta : L \to Y$ be a mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is an ordered filter of $\boldsymbol{Y}$, then $\zeta^{-1}(H)$ is an ordered filter of $\boldsymbol{L}$.*

(ii) *Suppose that ζ is surjective and satisfies* (36)*. If F is an ordered filter of $\boldsymbol{L}$, then $\zeta(F)$ is an ordered filter of $\boldsymbol{Y}$.*

Theorem 13. *Let $\zeta : L \to Y$ be an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$.*

(i) *If H is a subset of Y satisfying (18), then $\zeta^{-1}(H)$ containing* $\ker(\zeta)$ *is an ordered filter of* $\boldsymbol{L}$.

(ii) *Suppose that ζ is surjective. If F is a subset of L containing* $\ker(\zeta)$ *and satisfying (18), then $\zeta(F)$ is an ordered filter of* $\boldsymbol{Y}$.

Proof. (i) Let $H \subseteq Y$ satisfy (18). Clearly $e_L \in \zeta^{-1}(G)$. Let $w, c \in L$ be such that $w \in \zeta^{-1}(H)$ and $e_L \leq_L w \rightarrow_L c$. Then $\zeta(w) \in H$, and so $e_Y \leq_Y \zeta(w)$ by (18). Moreover

$$e_Y \leq_Y \zeta(w) \rightarrow_Y \zeta(c)$$

by the O-homomorphism of ζ, and so $e_Y \leq_Y \zeta(c)$ by $e_Y = \zeta(e_L)$, (5) and (6). Hence $c \in \ker(\zeta) \subseteq \zeta^{-1}(H)$. Therefore $\zeta^{-1}(H)$ is an ordered filter of $\mathbf{L}$.

(ii) Let ζ be surjective, and $F \subseteq L$ contain $\ker(\zeta)$ and satisfy (18). Clearly $e_Y \in \zeta(F)$. Let $c, u \in Y$ be such that $c \in \zeta(F)$ and $e_Y \leq_Y c \rightarrow_Y u$. As above, there exist $w \in F$ and $c \in L$ such that $\zeta(w) = c \in \zeta(F)$ and $\zeta(c) = u$, and so $e_Y \leq_Y \zeta(w) \rightarrow_Y \zeta(c)$. Moreover, $e_L \leq_L e_L \rightarrow_L w$ by (18) and (7), and so

$$e_Y \leq_Y \zeta(e_L) \rightarrow_Y \zeta(w) = \zeta(w)$$

by the O-homomorphism of ζ, $e_Y = \zeta(e_L)$ and (7). Then as in (i) we have $e_Y \leq_Y \zeta(c)$, and so $c \in \ker(\zeta) \subseteq F$. Hence $u = \zeta(c) \in \zeta(F)$, and therefore $\zeta(F)$ is an ordered filter of $\mathbf{Y}$. □

Corollary 6. *Let $\zeta : L \rightarrow Y$ be a surjective mapping from an OBCI-algebra $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$ such that $e_Y = \zeta(e_L)$. If F is an ordered filter of $\boldsymbol{L}$, contains* $\ker(\zeta)$ *and satisfies (18), then $\zeta(F)$ is an ordered filter of $\boldsymbol{Y}$.*

Using Corollary 4 and Corollary 6, we can prove the following two theorems.

Theorem 14. *Let $\zeta : L \rightarrow Y$ be an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \rightarrow_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$ such that ζ is surjective and satisfies $e_Y = \zeta(e_L)$. If we consider two sets:*

$$\mathcal{F} := \{F \mid F \text{ is a filter of } L \text{ containing } \ker(\zeta) \text{ and satisfying } (18)\},$$
$$\mathcal{G} := \{H \mid H \text{ satisfying } (35) \text{ is a filter of } Y\},$$

then there exists a bijective function

$$\varrho : \mathcal{F} \rightarrow \mathcal{G},\ F \mapsto \varrho(F) \tag{37}$$

such that $\varrho^{-1}(H) = \zeta^{-1}(H)$.

Proof. We first show that

$$(\forall F \in \mathcal{F})(\zeta^{-1}(\zeta(F)) = F). \tag{38}$$

Let $F \in \mathcal{F}$. It is clear that $F \subseteq \zeta^{-1}(\zeta(F))$. Let $w \in \zeta^{-1}(\zeta(F))$. Then $\zeta(w) \in \zeta(F)$ and so $\zeta(w) = \zeta(v)$ for some $v \in F$. Then since $v \in F$, we have $e_L \leq_L v = e_L \rightarrow_L v$ by (18) and (7). Moreover, since $e_L \leq_L w \rightarrow_L w$ by (3), we obtain

$$e_Y \leq_Y \zeta(e_L) \rightarrow_Y \zeta(v)$$

by the O-mapping of ζ, and so $e_Y \leq_Y \zeta(v)$ by $e_Y = \zeta(e_L)$ and (5). Also,

$$e_Y \leq_Y \zeta(w) \to_Y \zeta(w) = \zeta(v) \to_Y \zeta(w)$$

by the O-map. Hence $e_Y \leq_Y \zeta(w)$ by (5) and (6), and so $w \in \ker(\zeta)$ by (27). Thus, since $\ker(\zeta) \subseteq F$, we get $w \in F$. This shows that $\zeta^{-1}(\zeta(F)) \subseteq F$, and therefore (38) is valid. Using Corollary 4(ii), we can consider the mapping $\varrho : \mathcal{F} \to \mathcal{G},\ F \mapsto \varrho(F)$ given by $\varrho(F) = \zeta(F)$ for all $F \in \mathcal{F}$. Suppose that $\varrho(F_1) = \varrho(F_2)$ for all $F_1, F_2 \in \mathcal{F}$. Then $\zeta(F_1) = \zeta(F_2)$, which implies from (38) that

$$F_1 = \zeta^{-1}(\zeta(F_1)) = \zeta^{-1}(\zeta(F_2)) = F_2.$$

Hence ϱ is one-to-one. Since ζ is a surjective O-map, we know that $H = \zeta(\zeta^{-1}(H))$ and $\zeta^{-1}(H)$ is a filter of $\mathbf{L}$ containing $\ker(\zeta)$ and satisfying (18) for all $H \in \mathcal{G}$, i.e., $\zeta^{-1}(H) \in \mathcal{F}$, and $\varrho(\zeta^{-1}(H)) = H$. Thus ϱ is onto, and therefore ϱ is a bijective function and $\varrho^{-1}(H) = \zeta^{-1}(H)$. □

Theorem 15. *Let $\zeta : L \to Y$ be an O-mapping from an OBCI-algebra $\boldsymbol{L} := (L, \to_L, e_L, \leq_L)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ such that ζ is surjective and satisfies $e_Y = \zeta(e_L)$. If we consider two sets:*

$$\mathcal{F} := \{F \mid F \text{ is an ordered filter of } L \text{ containing } \ker(\zeta) \text{ and satisfying } (18)\},$$
$$\mathcal{G} := \{H \mid H \text{ is an ordered filter of } Y\},$$

then there exists a bijective function ϱ satisfying (37) as in Theorem 14.

Proof. The proof is almost the same as the proof in Theorem 14. We just note that using Corollary 6, we can consider the mapping $\varrho : \mathcal{F} \to \mathcal{G},\ F \mapsto \varrho(F)$ given by $\varrho(F) = \zeta(F)$ for all $F \in \mathcal{F}$. □

3.4 Kernels and direct products

It is henceforth assumed that $\mathbf{L}_1 := (L_1, \to_{L1}, e_{L1}, \leq_{L1})$, $\mathbf{L}_2 := (L_2, \to_{L2}, e_{L2}, \leq_{L2})$, $\mathbf{Y}_1 := (Y_1, \to_{Y1}, e_{Y1}, \leq_{Y1})$ and $\mathbf{Y}_2 := (Y_2, \to_{Y2}, e_{Y2}, \leq_{Y2})$ are OBCI-algebras.

Definition 9. *Let $\zeta_1 : L_1 \to Y_1$ and $\zeta_2 : L_2 \to Y_2$ be maps from $\boldsymbol{L}_1$ to $\mathbf{Y}_1$ and from $\boldsymbol{L}_2$ to $\mathbf{Y}_2$, respectively. Consider direct product OBCI-algebras:*

$$\boldsymbol{L} = \boldsymbol{L}_1 \times \boldsymbol{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L) \text{ and } \boldsymbol{Y} = \boldsymbol{Y}_1 \times \boldsymbol{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y).$$

A map $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ is said to be a direct product O-map if it is an O-map.

Theorem 16. *Let $\zeta_1 : L_1 \to Y_1$ and $\zeta_2 : L_2 \to Y_2$ be O-maps from $\boldsymbol{L}_1$ to $\boldsymbol{Y}_1$ and from $\boldsymbol{L}_2$ to $\boldsymbol{Y}_2$, respectively. Let $\boldsymbol{L} = \boldsymbol{L}_1 \times \boldsymbol{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$ and $\boldsymbol{Y} = \boldsymbol{Y}_1 \times \boldsymbol{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y)$ be direct product OBCI-algebras. Then a map $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ such that*

$$\zeta : \boldsymbol{L} \to \boldsymbol{Y},\ (x_1, x_2) \mapsto (\zeta_1(x_1), \zeta_2(x_2)) \tag{39}$$

is a direct product O-map.

Proof. Let a map $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ satisfy (39). We need to prove

$$\mathbf{e}_X \ll_L (x_1, x_2) \Rightarrow_L (c_1, c_2) \;\Rightarrow\; \mathbf{e}_Y \ll_Y \zeta(x_1, x_2) \Rightarrow_Y \zeta(c_1, c_2). \tag{40}$$

Assume that $\mathbf{e}_X \ll_L (x_1, x_2) \Rightarrow_L (c_1, c_2)$. Then $(x_1, x_2) \ll_L (c_1, c_2)$ by (5), and so

$$e_{L1} \leq_{L1} x_1 \to_{L1} c_1 \text{ and } e_{L2} \leq_{L2} x_2 \to_{L2} c_2$$

by the definition of $\ll_L$ and (5). Then

$$e_{Y1} \leq_{Y1} \zeta_1(x_1) \to_{Y1} \zeta_1(c_1) \text{ and } e_{Y2} \leq_{Y2} \zeta_2(x_2) \to_{Y2} \zeta_2(c_2)$$

by the O-maps of ζ_1 and ζ_2, and so

$$(\zeta_1(x_1), \zeta_2(x_2)) \ll_Y (\zeta_1(c_1), \zeta_2(c_2))$$

by (5) and the definition of $\ll_Y$. Hence

$$\mathbf{e}_Y \ll_Y (\zeta_1(x_1), \zeta_2(x_2)) \Rightarrow_Y (\zeta_1(c_1), \zeta_2(c_2))$$

by (5), and so

$$\mathbf{e}_Y \ll_Y \zeta(x_1, x_2) \Rightarrow_Y \zeta(c_1, c_2)$$

by (39). Hence (40) holds true, and therefore ζ is a direct product O-map. □

Definition 10. *Let $\zeta_1 : L_1 \to Y_1$ and $\zeta_2 : L_2 \to Y_2$ be mappings from $\boldsymbol{L}_1$ to $\boldsymbol{Y}_1$ and from $\boldsymbol{L}_2$ to $\boldsymbol{Y}_2$, respectively. Let $\boldsymbol{L} = \boldsymbol{L}_1 \times \boldsymbol{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$ and $\boldsymbol{Y} = \boldsymbol{Y}_1 \times \boldsymbol{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y)$ be direct product OBCI-algebras, and a map $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ satisfy (39). A subset A of $L_1 \times L_2$ such that $A = A_1 \times A_2$, where $A_1 \subseteq L_1$ and $A_2 \subseteq L_2$, is said to be the direct product kernel of ζ, denoted by $\ker_{DP}(\zeta)$, if both A_1 and A_2 satisfy (27).*

Theorem 17. *Let $\zeta_1 : L_1 \to Y_1$ and $\zeta_2 : L_2 \to Y_2$ be mappings from $\boldsymbol{L}_1$ to $\boldsymbol{Y}_1$ and from $\boldsymbol{L}_2$ to $\boldsymbol{Y}_2$, respectively; $\boldsymbol{L} = \boldsymbol{L}_1 \times \boldsymbol{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$ and $\boldsymbol{Y} = \boldsymbol{Y}_1 \times \boldsymbol{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y)$ be direct product OBCI-algebras; and a map $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ satisfy (39). Then $\ker(\zeta_1) \times \ker(\zeta_2)$ is the direct product kernel of ζ.*

Proof. Let $x_1 \in L_1$ and $x_2 \in L_2$ be such that $x_1 \in \ker(\zeta_1)$ and $x_2 \in \ker(\zeta_2)$, and $\mathbf{e_Y}$ be (e_{Y1}, e_{Y2}). It suffices to show that

$$(x_1, x_2) \in \ker(\zeta_1) \times \ker(\zeta_2) \;\Leftrightarrow\; \mathbf{e_Y} \ll_Y \zeta(x_1, x_2). \tag{41}$$

It can be verified as follows: We have

$$(x_1, x_2) \in \ker(\zeta_1) \times \ker(\zeta_2) = (x_1 \in \ker(\zeta_1), x_2 \in \ker(\zeta_2))$$

since $\times$ is the Cartesian product;

$$(x_1 \in \ker(\zeta_1), x_2 \in \ker(\zeta_2)) \;\Leftrightarrow\; (e_{Y1} \leq_{Y1} \zeta_1(x_1), e_{Y2} \leq_{Y2} \zeta_2(x_2))$$

by (27);

$$(e_{Y1} \leq_{Y1} \zeta_1(x_1), e_{Y2} \leq_{Y2} \zeta_2(x_2)) \;\Leftrightarrow\; (e_{Y1}, e_{Y2}) \ll_Y (\zeta_1(x_1), \zeta_2(x_2))$$

by the definition of $\ll_Y$; and

$$(e_{Y1}, e_{Y2}) \ll_Y (\zeta_1(x_1), \zeta_2(x_2)) \;\Leftrightarrow\; \mathbf{e_Y} \ll_Y \zeta(x_1, x_2)$$

by (39). Hence (41) holds true, and therefore $\ker(\zeta_1) \times \ker(\zeta_2)$ is the direct product kernel $\ker_{DP}(\zeta)$ of ζ. □

Corollary 7. *Let $\zeta_1 : L_1 \to Y_1$ and $\zeta_2 : L_2 \to Y_2$ be O-maps from $\boldsymbol{L}_1$ to $\boldsymbol{Y}_1$ and from $\boldsymbol{L}_2$ to $\boldsymbol{Y}_2$, respectively; $\boldsymbol{L} = \boldsymbol{L}_1 \times \boldsymbol{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$ and $\boldsymbol{Y} = \boldsymbol{Y}_1 \times \boldsymbol{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y)$ be direct product OBCI-algebras; and a direct product O-mapping $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ satisfy (39). Then $\ker(\zeta_1) \times \ker(\zeta_2)$ is the direct product kernel of ζ.*

Theorem 18. *Let $\zeta_1 : L_1 \to Y_1$ and $\zeta_2 : L_2 \to Y_2$ be mappings from $\boldsymbol{L}_1$ to $\boldsymbol{Y}_1$ and from $\boldsymbol{L}_2$ to $\boldsymbol{Y}_2$, respectively; $\boldsymbol{L} = \boldsymbol{L}_1 \times \boldsymbol{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$ and $\boldsymbol{Y} = \boldsymbol{Y}_1 \times \boldsymbol{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y)$ be direct product OBCI-algebras; and a map $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ satisfy (39). The direct product kernel $\ker_{DP}(\zeta)$ of ζ can be expressed as $\ker_{DP}(\zeta) = \ker(\zeta_1) \times \ker(\zeta_2)$ of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$, respectively.*

Proof. Let $\ker_{DP}(\zeta)$ be the kernel of $\mathbf{L}_1 \times \mathbf{L}_2$. Consider the projections $\pi_{L_1} : L_1 \times L_2 \to L_1$ and $\pi_{L_2} : L_1 \times L_2 \to L_2$. Let $\ker(\zeta_1) := \pi_{L_1}(\ker_{DP}(\zeta))$ and $\ker(\zeta_2) := \pi_{L_2}(\ker_{DP}(\zeta))$, and $x_1 \in L_1$ and $x_2 \in L_2$ be such that $x_1 \in \ker(\zeta_1)$ and $x_2 \in \ker(\zeta_2)$, respectively. We prove

$$(x_1, x_2) \in \ker_{DP}(\zeta) \;\Leftrightarrow\; x_1 \in \ker(\zeta_1), x_2 \in \ker(\zeta_2).$$

Since $\ker_{DP}(\zeta)$ is the kernel of $L_1 \times L_2$, we have

$$(x_1, x_2) \in \ker_{DP}(\zeta) \;\Leftrightarrow\; (e_{Y1}, e_{Y2}) = \mathbf{e}_Y \ll_Y \zeta(x_1, x_2)$$

by (27), and so

$$(e_{Y1}, e_{Y2}) \ll_Y \zeta(x_1, x_2) \;\Leftrightarrow\; (e_{Y1}, e_{Y2}) \ll_Y (\zeta_1(x_1), \zeta_2(x_2))$$

by (39). Hence

$$(e_{Y1}, e_{Y2}) \ll_Y (\zeta_1(x_1), \zeta_2(x_2)) \;\Leftrightarrow\; e_{Y1} \leq_{Y1} \zeta_1(x_1), e_{Y2} \leq_{Y2} \zeta_2(x_2)$$

by the definition of $\ll_Y$, and so

$$e_{Y1} \leq_{Y1} \zeta_1(x_1), e_{Y2} \leq_{Y2} \zeta_2(x_2) \;\Leftrightarrow\; x_1 \in \ker(\zeta_1), x_2 \in \ker(\zeta_2)$$

by (27). Therefore, $(x_1, x_2) \in \ker_{DP}(\zeta)$ if and only if $x_1 \in \ker(\zeta_1)$ and $x_2 \in \ker(\zeta_2)$, and so it holds that $\ker_{DP}(\zeta) = \ker(\zeta_1) \times \ker(\zeta_2)$. □

Corollary 8. *Let $\zeta_1 : L_1 \to Y_1$ and $\zeta_2 : L_2 \to Y_2$ be O-maps from $\boldsymbol{L}_1$ to $\boldsymbol{Y}_1$ and from $\boldsymbol{L}_2$ to $\boldsymbol{Y}_2$, respectively; $\boldsymbol{L} = \boldsymbol{L}_1 \times \boldsymbol{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$ and $\boldsymbol{Y} = \boldsymbol{Y}_1 \times \boldsymbol{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y)$ be direct product OBCI-algebras; and an O-mapping $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ satisfy (39). The direct product kernel $\ker_{DP}(\zeta)$ of ζ can be expressed as $\ker_{DP}(\zeta) = \ker(\zeta_1) \times \ker(\zeta_2)$ of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$, respectively.*

The following examples illustrate Theorem 18 and Corollary 8.

Example 9. Let $\zeta_1 : L_1 \to Y_1$ be the map $\zeta : L \to Y$ in Example 2 and $\zeta_2 : L_2 \to Y_2$ be the O-mapping $\zeta : L \to Y$ in Example 1. Let $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$ and $\mathbf{Y} = \mathbf{Y}_1 \times \mathbf{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y)$ be direct product OBCI-algebras, and a map $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ satisfy (39). Then ζ is not an O-mapping since (40) does not hold. For this, we verify

$$(\tfrac{2}{3}, e) = \mathbf{e}_X \ll_L (\tfrac{1}{3}, e) \Rightarrow_L (\tfrac{2}{3}, e),$$
$$(e, e) = \mathbf{e}_Y \not\ll_Y \zeta(\tfrac{1}{3}, e) \Rightarrow_Y \zeta(\tfrac{2}{3}, e).$$

It can be verified as follows:

$$(\tfrac{2}{3}, e) \ll_L (\tfrac{1}{3}, e) \Rightarrow_L (\tfrac{2}{3}, e) \text{ if and only if}$$
$$\tfrac{2}{3} \leq_{L1} \tfrac{1}{3} \rightarrow_{L1} \tfrac{2}{3} = \tfrac{2}{3} \text{ and } e \leq_{L2} e \rightarrow_{L2} e = e.$$

However, $(e, e) \not\ll_Y \zeta(\frac{1}{3}, e) \Rightarrow_Y \zeta(\frac{2}{3}, e)$ since

$$e = e_{Y1} \not\leq_{Y1} \zeta_1(\tfrac{1}{3}) \rightarrow_{Y1} \zeta_1(\tfrac{2}{3}) = \partial \rightarrow_{Y1} e = \partial \text{ and}$$
$$e = e_{Y2} \leq_{Y2} \zeta_2(e) \rightarrow_{Y2} \zeta_2(e) = e \rightarrow_{Y2} e = e.$$

Hence

$$\mathbf{e}_X \ll_L (\tfrac{1}{3}, e) \Rightarrow_L (\tfrac{2}{3}, e) = \mathbf{e}_X \text{ but}$$
$$\mathbf{e}_Y \not\ll_Y \zeta(\tfrac{1}{3}, e) \Rightarrow_Y \zeta(\tfrac{2}{3}, e) = (\partial, e).$$

Note that $\ker(\zeta_1) = \{1, \frac{2}{3}\}$ and $\ker(\zeta_2) = \{e\}$. Then

$$\ker(\zeta_1) \times \ker(\zeta_2) = \{(x_1, x_2) \in L_1 \times L_2 \mid x_1 \in \ker(\zeta_1),\ x_2 \in \ker(\zeta_2)\}$$

is the kernel of $\mathbf{L}_1 \times \mathbf{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$, where $\mathbf{e}_X = (\frac{2}{3}, e)$.

Example 10. Let $\zeta_1 : L_1 \to Y_1$ be the O-mapping $\zeta : L \to L$ in Example 4 and $\zeta_2 : L_2 \to Y_2$ be the O-mapping $\zeta : L \to Y$ in Example 1. Let $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$ and $\mathbf{Y} = \mathbf{Y}_1 \times \mathbf{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y)$ be direct product OBCI-algebras, and a map $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ satisfy (39). Then ζ is an O-mapping by Theorem 16. Note that $\ker(\zeta_1) = \{1, e\}$ and $\ker(\zeta_2) = \{e\}$. Then $\ker(\zeta_1) \times \ker(\zeta_2)$ is the kernel of $\mathbf{L}_1 \times \mathbf{L}_2$ as in Example 9.

Theorem 19. *Let $\zeta_1 : L_1 \to Y_1$ and $\zeta_2 : L_2 \to Y_2$ be mappings from $\boldsymbol{L}_1$ to $\mathbf{Y}_1$ and from $\boldsymbol{L}_2$ to $\mathbf{Y}_2$, respectively; $\boldsymbol{L} = \boldsymbol{L}_1 \times \boldsymbol{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$ and $\boldsymbol{Y} = \boldsymbol{Y}_1 \times \boldsymbol{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y)$ be direct product OBCI-algebras; and a map $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ satisfy (39). Given subsets K_{L_1} and K_{L_2} of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$, respectively, define two sets:*

$$\mathcal{K}^{e_{Y2}} := \{(x_1, x_2) \in L_1 \times L_2 \mid x_1 \in K_{L_1},\ e_{Y2} \leq_{Y2} \zeta(x_2)\}.$$

$$\mathcal{K}^{e_{Y1}} := \{(x_1, x_2) \in L_1 \times L_2 \mid e_{Y1} \leq_{Y1} \zeta(x_1),\ x_2 \in K_{L_2}\}.$$

If K_{L_1} and K_{L_2} are kernels of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$, respectively, then $\mathcal{K}^{e_{Y2}}$ and $\mathcal{K}^{e_{Y1}}$ are kernels of $\boldsymbol{L}_1 \times \boldsymbol{L}_2$.

Proof. Let K_{L_1} and K_{L_2} be kernels of $\mathbf{L}_1$ and $\mathbf{L}_2$, respectively. First we verify that $\mathcal{K}^{e_{Y2}}$ is the kernel of $\mathbf{L}_1 \times \mathbf{L}_2$. It is clear that $\mathbf{e_L} = (e_{L1}, e_{L2}) \in \mathcal{K}^{e_{Y2}}$. Given $(x_1, x_2) \in L_1 \times L_2$, we have to verify that

$$(x_1, x_2) \in \mathcal{K}^{e_{Y2}} \;\Leftrightarrow\; \mathbf{e_Y} \ll_Y \zeta(x_1, x_2). \tag{42}$$

Assume that $(x_1, x_2) \in \mathcal{K}^{e_{Y2}}$. Then $x_1 \in K_{L_1}$ and $e_{Y2} \leq_{Y2} \zeta(x_2)$ by the definition of $\mathcal{K}^{e_{Y2}}$, and so $e_{Y1} \leq_{Y1} \zeta(x_1)$ by (27). Thus

$$\mathbf{e_Y} \ll_Y (\zeta_1(x_1), \zeta_2(x_2))$$

by the definition of $\ll_Y$, and so

$$\mathbf{e_Y} \ll_Y \zeta(x_1, x_2)$$

by (39). Assume that $\mathbf{e_Y} \ll_Y \zeta(x_1, x_2)$. Analogously we can prove $(x_1, x_2) \in \mathcal{K}^{e_{Y2}}$. Therefore $\mathcal{K}^{e_{Y2}}$ is the kernel of $\mathbf{L}_1 \times \mathbf{L}_2$.

In a similar manner, we can verify that $\mathcal{K}^{e_{Y1}}$ is the kernel of $\mathbf{L}_1 \times \mathbf{L}_2$. □

Corollary 9. *Let $\zeta_1 : L_1 \to Y_1$ and $\zeta_2 : L_2 \to Y_2$ be O-maps from $\boldsymbol{L}_1$ to $\boldsymbol{Y}_1$ and from $\boldsymbol{L}_2$ to $\boldsymbol{Y}_2$, respectively; $\boldsymbol{L} = \boldsymbol{L}_1 \times \boldsymbol{L}_2 := (L_1 \times L_2, \Rightarrow_L, \mathbf{e}_X, \ll_L)$ and $\boldsymbol{Y} = \boldsymbol{Y}_1 \times \boldsymbol{Y}_2 := (Y_1 \times Y_2, \Rightarrow_Y, \mathbf{e}_Y, \ll_Y)$ be direct product OBCI-algebras; and an O-mapping $\zeta : L_1 \times L_2 \to Y_1 \times Y_2$ satisfy (39). Given subsets K_{L_1} and K_{L_2} of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$, respectively, define $\mathcal{K}^{e_{Y2}}$ and $\mathcal{K}^{e_{Y1}}$ as in Theorem 19. If K_{L_1} and K_{L_2} are kernels of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$, respectively, then $\mathcal{K}^{e_{Y2}}$ and $\mathcal{K}^{e_{Y1}}$ are kernels of $\boldsymbol{L}_1 \times \boldsymbol{L}_2$.*

Note that $\mathcal{K}^{e_{Y2}}$ and $\mathcal{K}^{e_{Y1}}$ in Theorem 19 and Corollary 9 are the same in the sense that for $(x_1, x_2) \in L_1 \times L_2$,

$$(x_1, x_2) \in \mathcal{K}^{e_{Y2}} \;\Leftrightarrow\; (x_1, x_2) \in \mathcal{K}^{e_{Y1}}.$$

4 Conclusion

This paper introduced the notion of ordered maps of OBCI-algebras as order-preserving maps of OBCI-algebras. This paper moreover defined the notion of kernels of OBCI-algebras and studied properties associated with (ordered) subalgebras, (ordered) filters and direct products of OBCI-algebras.

We still have some future works or open problems. First, we need to study ordered maps of OBCI-algebras in a more specific context. For instance, we can deal with ordered maps and kernels of OBCI-algebras related to (ordered) Y-filters, (ordered) R-filters and (ordered) J-filters. Second, we need to introduce more order related concepts such as ordered kernels of OBCI-algebras. As OBCI-algebras are a generalization of BCI-algebras with an underlying partial order, we can introduce a similar generalization of the kernel as an ordered kernels of OBCI-algebras.

Acknowledgements

The authors would like to thank the referee for careful reading.

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