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The co-prime power order graph of a finite group

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Abstract. In this study, we generalized the co-prime graph of a finite group called the co-prime power order graph of a finite group. It is denoted by β_G, and its vertex set is G, such that two distinct vertices x and y are adjacent if and only if $\gcd(|x|,|y|) = p^n$, where p is a prime number, and $n \in \mathbb{Z}^+ \cup \{0\}$. We characterized complete graphs and planar graphs on the co-prime power order graphs, and investigated some properties of graph β_G for some groups such as cyclic groups, dihedral groups, and the generalized quaternion groups, and obtained the vertex-connectivity among them. Finally, we characterized some induced subgraphs of co-prime power order graph for some finite groups.

Keywords: Cyclic group, Dihedral group, Generalized quaternion group, Complete graph, Planar graph, Co-prime order graph, Induced graph, Vertex-connectivity.
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1 Introduction

Creating graphs from numerous algebraic structures such as groups or semigroups is not a new work, but the new is how to combine and compare the relationship between properties of the groups and their elements on the one hand, and the created graph from them on the other hand. During this study, the graphs are simple and finite. The co-prime graph of a finite group G has been introduced in the literature for a considerable time. In [11], Sattanathan and Kala introduced it. They called it the order prime graph of a group G, which was defined as a graph $\Gamma_G = (\mathrm{V}(\Gamma_G), \mathrm{E}(\Gamma_G))$ when $\mathrm{V}(\Gamma_G) = G$, and $\mathrm{a}, \mathrm{b} \in G$ are adjacent if and only if $\gcd(|a|,|b|) = 1$. At a subsequent time, in [7], Ma et al. represented and recalled the order prime graph as the co-prime graph and worked on many properties of it. The co-prime graph was studied in [2,12]. In [1], many properties of the co-prime graph of finite cyclic and dihedral groups were studied. In [10], they introduced a new graph known as co-prime order graph of

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a finite group G which was defined as a graph $\theta(G) = (V(\theta(G)), E(\theta(G)))$ when $V(\theta(G)) = G$ and a, b $\in G$ are adjacent if and only if $\gcd(|a|, |b|) = 1$ or a prime number. The co-prime order graph of a finite group was studied extensively in [5], and obviously, the readers can find many results through this reference. They investigated the complete and planar co-prime order graph with respect to the order of elements in a finite group and the order of itself furthermore. Also they investigated the vertex-connectivity of the co-prime order graph of finite groups or subgroups of them. Especially, they worked on cyclic groups, dihedral groups and generalized quaternion groups. In [8–10], the readers can see the definition of p - group, and its properties while the readers can see the definitions of complete graph, planar graph, and their properties in [13, 14].

Obviously, the co-prime power order graph is a generalization of the co-prime order graph, and we introduced it in this paper. We identify some properties of the co-prime power order graph β_G by using algebraic properties of finite group G. So, we study when the graph β_G should be complete and planar. Also, we study the connectivity of the graph β_G.

This study has been arranged as follows: We have represented the preliminary definitions, propositions, lemmas and theorems in section 2 which have been used throughout this paper. In Section 3, we introduced the co-prime power order graph of a finite group G, denoted by β_G, and studied on complete and planar co-prime power order graph of cyclic groups, dihedral groups and generalized quaternion groups. In Section 4, we investigated the vertex-connectivity of cyclic groups, dihedral groups and generalized quaternion groups. Finally, we characterized the induced subgraph of the cyclic group, dihedral group, and the generalized quaternion group.

2 Preliminaries

In this section we review some preliminary definitions and notations in groups and graph theory that have been used throughout the paper. A dihedral group of order $2n$, denoted by D_n is defined as: $D_n =< a, b \mid a^n = b^2 = e$, and $bab = a^{-1} >$ [4, 6]. In a group, any element of order 2 is called involution [5]. A generalized Quaternion group denoted by Q_{4n} is defined as:
$Q_{4n} =< a, b \mid a^n = b^2, a^{2n} = b^4 = 1, bab = a^{-1} >$ [4]. A graph is called a connected graph if and only if there is a path between any two vertices [13, 14]. A graph is called a planar graph if it can be drawing with no crossing edges [13, 14]. A graph Γ is called complete graph if and only if every two vertices are adjacent, and a complete graph with n vertices is denoted by K_n [13, 14]. A group G is called a cyclic group if it is generated by an element. A cyclic group of order n is isomorphic to $\mathbb{Z}_n$ [5, 13, 14]. A group G is called p-group if and only if every element of it has a prime power order [4, 6, 8]. A group G is called P-group if every nontrivial element of it has a prime order [13, 14]. Let $Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq \cdots$ be an ascending series of G such that

i) $Z_0(G) = \{e\}$;

ii) $Z_1(G) = Z(G)$;

iii) $Z_i(G)/Z_{i-1}(G) = Z\left(G/Z_{i-1}(G)\right), i \geq 1.$

Then $Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq \cdots$ is called the ascending central series [6, 8]. A group G is called nilpotent group if it has an ascending central series such that $Z_n = G$, for a number

$n \in \mathbb{Z}^+$ [6,8].
A positive integer number n is a composite number if it has at least two nontrivial divisors except itself [3]. Let $n \in \mathbb{Z}^+$, then factorization n as $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, for some $k \in \mathbb{Z}^+$, and prime numbers $p_1, p_2, \ldots, p_k$ is called canonical factorization. The set of all elements of a group G of order 1 or a prime number is denoted by P_G [5].
The induced subgraph $< P_G >$ is a clique of $\theta(\mathrm{G})$, and $\deg(x) = n-1$, for all $x \in P_G, |G| = n$ [5]. A vertex x in a graph Γ is called isolated if and only if $\deg(x) = 0$. A graph is planar if and only if it does not have $K_5, K_{3,3}$ or a subdivision of $K_5, K_{3,3}$ as a subgraph (it is called Kuratowski's theorem) [5]. Let Γ be a connected graph, then a subset $S \subseteq V(\Gamma)$ is called a vertex cut if $\Gamma - S$ is disconnected or has one vertex. The vertex-connectivity of a graph Γ is denoted by $\kappa(\Gamma)$ that is equal to the minimum size of a vertex cut.

3 Complete and planar co-prime power order graphs.

In this section, we study the complete graphs and planar graphs of the co-prime power order graphs, and discuss them for some groups. Furthermore, we investigate some properties of these groups and their relationship with the co-prime power order graph.

Definition 1. *Let G be a group, and $|G| > 2$. The co-prime power order graph of G denoted by β_G is the graph with vertex set G, such that two distinct elements $x, y \in G$ are adjacent if and only if* $\gcd(|x|, |y|) = p^n, p$ *is a prime number,* $n \in \mathbb{Z}^+ \cup \{0\}$.

Definition 2. *A group G with $|G| > 2$ is called ϕ-group if any nontrivial element of G has a prime power order.*

Clearly, any ϕ-group is a P*-group, but the inverse of this result is not true. For example, by a simple discussion, we can see that D_4 is a ϕ-group, but it is not a P-group.*

Definition 3. *Let G be a group, and $|G| > 2$. The subset $\{x \in G || x |= p^\alpha$, for a prime p, and $\alpha \geq 0\}$ of G is denoted by $P(G)$.*

Theorem 1. *The co-prime power order graph of a group G is complete if and only if G is a ϕ-group.*

Proof. Suppose that β_G is complete for a group G. Then every two distinct elements $x, y \in G$ are adjacent. It follows that $\gcd(|x|, |y|) = p^n$, for a prime number $p, n \in \mathbb{Z}^+ \cup \{0\}$.

If G is not ϕ-group, then there exists an element $x \in G$ such that $|x| \neq p^n$, for a prime number p, $n \in \mathbb{Z}^+ \cup \{0\}$. Now, by using $\gcd\left(|x|, \left|x^{-1}\right|\right) = |x|$, we conclude that x, and x^{-1} are not adjacent and get a contradiction. Hence, G is a ϕ-group.

Conversely, suppose that G is a ϕ-group, then for every $x \in G, |x| = p^n$, for a prime number p. Thus for every two elements $x, y \in G, \gcd(|x|, |y|) = p^n$, for a prime number $p, n \in \mathbb{Z}^+ \cup \{0\}$. Therefore β_G is a complete graph. $\square$

Lemma 1. *Let $G = D_n$, then G is a ϕ-group if and only if $n = p^m, p$ is a prime, and $m \geq 1$.*

Proof. Let $D_n =< a, b \mid a^n = b^2 = 1, bab = a^{-1} >$. If $n = p^m$, for a prime number p, and $m \geq 1$, then $|x| = 1, 2$ or $p^k, 1 \leq k \leq m$, for every $x \in D_n$. Hence, D_n is a ϕ-group.

Conversely, suppose that D_n is a ϕ-group, and $n \neq p^m$, for any prime number p, and $m \geq 1$. We know that $|a| = n$. It follows that D_n is not a ϕ-group, a contradiction. Therefore, $n = p^m$, for any prime number p, and $m \geq 1$. □

Now, by Theorem 1 and Lemma 1, we have the following result.

Theorem 2. *Let $G = D_n$, then β_G is complete if and only if $n = p^m$, p is a prime, and $m \geq 1$.*

Lemma 2. *Let $G = Q_{4n}$, then G is a ϕ-group if and only if $n = 2^m, m \geq 1$.*

Proof. If $n = 2^m$, then $G = Q_{4n}$ is a $2-$ group, and it is a ϕ-group.
Conversely, suppose that $Q_{4n} =< a, b \mid a^n = b^2, a^{2n} = b^4 = 1$, bab $= a^{-1} >$ is a ϕ-group, and $n \neq 2^m, \forall m \in \mathbb{Z}^+$, then $|a| = 2n$, and $2n \neq p^m$, for any prime number $p, m \in \mathbb{Z}^+$.
Therefore, Q_{4n}, is not a ϕ-group. It is a contradiction. Hence $n = 2^m$. □

Now by Lemma 2 and Theorem 1, we get the following theorem.

Theorem 3. *Let $G = Q_{4n}$. Then β_G is complete if and only if $n = 2^m, m \in \mathbb{Z}^+$.*

Lemma 3. *Let G be a group of order n, and let $P(G)$ be the set of all elements $x \in G$ such that $|x| = p^m, p$ is a prime number, and $m \in \mathbb{Z}^+ \cup \{0\}$. Then for any vertex $a \in V(\beta_G)$ we have* $\deg(a) = n - 1$ *if and only if $a \in P(G)$.*

Proof. Let $a \in P(G)$, then $|a| = p^m$, for a prime number p, and $m \in \mathbb{Z}^+ \cup \{0\}$. It follows that for any $x \in G, \gcd(|a|, |x|) || a |= p^m$, then a and x are adjacent. Hence $\deg(a) = n - 1$. Conversely, let $\deg(a) = n-1$, and $a \notin P(G)$, then $|a| \neq p^m$ for any prime number p, and any $m \in \mathbb{Z}^+ \cup \{0\}$. Therefore, $\gcd\left(|a|, \left|a^{-1}\right|\right) = |a| \neq p^m$. So a and a^{-1} are not adjacent, and $\deg(a) < n - 1$, a contradiction. Thus $a \in P(G)$. □

From Lemma 2, we conclude that for a group G and an element $x \in G - P(G), P(G) \cup \{x\}$ is a clique of β_G.

Theorem 4. *The co-prime power order graph β_G of a group G is planar if and only if $|G| < 5$.*

Proof. If $|G| < 5$, then β_G is a graph of order less than 5. Hence, β_G is planar. Now let $|G| \geq 5$, then by a straight discussion we can see that if $|G| \geq 5$, and $G \neq \mathbb{Z}_6$, then $|P(G)| \geq 5$. Because the induced subgraph $\langle P(G) \rangle$ of β_G is a clique, β_G has a subgraph isomorphic to K_5. Now by Kurutowski's theorem, we conclude that β_G is not planar. □

Example 1. If $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, then the induced subgraph of β_G by $S = \{0, 1, 2, 3, 4\}$ is isomorphic to K_5. Hence, $\beta_{\mathbb{Z}_6}$ is not planar.

4 Vertex-connectivity of the co-prime power order graphs

In this section, we investigated the vertex-connectivity of the co-prime power order graph and obtain the vertex-connectivity $k(\beta_G)$ for some groups. We know that in a connected non-complete graph Γ of order n, if $\deg(x) = n-1$, for a vertex x in Γ, then every vertex cut contains x. By this fact, and Theorem 3.1 and Lemma 3.6, we have the following theorem

Theorem 5. *Let G be a group of order n, then the following statement holds:*

i) If G is a ϕ-group, then $k(\beta_G) = n-1$.

ii) If G is not a ϕ-group, then $k(\beta_G) \geq |P(G)|$.

From the elementary group theory we have the following lemma.

Lemma 4. *Let G be a finite group, and let N, M are two normal subgroup of G such that $NM = MN$, and $N \cap M = \{e\}$. Then $nm = mn$ for every $n \in N, m \in M$.*

Theorem 6. *Let G be a group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, such that $p_i, 1 \leq i \leq k$ are distinct prime numbers, $\alpha_i \geq 1$ and G has an element x of order $p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}, l_i \geq 1$, then $k(\beta_G) = |P(G)|$.*

Proof. Let $x \in G$ and $|x| = p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}, l_i \geq 1, 1 \leq i \leq k$.
If $y \in G - P(G)$, then $|y| = p_1^{t_{i1}} p_2^{t_{i2}} \dots p_j^{t_{ij}}$ for $2 \leq j \leq k, t_m \geq 1$.
So $\gcd(|x|, |y|) = p_1^{s_1} p_2^{s_2} \dots p_j^{s_{ij}}$ such that $s_m = \min(t_m, l_m)$. Hence, x and y are not adjacent.
It follows that x is an isolated vertex in $G - P(G)$. Now, by the fact that every vertex cut contains in $P(G)$, we conclude that $k(\beta_G) = |P(G)|$. $\square$

Now by Theorem 6, we have the following corollary.

Corollary 1. *For a number $n \in \mathbb{Z}^+$, we have $k(\beta_{\mathrm{z}_n}) = |P(\mathbb{Z}_n)|$.*

Theorem 7. *Let G be a nilpotent group of order n, such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$ is the canonical factorization of n, then we have*

$$k(\beta_G) = \begin{cases} p_1^{\alpha_1} - 1, & \text{when } t = 1 \\ 1 + \sum_{i=1}^{t} (p_i^{\alpha_i} - 1) & \text{when } t > 1 \end{cases}.$$

Proof. Let $t = 1$, then G is p_1- group, and it is a ϕ-group. By theorem 5, we obtain $k(\beta_G) = n - 1$. Now let $t > 1$, and $S_i, 1 \leq i \leq t$ be sylow p_i-subgroup of G.

Suppose that $x_i \in S_i$ is an element of order p_i. Set $x = x_1 x_2 \dots x_i$. Now by lemma 4 clearly, we can see that $|x| = |x_1||x_2| \dots |x_i| = p_1 p_2 \dots p_t$. By using Theorem 6 we obtain $k(\beta_G) = |P(G)|$.

The set $P(G)$ is a union of all sylow subgroups of G. Since G is a nilpotent group, there exists one sylow p_i-subgroup, $1 \leq i \leq t$ of G, named S_i. Hence $P(G) = \bigcup_{i=1}^{t} S_i$. Therefore, we have:

$$k(\beta_G) = |P(G)| = 1 + \sum_{i=1}^{t} (|S_i| - 1) = 1 + \sum_{i=1}^{t} (p_i^{\alpha_i} - 1).$$

$\square$

Corollary 2. *Let n be a positive integer, and $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ be its canonical factorization, then:*

$$k\left(\beta_{D_n}\right) = \begin{cases} 2n-1, & t = 1; \\ \sum_{i=1}^{t}\left(p_i^{\alpha_i}-1\right)+n+1, & t > 1. \end{cases}.$$

Proof. Let $D_n =< a, b \mid a^n = b^2 = 1, bab = a^{-1} >$. If $t = 1$, then by using Theorem 2, β_{D_n} is complete, and so $k\left(\beta_{D_n}\right) = 2n-1$. Now let $t > 1$, we have $P(G) = \left\{a^i b \mid 1 \leq i \leq n\right\} \cup P(< a\rangle\big)$. There exists an element $x \in< a\rangle$ of order $p_1 p_2 \ldots p_t$. The vertex x is an isolated vertex in $\beta_{D_n} - P\left(D_n\right)$. Hence $k\left(\beta_{D_n}\right) = \left|P\left(D_n\right)\right|$. Because $\langle a\rangle$ is nilpotent, then by Theorem 7,

$$|P(< a >)| = 1 + \sum_{i=1}^{t}\left(p_i^{\alpha_i}-1\right).$$

Therefore, $k\left(\beta_{D_n}\right) = \left|\left\{a^i b \mid 1 \leq i \leq n\right\}\right| + 1 + \sum_{i=1}^{t}\left(p_i^{\alpha_i}-1\right) = \sum_{i=1}^{t}\left(p_i^{\alpha_i}-1\right) + n + 1.$ □

Theorem 8. *Suppose that m is a positive integer, and $m \geq 2$ such that $2m = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ is a canonical factorization of $2m$, for prime numbers $p_i, 1 \leq i \leq t$, then we have*

$$k\left(\beta_{Q_{4m}}\right) = \begin{cases} \left|P\left(\beta_{Q_{\Delta m}}\right)\right| & , m \text{ is even }; \\ 2m+2+\sum_{i=1}^{t}\left(p_i-1\right), & m \text{ is odd} \end{cases}$$

Proof. Let $Q_{4m} =< a, b \mid a^m = b^2, a^{2m} = b^4 = 1, bab = a^{-1} >$. Let m be an even number, then $\left|Q_{4m}\right| = 4m = 2^{\alpha_0+1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$, for prime numbers $p_i, 1 \leq i \leq 2m$ such that $\alpha_0 \geq 2$. The order of a is $p_1^{s_1} p_2^{s_2} \ldots p_k^{s_k}$, then by Theorem 6 we obtain $k\left(\beta_{Q_{4m}}\right) = \left|P\left(Q_{4m}\right)\right|$. Now, suppose that m be an odd number, then we have $|a| = 2m$, and $|\langle a\rangle| = 2m = 2p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$, for prime numbers $p_j, 1 \leq j \leq t$. We can note that $P\left(Q_{4m}\right) = \left\{a^i b \mid 1 \leq i \leq 2m\right\} \cup P(< a >)$. There exists an element $y \in \langle a\rangle$, with $|y| = 2p_1 p_2 \ldots p_t, t \geq 2$. So, the vertex y is an isolated vertex in $\beta_{Q_{4m}} - P\left(\beta_{Q_{4m}}\right)$. We know $\left|\left\{a^i b \mid 1 \leq i \leq 2m\right\}\right| = 2m$, and by Theorem 7,

$$|P(< a >)| = 2 + \sum_{i=1}^{t}\left(p_i-1\right).$$

Then by Theorem 6 we obtain that

$$k\left(\beta_{Q_{4m}}\right) = \left|P\left(\beta_{Q_{4m}}\right)\right| = 2m + 2 + \sum_{i=1}^{t}\left(p_i^{\alpha_i}-1\right).$$

□

5 Induced graph of the co-prime power order graph

In this section, we investigate the induced subgraph $\beta_G[G - P(G)]$ of the co-prime power order graph β_G for some group G such as cyclic groups, dihedral groups, and generalized quaternion groups, and all result that we have got them are dependent on the order of group's elements. We arrange our result as the following theorems.

Definition 4. *Suppose that β_G is the co-prime power order graph of a group G, and $S \subseteq G$, then the induced graph by S in β_G, denoted by $\beta_G[S]$ is a subgraph of β_G such that the vertex set is S and two vertices $x, y \in S$ are adjacent in $\beta_G[S]$ if x and y are adjacent in β_G.*

Theorem 9. *Suppose that β_G is the co-prime power order graph of G, then the induced subgraph $\beta_G[G - P(G)]$ is a null graph if and only if $|G| = p^\alpha q^\delta$, for prime numbers p, q, and positive integer α, δ.*

Proof. Suppose that $|G| = n = p^\alpha q^\delta$ for prime numbers p, q, and positive integer $\alpha, \delta. P(G) = \{x || x |= p^{\alpha_i} \text{ or} q^{\ \delta_i}\}$. There are vertices $x, y \notin P(G)$ such that $|x| = p_1^{\alpha_1} p_2^{\delta_1}, |y| = p_1^{\alpha_2} p_2^{\delta_2}$. It follows that $\gcd(|x|, |y|) = p^{\min\{\alpha_1, \alpha_2\}} q^{\min\{\delta_1, \delta_2\}}$. Hence, x and y are not adjacent. Therefore, $\beta_G[G-\ P(G)]$ is a null graph. Conversely, suppose that $\beta_G[G - P(G)]$ is a null. Then any two vertices $x, y \in \beta_G[G - P(G)]$ are not adjacent. Now, by contrary, suppose that $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} {p_3}^{\alpha_3}$, for distinct prime numbers p_1, p_2, p_3. There are vertices $x, y \in \beta_G[G - P(G)]$ such that $|x| = p_1^{\alpha_1} p_2^{\alpha_2}$, and $|y| = p_1^{\alpha_1} p_3^{\alpha_3}$. It follows $\gcd(|x|, |y|) = {p_1}^{\alpha_1}$, which implies x and y are adjacent, a contradiction. Hence $n = p^\alpha q^\delta$. □

Theorem 10. *Suppose that $\beta_{\mathbb{Z}_n}$ is the co-prime power order graph of $\mathbb{Z}_n$, then the induced subgraph $\beta_{\mathbb{Z}_n} [\mathbb{Z}_n - P(\mathbb{Z}_n)]$ is a null if and only if $n = p^\alpha q^\delta$, for prime numbers p, q, and positive integer α, δ.*

Proof. Suppose that $\beta_{\mathbb{Z}_n} [\mathbb{Z}_n - P(\mathbb{Z}_n)]$ is a null graph. Then any two vertices $x, y \in \beta_{\mathbb{Z}_n} [\mathbb{Z}_n - P(\mathbb{Z}_n)]$ are not adjacent. Now, by contrary, suppose that $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} {p_3}^{\alpha_3}$, for distinct prime numbers p_1, p_2, p_3. There are vertices $x, y \in \beta_{\mathbb{Z}_n} [\mathbb{Z}_n - P(\mathbb{Z}_n)]$ such that $|x| = p_1^{\alpha_1} p_2^{\alpha_2}$, and $|y| = p_1^{\alpha_1} p_3^{\alpha_3}$. It follows $\gcd(|x|, |y|) = {p_1}^{\alpha_1}$, which implies x and y are adjacent, a contradiction. Conversely, by Theorem 9 $\beta_{\mathbb{Z}_n} [\mathbb{Z}_n - P(\mathbb{Z}_n)]$ is a null graph. □

Theorem 11. *Let β_{D_n} be the co-prime power order graph of dihedral group $D_n =< a, b \mid a^n = b^2 = e$, and $bab = a^{-1} >$, then the induced subgraph $\beta_{D_n} [D_n - P(D_n)]$ is a null graph if and only if $n = p^\alpha q^\gamma$, for prime numbers p, q, and positive integer α, γ.*

Proof. Let $\beta_{D_n} [D_n - P(D_n)]$ be a null graph. Then any two vertices $x, y \in \beta_{D_n} [D_n - P(D_n)]$ are not adjacent. Now by contrary, assume that $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} {p_3}^{\alpha_3}$, for prime numbers p_1, p_2, p_3, then there are vertices $x = a^{p_3 \alpha_3}, y = a^{p_2 \alpha_2} \in D_{D_n} [D_n - P(D_n)]$ such that $|x| = p_1^{\alpha_1} p_2^{\alpha_2}$, and $|y| = p_1^{\alpha_1} p_3^{\alpha_3}$. It follows $\gcd(|x|, |y|) = p_1^{\alpha_1}$, which implies x and y are adjacent, a contradiction. Conversely, by Theorem 9 $\beta_{D_n} [D_n - P(D_n)]$ is a null graph. □

Theorem 12. *Let $G = Q_{4n} =< a, b \mid a^n = b^2, a^{2n} = b^4 = 1$ and $bab = a^{-1} >$, and $\beta_{Q_{4n}}$ is the co-prime power order graph of Q_{4n}. Then the induced subgraph $\beta_{Q_{4n}} [Q_{4n} - P(Q_{4n})]$ is a null graph if and only if $n = p^\alpha q^\gamma$, for prime numbers p, q, and positive integer α, γ.*

Proof. Let $\beta_{Q_{4n}} [Q_{4n} - P(Q_{4n})]$ be a null graph. Then any two vertices $x, y \in \beta_{Q_{4n}} [Q_{4n} - P(Q_{4n})]$ are not adjacent. Now by contrary, assume that $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} {p_3}^{\alpha_3}$, for prime numbers p_1, p_2, p_3, then there are vertices $x, y \in \beta_{Q_{4n}} [Q_n - P(Q_{4n})]$ such that $|x| = p_1^{\alpha_1} p_2^{\alpha_2}$, and $|y| = p_1^{\alpha_1} p_3^{\alpha_3} = |x|$ and $|y|$ are divide n. It follows that $\gcd(|x|, |y|) = {p_1}^{\alpha_1}$, which implies x and y are adjacent, a contradiction. Conversely, by Theorem 9 $\beta_{Q_{4n}} [Q_{4n} - P(Q_{4n})]$ is a null graph. □

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