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Weak u-S-projective modules and dimensions

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Abstract. The primary focus of this paper is to introduce and investigate a fresh category of projective modules, referred to as weak u-S-projective modules (w-u- is an abbreviation for weak uniformly). These novel modules are utilized for characterizing u-S-von Neumann regular rings. Additionally, the paper investigates a new type of rings, named u-S-semihereditary rings. This leads to the introduction of the weak u-S-projective dimensions of modules and weak u-S-global dimension of rings in this paper.

Keywords: u-S-projective module, u-S-flat module, u-S-torsion, u-S-exact sequence, u-S-von Neumann regular ring, u-S-semihereditary.
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1 Introduction

Throughout this paper, we denote by A a commutative ring with identity, M an A-module, and S a multiplicative subset of A, where $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S$ and $s_2 \in S$. The study of commutative rings in terms of multiplicative sets started with Anderson and Dumitrescu [1], who introduced the concept of S-Noetherian rings. A ring A is called S-Noetherian if for any ideal J of A, there exists a finitely generated sub-ideal J' of J such that $tJ \subseteq J'$ for some fixed $t \in S$. However, the element $t \in S$ in the definition of S-Noetherian rings is not generally "uniform," which complicates the study of S-Noetherian rings using module-theoretic methods. To address this issue, Qi et al. [5] introduced the concept of uniformly S-Noetherian rings, which are S-Noetherian rings in which the choice of t is fixed. A ring A is called a coherent ring if any finitely generated ideal is finitely presented, and this concept is another important type of ring defined by a finiteness condition. Coherent rings have been studied by many algebraists in terms

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of various modules. To extend coherent rings by multiplicative sets, Bennis et al. [4] introduced the notions of S-coherent rings and c-S-coherent rings. Recently, Zhang [13] introduced the concept of uniformly S-coherent rings, which are "uniform" versions of S-coherent rings.

In this paper, we require a quick review of uniformly torsion theory. According to [8], an A-module M is considered a u-S-torsion module (with respect to s) if there exists an element $t \in S$ such that $tM = 0$. A sequence $0 \to M \xrightarrow{f} M' \xrightarrow{g} M'' \to 0$ is labeled u-S-exact (at M') if there is an element $t \in S$ such that $t\,\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $t\,\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. A long sequence $\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$ is deemed u-S-exact if for any i there exists an element $t \in S$ such that $t\,\mathrm{Ker}(f_{i+1}) \subseteq \mathrm{Im}(f_i)$ and $t\,\mathrm{Im}(f_i) \subseteq \mathrm{Ker}(f_{i+1})$. A u-S-exact sequence $0 \to M' \to M \to M'' \to 0$ is considered a short u-S-exact sequence. A homomorphism $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism, u-S-isomorphism) if $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$, $0 \to M \xrightarrow{f} N \to 0$) is u-S-exact. One can verify that a homomorphism $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism, u-S-isomorphism) if and only if $\mathrm{Ker}(f)$ (resp., $\mathrm{CoKer}(f)$, both $\mathrm{Ker}(f)$ and $\mathrm{CoKer}(f)$) is a u-S-torsion module. Suppose M and N are A-modules, then M is said to be u-S-isomorphic to N if there exists a u-S-isomorphism $f : M \to N$.

Following [1], an A-module M is called S-finite provided that there is an element $s \in S$ and a finitely generated A-module F such that $sM \subseteq F \subseteq M$. Trivially, S-finite modules are generalizations of finitely generated modules. For generalizing finitely presented A-modules, Bennis et al. [4] introduced the notions of S-finitely presented modules and c-S-finitely presented modules. Following [4], an A-module M is called S-finitely presented provided that there exists an exact sequence of A-modules $0 \to K \to F \to M \to 0$ with K S-finite and F finitely generated free. Certainly, an A-module M is S-finitely presented if and only if there exists an exact sequence of A-modules $0 \to T_1 \to N \to M \to 0$ with N finitely presented and $sT_1 = 0$ for some $s \in S$. Following [4], an A-module M is called c-S-finitely presented provided that there exists a finitely presented submodule N of M such that $sM \subseteq N \subseteq M$ for some $s \in S$. Trivially, an A-module M is called c-S-finitely presented if and only if there exists an exact sequence of A-modules $0 \to N \to M \to T_2 \to 0$ with N finitely presented and $sT_2 = 0$ for some $s \in S$. Recently, Zhang [13] introduced and studied the notion of uniformly S-finitely presented modules which generalize both S-finitely presented modules and c-S-finitely presented modules. An A-module M is called u-S-finitely presented (abbreviates uniformly S-finitely presented) provided that there is an exact sequence $0 \to T_1 \to F \to M \to T_2 \to 0$ with F finitely presented and $sT_1 = sT_2 = 0$.

In [11], the author introduced the class of u-S-projective modules. An A-module P is called uniformly S-projective (u-S-projective) provided that the induced sequence $0 \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, M') \to \mathrm{Hom}_A(P, M'') \to 0$ is u-S-exact for any u-S-short exact sequence $0 \to M \to M' \to M'' \to 0$. The class of u-S-projective modules can be seen as a "uniform" generalization of that of projective modules, since an A-module P is u-S-projective if and only if $\mathrm{Ext}^1_A(P, M)$ is u-S-torsion for any A-module M.

In [12], the authors introduced and studied the u-S-projective dimensions of modules and rings. They defined the u-S-projective dimension u-S-$\mathrm{pd}_A(M)$ of an A-module M to be the length of the shortest u-S-projective u-S-resolution of M. We characterize u-S-projective dimensions of A-modules using the uniform torsion property of the Ext functors in [12, Proposition 2.4]. The u-S-global dimension u-S-gl.dim(A) of a commutative ring A is defined to be the supremum of u-S-projective dimensions of all A-modules.

$$u\text{-}S\text{-gl.dim}(A) = \sup\{u\text{-}S\text{-pd}_A(M) \mid \ M \text{ is an } A\text{-module}\}.$$

Zhang [8] introduced the class of u-S-flat modules F for which the functor $F \otimes_A -$ preserves u-S-exact sequences. The class of u-S-flat modules can be seen as a "uniform" generalization of that of flat modules, since an A-module F is u-S-flat if and only if $\text{Tor}_1^A(F, M)$ is u-S-torsion for any A-module M. In [10], the author introduced the u-S-flat dimensions of modules and rings. Let A be a ring, S a multiplicative subset of A and n be a positive integer. We say that an A-module has a u-S-flat dimension less than or equal to n, u-S-fd$_A(M) \leq n$, if $\text{Tor}_{n+1}^A(M, N)$ is u-S-torsion A-module for all A-modules N. Hence, the u-S-weak global dimension of A is defined to be

$$u\text{-}S\text{-w.gl.dim}(A) = \sup\{u\text{-}S\text{-fd}_A(M) \mid \ M \text{ is an } A\text{-module}\}.$$

Zhang [8] defined the u-S-von Neumann regular ring as follows: Let A be a ring and S a multiplicative subset of A. A is called a u-S-von Neumann regular ring provided there exists an element $s \in S$ satisfies that for any $a \in A$ there exists $r \in A$ such that $sa = ra^2$. Thus, by [8, Theorem 3.13], A is a u-S-von Neumann regular ring if and only if every A-module is u-S-flat.

In Section 2, we introduce the concept of weak u-S-projective modules and study some characterizations of such modules. We prove that a ring A is u-S-von Neumann regular if and only if every u-S-finitely presented A-module is weak u-S-projective, also we prove that if an A-module F is weak u-S-projective, then F_S is free over A_S. Furthermore, we introduce and study a new class of rings called u-S-semihereditary rings. We prove that a ring A is u-S-semihereditary if and only if A is u-S-coherent and u-S-w.gl.dim$(A) \leq 1$.

In Section 3, we introduce and study the weak u-S-projective dimensions of modules and the weak u-S-global dimension of rings. We prove that a ring A is u-S-semihereditary if and only if w-u-S-w.gl.dim$(A) = 0$ if and only if every ideal of A is u-S-flat.

2 weak u-S-projective modules

In this section, we introduce a class of modules called weak u-S-projective modules, study their properties and characterize them. We begin this section with the following results which we will need in this paper.

Throughout the paper, $\mathcal{U}_S^\dagger$ denote the class of S-torsion-free A-modules N with the property that $\text{Ext}_A^i(M, N) = 0$ for all u-S-projective A-modules M and for all integers $i \geq 1$. Clearly, every S-torsion-free injective A-module belongs to $\mathcal{U}_S^\dagger$.

Remark 1. *If S is composed of units, we have every A-module belongs to $\mathcal{U}_S^\dagger$, since every u-S-projective A-module is projective and every A-module is S-torsion-free.*

Proposition 1. *1. Let $\{M_i\}_{i\in I}$ be a family of S-torsion-free A-modules. Then $\Pi_{i\in I} M_i \in \mathcal{U}_S^\dagger$ if and only if $M_i \in \mathcal{U}_S^\dagger$ for all $i \in I$.*

2. *If* $X \in \mathcal{U}_S^{\dagger}$*, then* $\mathrm{Ext}_A^i(T, X) = 0$ *for all u-S-torsion A-module T and for all integer* $i \geq 1$.

3. *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of A-modules with* $M' \in \mathcal{U}_S^{\dagger}$*. Then* $M \in \mathcal{U}_S^{\dagger}$ *if and only if so is* M''.

Proof. (1) Follows from [6, Theorem 3.3.9 and Example 1.6.11(6)].

(2) Trivial, by [11, Corollary 2.11].

(3) Let $M' \in \mathcal{U}_S^{\dagger}$, then for any u-S-torsion A-module T, there exists an exact sequence of A-module

$$\mathrm{Hom}_A(T, M') \to \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, M'') \to \mathrm{Ext}_A^1(T, M').$$

The left term is zero by [8, Proposition 2.5] and the right term is zero by (2). Hence by [8, Proposition 2.5] again, M is S-torsion-free if and only if so is M''. Moreover, for any u-S-projective A-module P and for any integer $i \geq 1$, we have

$$\mathrm{Ext}_A^i(P, M') \to \mathrm{Ext}_A^i(P, M) \to \mathrm{Ext}_A^i(P, M'') \to \mathrm{Ext}_A^{i+1}(P, M').$$

Hence $\mathrm{Ext}_A^i(P, P') = \mathrm{Ext}_A^{i+1}(P, M') = 0$ since $M' \in \mathcal{U}_S^{\dagger}$.
Thus, $\mathrm{Ext}_A^i(P, M) \cong \mathrm{Ext}_A^i(P, M'')$ which implies that $M \in \mathcal{U}_S^{\dagger}$ if and only if so is M''. □

Proposition 2. *Every* A_S*-module, as an A-module, is in* $\mathcal{U}_S^{\dagger}$.

Proof. Let N be an A_S-module, and let M be a u-S-projective A-module. By [11, Theorem 2.9], $\mathrm{Ext}_A^n(M, N)$ is u-S-torsion for any $n \geq 1$. Hence it is an S-torsion A-module for any $n \geq 1$. $\mathrm{Ext}_A^n(M, N)$ is an S-torsion-free A-module by [6, Example 1.6.12(2)] and since $\mathrm{Ext}_A^n(M, N)$ is an A_S-module. Consequently, we have $\mathrm{Ext}_A^n(M, N) = 0$ by [6, Example 1.6.13(5)]. Hence we conclude that $N \in \mathcal{U}_S^{\dagger}$. □

Proposition 3. *Let E be an S-torsion-free injective A-module. Then* $\mathrm{Hom}_A(M, E) \in \mathcal{U}_S^{\dagger}$ *for any A-module M.*

Proof. Let M be an A-module, and let E be an S-torsion-free injective A-module. By [6, Theorem 3.4.11], we have

$$\mathrm{Ext}_A^n(P, \mathrm{Hom}_A(M, E)) \cong \mathrm{Hom}_A(\mathrm{Tor}_n^A(P, M), E)$$

for any u-S-projective A-module P. Thus, P is a u-S-flat A-module by [11, Proposition 2.13]. Hence $\mathrm{Tor}_n^A(P, M)$ is u-S-torsion by [8, Theorem 3.2]. By [8, Proposition 2.5], $\mathrm{Hom}_A(\mathrm{Tor}_n^A(P, M), E) = 0$. Therefore, we have $\mathrm{Ext}_A^n(P, \mathrm{Hom}_A(M, E)) = 0$, which implies that $\mathrm{Hom}_A(M, E) \in \mathcal{U}_S^{\dagger}$. □

Next, we will introduce a new class of modules called the weak uniformly S-projective modules.

Definition 1. *An A-module M is said to be w-u-S-projective (abbreviates weak uniformly S-projective) if* $\mathrm{Ext}_A^1(M, N) = 0$ *for any* $N \in \mathcal{U}_S^{\dagger}$.

Clearly the following containments hold.

$$\{\text{projective}\} \subseteq \{u\text{-}S\text{-projective}\} \subseteq \{w\text{-}u\text{-}S\text{-projective}\}.$$

Remark 2.

1. If S consists of units, it is easy to see that the three classes of modules previous coincide.
2. Using [11, Theorem 3.5], it is easy to see that every w-u-S-projective is u-S-projective over a u-S-semisimple ring.
3. Every projective module is w-u-S-projective but the converse is not true in general by [11, Example 3.11].

The following proposition summarizes some of the properties of weak uniformly S-projective modules.

Proposition 4. *The following statements hold for any ring A and multiplicative subset S of A:*

1. *An A-module M is w-u-S-projective if and only if $\mathrm{Ext}_A^j(M,N)=0$ for any $N \in \mathcal{U}_S^{\dagger}$ and any $j \geq 1$.*
2. *The class of all w-u-S-projective modules is closed under direct sums and under direct summands.*
3. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules with M'' is w-u-S-projective. Then M' is w-u-S-projective if and only if so is M.*

Proof. (1) Suppose M is a w-u-S-projective module, and let $N \in \mathcal{U}S^{\dagger}$. If $j=1$, the result is trivial by definition of w-u-S-projective. For any $j>1$, by Proposition 1, there exists an exact sequence of A-modules

$$0 \to N \to I_0 \to I_1 \to \cdots \to I_{j-2} \to L \to 0$$

where $I_0, \ldots, I_{j-2}$ are S-torsion-free injective and $L \in \mathcal{U}_S^{\dagger}$. Thus, we have $\mathrm{Ext}_A^j(M,N) \cong \mathrm{Ext}_A^1(M,L)=0$. The converse is obvious.

(2) The statement follows from [6, Theorem 3.3.9].

(3) Let $N \in \mathcal{U}_S^{\dagger}$, and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence with M'' being w-u-S-projective. By the long exact sequence of Ext's associated to this short exact sequence, we get

$$\mathrm{Ext}_A^1(M'',N) \to \mathrm{Ext}_A^1(M,N) \to \mathrm{Ext}_A^1(M',N) \to \mathrm{Ext}_A^2(M'',N).$$

Since M'' is w-u-S-projective, we have $\mathrm{Ext}_A^1(M'',N)=\mathrm{Ext}_A^2(M'',N)=0$ by (1). Hence $\mathrm{Ext}_A^1(M,N) \cong \mathrm{Ext}_A^1(M',N)$, which implies that M' is w-u-S-projective if and only if M is. □

Corollary 1. *Let $0 \to L \to F \to M \to 0$ be a u-S-exact sequence of A-modules with F is w-u-S-projective. Then for any $N \in \mathcal{U}_S^{\dagger}$ and integre $n \geq 1$, $\mathrm{Ext}_A^n(L,N)$ is u-S-isomorphic to $\mathrm{Ext}_A^{n+1}(M,N)$.*

Proof. By [9, Theorem 1.4] and Proposition 4. □

The following example shows that w-u-S-projective modules may not necessarily be u-S-projective in general.

Example 1. Let $A = \mathbb{Z}$ be the ring of integers, p a prime in $\mathbb{Z}$ and $S = \{p^n | n \in \mathbb{N}\}$. Since $\mathbb{Z}/\langle p^n \rangle$ is u-S-torsion, and so is w-u-S-projective. Thus, by Proposition 4, the A-module $N = \bigoplus_{n=1}^{\infty} \mathbb{Z}/\langle p^n \rangle$ is w-u-S-projective. However, we claim that it is not u-S-projective. Indeed, first, we note that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/\langle p^n \rangle, \mathbb{Z}/\langle p^m \rangle) \cong \mathbb{Z}/\langle p^{\min\{m,n\}} \rangle$. So we have

$$\mathrm{Ext}^1_{\mathbb{Z}}(N, N) \cong \prod_{n\in\mathbb{N}} (\bigoplus_{m\in\mathbb{N}} \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/\langle p^n \rangle, \mathbb{Z}/\langle p^m \rangle)) \cong \prod_{n\in\mathbb{N}} (\bigoplus_{m\in\mathbb{N}} \mathbb{Z}/\langle p^{\min\{m,n\}} \rangle).$$

Note that the abelian group $\prod_{n\in\mathbb{N}} (\bigoplus_{m\in\mathbb{N}} \mathbb{Z}/\langle p^{\min\{m,n\}} \rangle)$ contains a subgroup $\prod_{n\in\mathbb{N}} \mathbb{Z}/\langle p^n \rangle$. Since $\prod_{n\in\mathbb{N}} \mathbb{Z}/\langle p^n \rangle$ is not u-S-torsion, $\mathrm{Ext}^1_{\mathbb{Z}}(\bigoplus_{n=1}^{\infty} \mathbb{Z}/\langle p^n \rangle, \bigoplus_{n=1}^{\infty} \mathbb{Z}/\langle p^n \rangle)$ is also not u-S-torsion. Consequently, $\bigoplus_{n=1}^{\infty} \mathbb{Z}/\langle p^n \rangle$ is not u-S-projective.

The following proposition gives some characterizations of w-u-S-projective modules.

Proposition 5. *The following statements are equivalent for any A-module M:*

1. *M is w-u-S-projective,*
2. *$M \otimes F$ is w-u-S-projective for any projective A-module F,*
3. *$\mathrm{Hom}_A(F, M)$ is w-u-S-projective for any finitely generated projective A-module F.*

Proof. (1) $\Rightarrow$ (2) Let F be a projective A-module. For any A-module $N \in \mathcal{U}^{\dagger}_S$. By [6, Theorem 3.3.10], we have $\mathrm{Ext}^1_A(F \otimes M, N) \cong \mathrm{Hom}_A(F, \mathrm{Ext}^1_A(M, N))$. Since M is w-u-S-projective, $\mathrm{Ext}^1_A(M, N) = 0$, and $\mathrm{Ext}^1_A(F \otimes M, N) = 0$. Hence $F \otimes M$ is a w-u-S-projective A-module.

(1) $\Rightarrow$ (3) Let $N \in \mathcal{U}^{\dagger}_S$, for any finitely generated projective A-module F. By [6, Theorem 3.3.12], we have $F \otimes \mathrm{Ext}^1_A(M, N) \cong \mathrm{Ext}^1_A(\mathrm{Hom}_A(F, M), N)$. Since M is w-u-S-projective, so $\mathrm{Ext}^1_A(M, N) = 0$. Hence $\mathrm{Ext}^1_A(\mathrm{Hom}_A(F, M), N) = 0$, which implies that $\mathrm{Hom}_A(F, M)$ is a w-u-S-projective A-module.

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) Follow by letting $F = A$. □

Proposition 6. *Let $A = A_1 \times A_2$ be a direct product of rings A_1 and A_2 and $S = S_1 \times S_2$ be a direct product of multiplicative subsets of A_1 and A_2. Then M is a w-u-S-projective A-module if and only if M_i is a w-u-S_i-projective A_i-module for each $i = 1, 2$.*

Proof. Suppose M be a w-u-S-projective A-module so $M = M_1 \times M_2$. Let N be an A_1-module and $N \in \mathcal{P}^{\dagger}_{S_1}$. Then we have $0 = \mathrm{Ext}^1_A(M, N \times 0) \cong \mathrm{Ext}^1_{A_1}(M_1, N)$. Consequently, M_1 is a w-u-S_1-projective A_1-module. Similarly, M_2 is a w-u-S_2-projective A_2-module. Now, suppose that M_i is a w-u-S_i-projective A_i-module for each $i = 1, 2$. Let N be an A-module and $N \in \mathcal{U}^{\dagger}_S$, so $N = N_1 \times N_2$. Hence $\mathrm{Ext}^1_A(M, N) \cong \mathrm{Ext}^1_{A_1}(M_1, N_1) \times \mathrm{Ext}^1_{A_2}(M_2, N_2) = 0$. Thus, M is a w-u-S-projective A-module. □

Proposition 7. *If M is w-u-S-projective, then M_S is projective over A_S. The converse holds, if M is u-S-finitely presented and S is finite.*

Proof. Let N be an A_S-module and let $0 \to L \to F \to M \to 0$ be an exact sequence with F free. Hence by Proposition 2, $N \in \mathcal{U}_S^{\dagger}$ and so $\mathrm{Ext}_A^1(M,N) = 0$. Consider the following diagramme with exact rows

$$\begin{array}{ccccccc} \mathrm{Hom}_{A_S}(F_S,N) & \longrightarrow & \mathrm{Hom}_{A_S}(L_S,N) & \longrightarrow & \mathrm{Ext}^1_{A_S}(M_S,N) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}_A(F,N) & \longrightarrow & \mathrm{Hom}_A(L,N) & \longrightarrow & \mathrm{Ext}^1_A(M,N) & \longrightarrow & 0 \end{array}$$

By [6, Theorem 2.2.16], the first two vertical maps are isomorphisme. Thus, we have $\mathrm{Ext}^1_{A_S}(M_S,N) \cong \mathrm{Ext}^1_A(M,N) = 0$. Hence M_S is a projective over A_S. The converse, suppose that M_S is a projective A_S-module, so M_S is flat. Hence by [8, Proposition 3.8], we have M is a u-S-flat A-module and by [13, Proposition 2.8], M is u-s-projective and so w-u-s-projective. □

Corollary 2. *Let S be finite. Every w-u-S-projective module is u-S-flat.*

Proof. By Proposition 7 and [8, Proposition 3.8]. □

Proposition 8. *If P is a w-u-S-projective A-module and E is an S-torsion-free injective A-module. Then $\mathrm{Hom}_A(\mathrm{Tor}_n^A(P,M),E) = 0$.*

Proof. Let P be a w-u-S-projective A-module, E be S-torsion-free injective, and let M be an A-module. Hence by [6, Theorem 3.4.11], we have

$$\mathrm{Ext}^n_A(P,\mathrm{Hom}_A(M,E)) \cong \mathrm{Hom}_A(\mathrm{Tor}_n^A(P,M),E).$$

Since $\mathrm{Hom}_A(M,E) \in \mathcal{U}_S^{\dagger}$ by Proposition 3, we have $\mathrm{Ext}^n_A(P,\mathrm{Hom}_A(M,E)) = 0$. Hence $\mathrm{Hom}_A(\mathrm{Tor}_n^A(P,M),E) = 0$. □

Proposition 9. *Let S be finite, and M be an S-torsion-free A-module. The following assertions holds.*

1. *M_S/M is w-u-S-projective.*
2. *M is w-u-S-projective if and only if so is M_S.*

Proof. (1) Since M is a S-torsion-free A-module, M_S/M is S-torsion by [6, Example 1.6.13]. Hence by [8, Propositin 2.3], M_S/M is u-S-torsion. Thus, by [11, Corollary 2.11], M_S/M is u-S-projective which implies that is a w-u-S-projective A-module.

(2) Let $N \in \mathcal{U}_S^{\dagger}$, by (1) we have M_S/M is u-S-projective. Consider the following exact sequence $0 \to M \to M_S \to M_S/M$. Hence by Proposition 4, we have M is w-u-S-projective if and only if M_S is w-u-S-projective. □

From [2], an A-module M is said to be weak u-S-flat (w-u-S-flat) if $\mathrm{Tor}_1^A(A/I, M)$ is u-S-torsion for any ideal I of A.

Lemma 1. *Let S be finite, and M be a u-S-finitely presented A-module. The following assertions are equivalent.*

1. *M is u-S-projective,*
2. *M is w-u-S-projective,*
3. *M is w-u-S-flat,*
4. *M is u-S-flat,*
5. *M_S is a projective A_S-module.*

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) By Corollary 2.

(3) $\Rightarrow$ (4) By [2, Corollary 2.11].

(4) $\Leftrightarrow$ (5) Let M be a u-S-finitely presented A-module, so M_S is a finitely presented A_S-module by [13, Proposition 2.4]. Hence M_S is a flat A_S-module by [8, Corollary 3.6]. Thus, M_S is a projective A_S-module.

(5) $\Rightarrow$ (1) By Proposition 7. □

Proposition 10. *If M is w-u-S-projective then $\mathrm{Ext}_A^1(M, N) = 0$ for any A_S-module N. The converse holds if M is u-S-finitely presented and S is finite.*

Proof. Let N be an A_S-module, then by Proposition 2, $N \in \mathcal{U}_S^\dagger$. Hence $\mathrm{Ext}_A^1(M, N) = 0$ since M is w-u-S-projective. Conversitely, suppose that $\mathrm{Ext}_A^1(M, N) = 0$ for any A_S-module N, so M_S is a projective A_S-module. Hence by Lemma 1, M is w-u-S-projective. □

Next, we give a new characterizations of u-S-Von Neumann regular rings in terms of w-u-S-projective modules.

Proposition 11. *Let S be finite. The following assertions are equivalent.*

1. *Every u-S-finitely presented A-module is w-u-S-projective,*
2. *Every u-S-finitely presented A-module is w-u-S-flat,*
3. *A is u-S-Von Neumann regular.*

Proof. (1) $\Leftrightarrow$ (2) Follows from Lemma 1.

(2) $\Rightarrow$ (3) Let M be a finitely presented A-module, so M is w-u-S-flat by (2). Thus, M is u-S-flat by Lemma 1. Hence A is u-S-Von Neumann regular ring by [3, Proposition 3.19].

(3) $\Rightarrow$ (1) Let M be a u-S-finitely presented A-module. Hence M is u-S-flat since A is u-S-Von Neumann regular and so M is w-u-S-projective by Lemma 1. □

Recall that a ring A is said to be semihereditary if every finitely generated ideal of A is projective. It is proved in [6, Theorem 3.7.10], that a ring is semihereditary if and only if A is coherent and every ideal of A is flat.

In the next part, we introduce a new class of rings, which is a u-S-version of semihereditary rings.

Definition 2. *A ring A is called a u-S-semihereditary (abbreviates uniformly S-semihereditary) if every finitely generated ideal of A is u-S-projective.*

Obviously, every semisimple and semihereditary rings are u-S-semihereditary but the converse is not true in general (see Example 2). Also, every u-S-semisimple rings are u-S-semihereditary rings (see [11, Theorem 3.5]).

Lemma 2. *Let $\{M_i | i \in I\}$ be a direct system of u-S-flat sub-modules of M which are all with respect to some $s \in S$. Then $\varinjlim M_i$ is u-S-flat.*

Proof. Set $M = \varinjlim M_i$. Hence $s\operatorname{Tor}_1^A(F, \varinjlim M_i) \cong s(\varinjlim \operatorname{Tor}_1^A(F, M_i)) \cong \varinjlim(s\operatorname{Tor}_1^A(F, M_i)) = 0$ by [13, Lemma 4.1]. So, $\operatorname{Tor}_1^A(F, \varinjlim M_i)$ is u-S-torsion, which implies that $\varinjlim M_i$ is u-S-flat by [8, Theorem 3.2]. □

Proposition 12. *Let $A = A_1 \times A_2$ be direct product of rings A_1 and A_2, $S = S_1 \times S_2$ a multiplicative subset of A. Then A is u-S-semihereditary if and only if A_i is u-S_i-semihereditary for any $i = 1, 2$.*

Proof. This is straightforward. □

In the following example, we show that there exists a u-S-semihereditary ring but not semihereditary.

Example 2. Let A_1 be a semihereditary ring and A_2 a non-semihereditary ring. Denote by $A = A_1 \times A_2$. Then A is not a semihereditary. Set $S = S_1 \times S_2$, where $S_1 = \{1\}$ and $S_2 = \{0\}$. Hence A is u-S-semihereditary ring by Proposition 12 (A_2 is u-S-semihereditary because $0 \in S_2$ and so A_2 is u-S-semisimple ring).

Recall from [13], that an A-module M is called u-S-coherent (with respective to s) provided that there is $s \in S$ such that it is S-finite with respect to s and any finitely generated submodule of M is u-S-finitely presented with respective to s. A ring A is called u-S-coherent (with respective to s) if A itself is a u-S-coherent A-module with respective to s.

In the following result we give a new characterization of u-S-coherent.

Proposition 13. *The following statements are equivalent.*

1. *A is u-S-coherent ring,*
2. *Every finitely generated ideal of A is u-S-finitely presented with respect to some fixed $s \in S$,*
3. *Every finitely generated submodule of a free module is u-S-finitely generated with respect to some fixed $s \in S$.*

Proof. (3) ⇒ (2) This is trivial.

(2) ⇒ (1) By [13, Theorem 2.2].

(1) ⇒ (3) Let M be a finitely generated submodule of a free module F. So by [13, Theorem 3.2(3)], F is u-S-coherent. Hence M is u-S-finitely presented. □

In the following Proposition we characterize a ring with u-S-w.gl.dim$(A) \leq 1$.

Proposition 14. *The following statements are equivalent.*

1. *Every submodule of u-S-flat modules is u-S-flat,*
2. *Every submodule of flat modules is u-S-flat,*
3. *Every ideal of A is u-S-flat with respect to some fixed $s \in S$,*
4. *Every finitely generated ideal of A is u-S-flat with respect to some fixed $s \in S$.*
5. *u-S-w.gl.dim$(A) \leq 1$*

Proof. (1) ⇔ (2) ⇔ (5) By [10, Proposition 3.9].

(4) ⇒ (3) Let I be an ideal of A. Then $I = \cup B$ where B ranges over the set of all finitely generated subideal of I. Thus, I is u-S-flat by Lemma 2.

(2) ⇒ (3) ⇒ (4) Theses are obvious.

(3) ⇒ (5) Let I be an ideal of A, so is u-S-flat with respect to some fixed $s \in S$ by (3). Hence u-S-fd$_A(A/I) \leq 1$. By [10, Proposition 2.3], we have $s\,\mathrm{Tor}_2^A(A/I, A/J) = 0$ for all ideal J of A. Thus, u-S-w.gl.dim$(A) \leq 1$ by [10, Proposition 3.2]. □

Proposition 15. *Let A be a u-S-coherent ring and S be finite. Then*

$$u\text{-}S\text{-w.gl.dim}(A) \leq n \textit{ if and only if } u\text{-}S\text{-fd}_A(A/\mathfrak{m}) \leq n \textit{ for any } \mathfrak{m} \in \mathrm{Max}(A).$$

Proof. Let A be a u-S-coherent ring with respect to some $s \in S$. Then A_s is coherent ring by [13, Proposition 3.14]. Hence u-S-w.gl.dim$(A) \leq n$ if and only if w.gl.dim$(A_s) \leq n$ by [10, Corollary 3.5] if and only if $\mathrm{fd}_{A_s}(A_s/\mathfrak{m}A_s) \leq n$ for any $\mathfrak{m} \in \mathrm{Max}(A)$ by [7, Lemma 3.7] if and only if u-S-fd$_A(A/\mathfrak{m}) \leq n$ for any $\mathfrak{m} \in \mathrm{Max}(A)$ by [10, Corollary 2.6]. □

Recall from [5], that an A-module E is called u-S-injective provided that the induced sequence $0 \to \mathrm{Hom}_A(M'', E) \to \mathrm{Hom}_A(M, E) \to \mathrm{Hom}_A(M', E) \to 0$ is u-S-exact for any u-S-exact sequence $0 \to M' \to M \to M'' \to 0$.

Proposition 16. *An A-module N is S-torsion if and only if $\mathrm{Hom}_A(N, E) = 0$ for any u-S-injective S-torsion-free A-module E.*

Proof. Let N be a S-torsion A-module, so $\mathrm{Hom}_A(N, E) = 0$ by [8, Proposition 2.5].

Converstely, Set $T = \mathrm{tor}_S(N)$ and $C = N/T$. Thus, C is S-torsion-free by [6, Example 1.6.13]. Set $E = E(C)$, so E is S-torsion-free by [6, Exercise 2.34] and so E is u-S-injective S-torsion-free. Hence $\mathrm{Hom}_A(N, E) = 0$ by hypothesis. Since, $0 \to \mathrm{Hom}_A(C, E) \to \mathrm{Hom}_A(N, E)$ is exact, $\mathrm{Hom}_A(C, E) = 0$ and so the inclusion map $C \hookrightarrow E$ it the zero homomorphism. Hence $C = 0$ which implies that N is S-torsion. □

Recall from [4], that a ring A is called c-S-coherent, if every S-finite ideal of A is S-finitely presented.

Proposition 17. *Let S be finite. If A_S is a von Neumann regular ring, then A is u-S-coherent.*

Proof. Let A_S be a von Neumann regular ring, so A is c-S-coherent by [8, Corollary 3.11]. Hence by [13, Proposition 3.13], we have A is u-S-coherent. □

Corollary 3. *Let S be finite. Then any u-S-von Neumann regular ring is u-S-coherent.*

Proof. By [8, Corollary 3.14] and Proposition 17. □

Proposition 18. *Every u-S-semihereditary ring is u-S-coherent.*

Proof. Let J be a finitely generated ideal of A. Hence J is u-S-projective. Thus, J is u-S-finitely presented by [13, Proposition 2.8]. Hence A is u-S-coherent by Proposition 13. □

Corollary 4. *Every u-S-semihereditary ring is S-coheren (resp., c-S-coheren).*

Proof. By Proposition 18 and [13, Proposition 3.12]. □

Recall from [6], that a ring A is semihereditary if and only if A is coherent and $\text{w.gl.dim}(A) \leq 1$.

Next, we give a u-S-analogue of this result.

Proposition 19. *The following statements are equivalent.*

1. *A is u-S-semihereditary,*
2. *A is u-S-coherent and every finitely generated ideal of A is u-S-flat with respect to some fixed $s \in S$,*
3. *A is u-S-coherent and u-S-w.gl.dim$(A) \leq 1$.*

Proof. (1) ⇒ (2) By Proposition 18, we have every u-S-semihereditary ring is u-S-coherent. The secand part, let J is a finitely generated ideal of A, so J is u-S-projective. Hence J is u-S-flat by [11, Proposition 2.13].

(2) ⇒ (3) By Proposition 14.

(3) ⇒ (1) Let J be a finitely generated ideal of A, so J is u-S-flat by Proposition 14 and J is u-S-finitely presented by Proposition 13. Hence J is u-S-projective by [13, Proposition 2.8]. So A is u-S-semihereditary. □

3 Weak u-S-projective dimension of modules and weak u-S-global dimension of rings

In this section, we introduce and investigate the notion of weak u-S-projective dimension of modules and rings. We begin this section with the following definition.

Definition 3. *The weak u-S-projective dimension of M, denoted by* w-u-S-$\mathrm{pd}_A(M)$*, is the smallest integer* $n \geq 0$ *such that* $\mathrm{Ext}_A^{n+1}(M,N) = 0$ *for any* $N \in \mathcal{U}_S^{\dagger}$*. If no such integer exists, set* w-u-S-$\mathrm{pd}_A(M) = \infty$.

The weak u-S-global dimension of A is defined by

$$w\text{-}u\text{-}S\text{-gl.dim}(A) = \sup\{w\text{-}u\text{-}S\text{-pd}_A(M) : \ M \text{ is an } A\text{-module}\}$$

Clearly, w-u-S-$\mathrm{pd}_A(M) \leq u$-S-$\mathrm{pd}_A(M) \leq \mathrm{pd}_A(M)$, where $\mathrm{pd}_A(M)$ denotes the classical projective dimension of M, with equality when S is composed of units, also the equality when A is a semisimple ring. However, this inequality may be strict (see, Example 1 and [11, Example 3.11]. It is also obvious that an A-module M is w-u-S-projective if and only if w-u-S-$\mathrm{pd}_A(M) = 0$. Also, w-u-S-gl.dim$(A) \leq u$-S-gl.dim$(A) \leq$ gl.dim(A), where gl.dim(A) denotes the global dimension of A, with equality when S is composed of units or A is a semisimple ring. This inequality may be strict (see, Example 3 and [12, Example 3.5]).

The next result gives a description of the w-u-S-projective dimensions of modules.

Proposition 20. *The following statements are equivalent for any A-module M.*

1. w-u-S-$\mathrm{pd}_A(M) \leqslant n$,
2. $\mathrm{Ext}_A^{n+1}(M,N) = 0$ *for any* $N \in \mathcal{U}_S^{\dagger}$,
3. $\mathrm{Ext}_A^{n+i}(M,N) = 0$ *for any* $N \in \mathcal{U}_S^{\dagger}$ *and any* $i > 0$,
4. *If the sequence* $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ *is exact with* $P_0, \ldots, P_{n-1}$ *are w-u-S-projective A-modules, then* P_n *is w-u-S-projective,*
5. *If the sequence* $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ *is exact with* $P_0, \ldots, P_{n-1}$ *are projective A-modules, then* P_n *is w-u-S-projective,*
6. *There exists an exact sequence* $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ *where each* P_i *are w-u-S-projective.*

Proof. (3) ⇒ (2) ⇒ (1) and (4) ⇒ (5) ⇒ (6) These are obvious.

(6) ⇒ (3) We prove (3) by induction on n. For the case $n = 0$, (2) holds by Proposition 4 as M is a w-u-S-projective module. If $n > 0$, then there is an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ with all P_i are w-u-S-projective. Set $K_0 = \ker(P_0 \to M)$, we have two exact sequences $0 \to K_0 \to P_0 \to M \to 0$ and $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to K_0 \to 0$. Hence by induction we have, $\mathrm{Ext}_A^{n-1+i}(K_0,N) = 0$ for any $N \in \mathcal{U}_S^{\dagger}$ and any $i > 0$. Thus, $\mathrm{Ext}_A^{n+i}(M,N) = 0$.

(1) ⇒ (4) Let $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be an exact sequence with all P_i are w-u-S-projective ($i = 0, \ldots, n-1$). Set $K_0 = \ker(P_0 \to M)$ and $K_i = \ker(P_i \to P_{i-1})$, where ($i = 1, \ldots, n-1$). Hence $K_{n-1} = P_n$. Since all P_i, ($i = 0, \ldots, n-1$) are w-u-S-projevtive, $\mathrm{Ext}_A^1(P_n,N) \cong \mathrm{Ext}_A^{n+1}(M,N) = 0$ for any $N \in \mathcal{U}_S^{\dagger}$ by Proposition 4. Thus, P_n is a w-u-S-projective module. □

Corollary 5. $\mathrm{pd}_{A_S}(M_S) \leq w$-$u$-$S$-$\mathrm{pd}_A(M)$. *Moreover, if S is finite and M be u-S-finitely presented, then* w-u-S-$\mathrm{pd}_A(M) = \mathrm{pd}_{A_S}(M_S)$.

Proof. Let $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be an exact sequence, where $P_0, P_1, \ldots, P_{n-1}$ are projective A-modules. By localizing at S, we get an exact sequence of A_S-modules, $0 \to (P_n)_S \to (P_{n-1})_S \to \cdots \to (P_1)_S \to (P_0)_S \to (M)_S \to 0$. By Proposition 7, if P_n is w-u-S-projective, so $(P_n)_S$ is projective over A_S, and the converse by Proposition 7. □

Proposition 21. *Let $0 \to M'' \to M' \to M \to 0$ be an exact sequence of A-modules. If two of w-u-S-$\text{pd}_A(M'')$, w-u-S-$\text{pd}_A(M')$ and w-u-S-$\text{pd}_A(M)$ are finite, so is the third. Moreover,*

1. w-u-S-$\text{pd}_A(M'') \leq \max\{w\text{-}u\text{-}S\text{-pd}_A(M'),\ w\text{-}u\text{-}S\text{-pd}_A(M) - 1\}$.
2. w-u-S-$\text{pd}_A(M') \leq \max\{w\text{-}u\text{-}S\text{-pd}_A(M''),\ w\text{-}u\text{-}S\text{-pd}_A(M)\}$.
3. w-u-S-$\text{pd}_A(M) \leq \max\{w\text{-}u\text{-}S\text{-pd}_A(M'),\ w\text{-}u\text{-}S\text{-pd}_A(M'') + 1\}$.

Proof. Standard in homological algebra. □

Corollary 6. *Let $0 \to M'' \to M' \to M \to 0$ be an exact sequence of A-modules. If M' is w-u-S-projective and w-u-S-$\text{pd}_A(M) > 0$, then*

$$w\text{-}u\text{-}S\text{-pd}_A(M) = w\text{-}u\text{-}S\text{-pd}_A(M'') + 1.$$

Proposition 22. *Let $\{M_i\}$ be a family of A-modules. Then*

$$w\text{-}u\text{-}S\text{-pd}_A(\oplus_i M_i) = \sup_i\{w\text{-}u\text{-}S\text{-pd}_A(M_i)\}.$$

Proof. The proof is straightforward. □

Proposition 23. *Let $n \geq 0$ be an integer. The following statements are equivalent.*

1. w-u-S-$\text{gl.dim}(A) \leq n$,
2. w-u-S-$\text{pd}_A(M) \leq n$ *for any finitely generated A-module M,*
3. w-u-S-$\text{pd}_A(A/I) \leq n$ *for any ideal I of A,*
4. $\text{id}_A(N) \leq n$ *for any $N \in \mathcal{U}_S^{\dagger}$.*

Consequently, we have

$$\begin{aligned} w\text{-}u\text{-}S\text{-gl.dim}(A) &= \sup\{w\text{-}u\text{-}S\text{-pd}_A(M) \mid M \text{ is a finitely generated } A\text{-module}\} \\ &= \sup\{w\text{-}u\text{-}S\text{-pd}_A(A/I) \mid I \text{ is an ideal of } A\} \\ &= \sup\{\text{id}_A(N) \mid N \in \mathcal{U}_S^{\dagger}\}. \end{aligned}$$

Proof. (1) ⇒ (2) and (2) ⇒ (3) These are obvious.

(3) ⇒ (4) Let $N \in \mathcal{U}_S^{\dagger}$. For any ideal I of A, $\text{Ext}_{n+1}^A(A/I, N) = 0$, so $\text{id}_A(N) \leq n$.

(4) ⇒ (1) Let M be an A-module. For any $N \in \mathcal{U}_S^{\dagger}$, we have $\text{Ext}_{n+1}^A(M, N) = 0$. Thus, w-u-S-$\text{pd}_A(M) \leq n$, which implies that w-u-S-$\text{gl.dim}(A) \leq n$. □

Corollary 7. *The following statements are equivalent.*

1. w-u-S-gl.dim$(A) = 0$,
2. *Every finitely generated A-module is w-u-S-projective,*
3. *A/I is w-u-S-projective for any ideal I of A,*
4. *Every A-module belong to $\mathcal{U}_S^{\dagger}$ is injective.*

Every u-S-von Neumann regular ring A has w-u-S-gl.dim$(A) = 0$. However, the converse is not true in general.

Example 3. Let $A = \mathbb{Z}$ be the ring of integers and $S = \mathbb{Z} - \{0\}$. Then A is not u-S-von Neumann regular since S is composed of regular elements and A is not von Neumann. But each $A/\langle s\rangle$ is u-S-torsion, so $\mathcal{U}_S^{\dagger}$ is equal to torsion-free injective A-modules. Hence w-u-S-gl.dim$(A) = 0$.

Corollary 8. *The following statements are equivalent.*

1. w-u-S-gl.dim$(A) \leq 1$,
2. *Every submodule of a w-u-S-projective A-module is w-u-S-projective,*
3. *Every submodule of a projective A-module is w-u-S-projective,*
4. *Every ideal of A is w-u-S-projective,*
5. $\mathrm{id}_A(N) \leq 1$ *for any* $N \in \mathcal{U}_S^{\dagger}$.

Recall from [2], The weak u-S-flat dimension of A is defined by

$$w\text{-}u\text{-}S\text{-w.gl.dim}(A) = \sup\{w\text{-}u\text{-}S\text{-fd}_A(M) : \text{ M is an } A\text{-module}\},$$

where w-u-S-fd$_A(M)$ denotes the weak u-S-flat dimension of M.

Proposition 24. *Let S be finite. Then*

$$w\text{-}u\text{-}S\text{-fd}_A(M) \leq u\text{-}S\text{-fd}_A(M) \leq w\text{-}u\text{-}S\text{-pd}_A(M).$$

Consequently,

$$w\text{-}u\text{-}S\text{-w.gl.dim}(A) \leq u\text{-}S\text{-w.gl.dim}(A) \leq w\text{-}u\text{-}S\text{-gl.dim}(A).$$

The equivalence holds, if M is a u-S-finitely presented A-module.

Proof. Follows from Corollary 2 and Lemma 1. □

Recall from [1], a ring A is said to be S-Noetherian if every ideal of A is S-finite.

Proposition 25. *Let A be an S-Noetherian and S be finite.*

$$u\text{-}S\text{-w.gl.dim}(A) = w\text{-}u\text{-}S\text{-w.gl.dim}(A) = \sup\{w\text{-}u\text{-}S\text{-pd}_A(A/I) \mid I \textit{ is an ideal of } A\}.$$

Proof. Let I be an ideal of A, so A/I is u-S-finitely presented by [13, Proposition 2.3]. Hence by Proposition 24, we have w-u-S-$\text{fd}_A(A/I) = u$-S-$\text{fd}_A(M) = w$-u-S-$\text{pd}_A(M)$ and so w-u-S-w.gl.dim(A) $= u$-S-w.gl.dim$(A) = w$-u-S-gl.dim(A). Thus, by Proposition 23, we have the result. □

Corollary 9. *Let A be an S-Noetherian ring and S be finite. The following are equivalent.*

1. *A is u-S-semihereditary ring,*
2. *A is u-S-von Neumann regular ring,*
3. *Every ideal of A is u-S-flat,*
4. *w-u-S-w.gl.dim$(A) = 0$.*

Proof. (2) ⇔ (4) By [10, Corollary 3.8] and Proposition 25.
(1) ⇒ (4) Since A u-S-semihereditary ring, so every finitely generated ideal is w-u-S-projective. Hence w-u-S-$\text{pd}_A(A/I) = 0$, and by Proposition 25, we have the result.
(2) ⇒ (3) By [8, Theorem 3.13].
(3) ⇒ (4) By Proposition 14 and Proposition 25.
(4) ⇒ (1) By Corollary 7, we have every finitely generated module is w-u-S-projective, and so A is u-S-semihereditary ring. □

Proposition 26. *Every u-S-semihereditary ring is S-Noetherian.*

Proof. Let I be an ideal of A, so A/I is w-u-S-projective since A is u-S-semihereditary. Hence by [13, Proposition 2.8], A/I is u-S-finitely presented and by [13, Theorem 2.2], I is S-finite. Hence A is S-Noetherian. □

Corollary 10. *Let S be finite. Every u-S-semihereditary ring is u-S-Noetherian.*

Proof. By Proposition 26 and [5, Proposition 2.4]. □

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