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$\mathcal{LA}$-semiperfect modules

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Abstract. Inspired in [7], we strengthened semiperfect modules to $\mathcal{LA}$-semiperfect modules by adding the concept of locally artinian submodule and we proved that the concept of these modules is not empty. Then we characterized these projective modules with the help of defining $\mathcal{LA}$-projective covers. Thus we achived the necessary and sufficient condition for the projective module between the notion of $\mathcal{LA}$-semiperfect modules and the notion of locally artinian supplemented modules.

Keywords: Locally artinian module, Semiperfect module, $\mathcal{LA}$-semiperfect module.
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1 Introduction

In the text, S always denotes an associative ring with an identity element, and modules are left as unital S-modules. $Rad(S)$ will represent the Jacobson radical of the ring S. If N is a small submodule of X, we will use notation $N \ll X$. A submodule N of X will point to $N \subseteq X$. $Rad(X)$ and $Soc(X)$ will indicate radical and socle X, respectively. A nonzero module X is *local* if the sum of all proper submodules of X is also a proper submodule of X. It is said to be *hollow* if each proper submodule of X is small in X. Note that local modules are hollow. A finitely generated hollow module is local. For modules X and P, if $f : P \longrightarrow X$ is an epimorphism such that $ker(f)$ is small in P, then f is named *cover*. A projective module P with a cover f on X is said to be a *projective cover* of X. With [1], every (finitely generated) module of a (semi-)perfect ring possess a projective cover. In [3], Kasch and Mares transferred their concept of (semi)perfect rings to modules. Semi-perfect modules were originally defined by Mares for projective modules, but have been extended to arbitrary modules in [4].

A S-module X is named *semiperfect* if each factor module of X possesses a projective cover. Mares gave a characterization of semiperfect modules through supplemented modules, [6]. A

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submodule K of X is named a *(weak) supplement* of N in X, if $X = N + K$ and $(N \cap K \ll X)$ $N \cap K \ll K$. The module X is *(weakly) supplemented* in case each submodule of X possesses a (weak) supplement in X. A submodule K of X possesses *ample supplements* in X in case each submodule T of X provided that $X = K + T$ contains a supplement of K in X. The module X is said to be *amply supplemented* in case each submodule of X possesses ample supplements in X [11]. A semiperfect module X is supplemented. Note that the converse assertion is true in case where the module is projective.

A module X is named *locally artinian* if each finitely generated submodule of X is artinian [11]. For a module X, we consider the following submodule of X as in [12],

$$Soc_s(X) = \sum\{\, N \ll X \mid N \text{ is simple } \}.$$

It is clear that $Soc_s(X) \subseteq Rad(X)$ and $Soc_s(X) \subseteq Soc(X)$.

In [5], a local module X is said to be *strongly local* if $Rad(X)$ is semisimple. A submodule K of X is named an *ss-supplement* of N in X, if $X = N + K$ and $N \cap K \subseteq Soc_s(K)$. The module X is said to be *ss-supplemented* if every submodule of X possesses an ss-supplement in X. A submodule K of X has *ample ss-supplements* in X if each submodule T of X provided that $X = K + T$ contains an ss-supplement of K in X. The module X is named *amply ss-supplemented* if each submodule of X possesses ample ss-supplements in X. The class of (amply) ss-supplemented modules was first studied by Kaynar et.al. in [5].

In [8], a module X is said to be *locally artinian supplemented* if each submodule L of X possesses a locally artinian supplement K in X, that is, K is a supplement of L in X provided that $L \cap K$ is locally artinian. X is named *amply locally artinian supplemented* if each submodule L of X possesses ample locally artinian supplements in X. Here a submodule L of X possesses ample locally artinian supplements in X if each submodule K of X includes a locally artinian supplement $K^{'}$ of L in X providing $X = L + K$. (Amply) locally artinian supplemented modules are defined as a proper generalization of (amply) ss-supplemented modules.

In section 2, we focused on the characterization theorems in projective modules by defining a $\mathcal{LA}$-semiperfect module, which is a special notion of semiperfect modules. It is obtained that a projective module X is $\mathcal{LA}$-semiperfect if and only if every proper submodule of X is contained in a maximal submodule and $\frac{X}{Y}$ possesses a locally artinian projective cover for every maximal submodule Y of X. It is proven that a projective module X is $\mathcal{LA}$-semiperfect if and only if each maximal submodule and each cyclic submodule possesses a locally artinian supplement, and each proper submodule is included in a maximal submodule. It is also proven that a projective module X is $\mathcal{LA}$-semiperfect if and only if it is locally artinian supplemented.

2 $\mathcal{LA}$-Semiperfect Modules

Definition 1. (a) *Let $\pi : P \longrightarrow X$ be an epimorphism. Then π is said to be* a $\mathcal{LA}$-projective cover of X *if P is projective, $ker(\pi) \ll P$ and $ker(\pi)$ is locally artinian.*
(b) *A module X is said to be $\mathcal{LA}$-*semiperfect *if each factor module of X possesses a $\mathcal{LA}$-projective cover.*
(c) *A ring S is said to be $\mathcal{LA}$-*semiperfect *if ${}_SS$ is a $\mathcal{LA}$-semiperfect module.*

Recall from [1] that $Rad\,(X) = Rad\,(S)X$ for a projective module X over a ring S. Moreover, if $X = N \oplus K$ with $N \subseteq Rad(P)$, then $N = 0$.

Lemma 1. *Let X be an S-module that possesses a $\mathcal{LA}$-projective cover. Then X is a projective $\frac{S}{Rad(S)}$-module.*

Proof. Let

$$0 \longrightarrow N \longrightarrow P \xrightarrow{\pi} X \longrightarrow 0$$

be a $\mathcal{LA}$-projective cover of X. So P is projective, $ker(\pi) \ll P$ and $ker(\pi)$ is locally artinian. Then we transform above sequence to induced sequence as below.

$$\frac{S}{Rad(S)} \otimes_S N \longrightarrow \frac{S}{Rad(S)} \otimes_S P \longrightarrow \frac{S}{Rad(S)} \otimes_S X \longrightarrow 0$$

Here, we obtained the induced sequence is also exact. Thus $X \cong \frac{S}{Rad(S)} \otimes_S X$ as $\frac{S}{Rad(S)}$-modules and $\frac{S}{Rad(S)} \otimes P$ is a projective $\frac{S}{Rad(S)}$-module. Since $N \subseteq Rad(P) = Rad(S)P$, then we have $\frac{S}{Rad(S)} \otimes_S N \longrightarrow \frac{S}{Rad(S)} \otimes_S P$ is the zero map, as required. □

Sandomierski has shown in [10] the necessary and sufficient condition that a ring S is semiperfect is that each simple left S-module possesses a projective cover. We obtain it in a stronger way than the proof method in Sandomierski's Theorem to projective modules as follows.

Theorem 1. *Let X be a projective S-module.Then the projective module X is $\mathcal{LA}$-semiperfect if and only if every proper submodule of X is included in a maximal submodule and $\frac{X}{Y}$ has a $\mathcal{LA}$-projective cover for each maximal submodule Y of X.*

Proof. Let X be a $\mathcal{LA}$-semiperfect module. Since $Rad(X) \ll X$ and $\frac{X}{Rad(X)}$ is locally artinian, $\frac{X}{Rad(X)}$ is semisimple by [11, 31.2. (3) (ii)]. So, each proper submodule of X is included in a maximal submodule. The necessity of the other statement is clear.

Conversely, we verify firstly $Rad(X) \ll X$. For if $Rad(X) + N = X$, where N is a proper submodule of X, we can include N in a maximal submodule of Y and so obtain $X = Rad(X) + N \subseteq Y$, a contradiction. Now we show that $A = \frac{X}{Rad(X)}$ is locally artinian. For this, it is enough to use [11, 31.2(3) (ii)] and show that A is semisimple. If $\frac{Y}{Rad(X)}$ is a maximal submodule of A, say B then $\frac{A}{B} \cong \frac{X}{Y}$ has a $\mathcal{LA}$-projective cover. Since $\frac{A}{B}$ is an $\frac{S}{Rad(S)}$-module, B is a direct summand of A. Now suppose $Soc(A) \neq A$. Then we obtain that $Soc(A)$ is a maximal submodule of A which is a direct summand. This contradiction ensures that A is semisimple and so A is locally artinian. Finally, we must show that idempotents in $Hom_S(\frac{X}{Rad(X)}, \frac{X}{Rad(X)})$ are induced by idempotents in $Hom_S(X, X)$. Let $\pi : X \longrightarrow \frac{X}{Rad(X)}$ denote the natural homomorphism. It suffices to show that if $A = \frac{X}{Rad(X)} = L \oplus K$, then we get $X = U \oplus V$, where $\pi(U) = L$ and $\pi(V) = K$.

Now, if $A = L \oplus K$, write $L = \oplus_{\alpha\in\Lambda} S_\alpha$ and $K = \oplus_{\beta\in\Psi} T_\beta$ where S_α and T_β are simple. As every S_α and T_β is a homomorphic image of X, they have $\mathcal{LA}$-projective covers by the hypothesis, say, $P_\alpha \xrightarrow{\psi_\alpha} S_\alpha \longrightarrow 0$ and $Q_\beta \xrightarrow{\varphi_\beta} T_\beta \longrightarrow 0$. From above the diagram $W = \oplus_{\alpha\in\Lambda} P_\alpha$

and $T = \oplus_{\beta\in\Psi} Q_\beta$, we have $W \oplus T$ is projective, $ker(\psi_\alpha) \ll P_\alpha$, $ker(\varphi_\beta) \ll Q_\beta$ and $ker(\psi_\alpha)$, $ker(\varphi_\beta)$ are locally artinian for each $\alpha \in \Lambda$ and $\beta \in \Psi$. So we have the following diagram.

$$\begin{tikzcd}
 & W \oplus T \arrow[dl, dotted] \arrow[d, "\psi\oplus\varphi"] & \\
X \arrow[r, two heads, "\pi"] & A \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

where $\psi = \prod_{\alpha\in\Lambda} \psi_\alpha$ and $\varphi = \prod_{\beta\in\Psi} \varphi_\beta$. Since $ker(\pi) = Rad(X) \ll X$ and $\frac{X}{Rad(X)}$ is locally artinian, $X \xrightarrow{\pi} A \longrightarrow 0$ is a projective cover of A. According to the uniqueness of projective covers, there exist the decomposition $W \oplus T = V \oplus P'$ where $V \subseteq ker(\psi \oplus \varphi)$ and isomorphism $\theta_{|_{P'}} : P' \to X$, where $\theta : V \oplus P' \to X$ is a epimorphism. Since $ker(\psi \oplus \varphi)$ is the sum of the kernels of each ψ_α and φ_β and so is included in $Rad(W \oplus T)$, we have $V \subseteq Rad(W \oplus T)$ and so, $V = 0$. As θ is an isomorphism, $X = \theta(W) \oplus \theta(T)$. Since $\pi(\theta(W)) = L$ and $\pi(\theta(T)) = K$, we have lifted the decomposition $A = L \oplus K$, as desired. $\square$

Since a ring S is a projective S-module, the following corollary is clearly obtained.

Corollary 1. *A ring S is $\mathcal{LA}$-semiperfect if and only if every simple left S-module has a $\mathcal{LA}$-projective cover.*

Corollary 2. *Let X be a finitely generated projective S-module. Then the following implications are equivalent:*

1. *X is $\mathcal{LA}$-semiperfect;*
2. *$\frac{X}{Y}$ possesses a $\mathcal{LA}$-projective cover for each maximal submodule Y of X.*

Recall from [3] that a projective module is semiperfect if and only if each submodule possesses a supplement. In order to obtain a strong result in [2], we need the following lemma which can be proven clearly by [11, 41.14 (2)].

Lemma 2. *Let X be a projective S-module and $X = U+V$ provided that U and V are submodules each of which is a locally artinian supplement of the other. Then $X = U \oplus V$.*

Corollary 3. *Let X be a projective S-module. X is $\mathcal{LA}$-semiperfect if and only if it satisfied directly the following both of cases:*

1. *Every maximal submodule and each cyclic submodule possesses a locally artinian supplement.*
2. *Every proper submodule is included in a maximal submodule.*

Proof. ($\Leftarrow$) Let N be a maximal submodule of X and K be a locally artinian supplement of N in X. Then $X = N + K$, $N \cap K \ll K$ and $N \cap K$ is locally artinian. If $x \in K \setminus N$, we get $Sx \subseteq K$ and $N + Sx = X$. So $K = Sx$. Hence let M be a locally artinian supplement of K, we claim that K is a locally artinian supplement of M, that is $M \cap K \ll K$. Obviously $M \cap K$ is locally artinian. Indeed, if $(M \cap K) + T = K$ then $X = (M \cap K) + T + N$. Since $M \cap K \ll M$ and for this reason $M \cap K \ll X$, this implies $X = T + N$ with $T \subseteq K$. It follows that $T = K$. It follows from Lemma 2 that $X = M \oplus K$ and K is projective. Thus the exact sequence $0 \longrightarrow N \cap K \xrightarrow{i} X \xrightarrow{\pi} \frac{X}{N} \longrightarrow 0$ is a $\mathcal{LA}$-projective cover of $\frac{X}{N}$. By Corollary 2, X is $\mathcal{LA}$-semiperfect.

($\Rightarrow$) If a projective module X is $\mathcal{LA}$-semiperfect, the condition (2) directly satisfies by Corollary 2. Moreover $\frac{X}{N}$ possesses a $\mathcal{LA}$-projective cover for each maximal submodule N of P. We denote this $\mathcal{LA}$-projective cover $\pi : K \longrightarrow \frac{X}{N}$ where K is a projective module, $ker(\pi)$ is locally artinian. Then we can complete the diagram the canonical epimorphism α,

$$\begin{tikzcd} & X \arrow[dl, dotted, "\beta"'] \arrow[d, "\alpha"] & \\ K \arrow[r, two heads, "\pi"] & \frac{X}{N} \arrow[r] & 0 \end{tikzcd}$$

commutatively by an $\beta : X \longrightarrow K$, β is surjective, hence it splits. So there is a $\theta : K \longrightarrow X$ with $\theta\beta = id_P$ and thus $\pi = \theta\beta\pi = \theta\alpha$. It follows that $X = N + \theta(K)$, $\theta(K)$ is a projective cover of $\frac{X}{N}$ and consequently $N \cap \theta(K) \ll \theta(K)$. As $ker(\pi)$ is locally artinian, $N \cap \theta(K)$ is locally artinian by [11, 31.2 (1)(i)]. Let U be a cyclic submodule of P. As X is $\mathcal{LA}$-semiperfect, $\frac{X}{U}$ possesses a $\mathcal{LA}$-projective cover. Appliying the above method to $\frac{X}{U}$ as above, we have U has a locally artinian supplement in X. □

Recall from [8, Lemma 2.22] that each projective supplemented module X with locally artinian $Rad(X)$ is locally artinian supplemented.

Theorem 2. *Let X be a projective S-module. Then the following implications are equivalent.*

1. *X is locally artinian supplemented;*
2. *each submodule of X possesses a locally artinian supplement which is a direct summand of X;*
3. *for each submodule N of X, P possesses the decomposition $X = X^{'} \oplus Y$ provided that $X^{'} \subseteq N$, $N \cap Y \ll P$ and $N \cap Y$ is locally artinian;*
4. *X is $\mathcal{LA}$-semiperfect.*

Proof. (1) $\Rightarrow$ (3) Let $N \subseteq X$. By the hypothesis, N possesses a locally artinian supplement, say T in X. It follows that $X = N + T$, $N \cap T \ll T$ and $N \cap T$ is locally artinian. Since $X = N + T$ is projective, [8, Proposition 2.20] and [11, 41.14(2)], there is the decomposition $X = X^{'} \oplus T$ for some submodule $X^{'}$ of N, as required.

(3) $\Rightarrow$ (4) Let $N \subseteq X$. By (3), X has a decomposition $X = X^{'} \oplus T$ provided that $X^{'} \subseteq N$, $N \cap T \ll T$ and $N \cap T$ is locally artinian. Thus T is a locally artinian supplement of N in X and it is a projective as a direct summand of the projective module X. Consider the epimorphism $f : T \longrightarrow \frac{X}{N}$ via $f(a) = a + N$ for each $a \in T$. Since $ker(f) = N \cap T \ll T$ and $ker(f)$ is locally artinian, T is a $\mathcal{LA}$-projective cover of $\frac{X}{N}$. So X is $\mathcal{LA}$-semiperfect.

(4) $\Rightarrow$ (2) For a submodule $N \subseteq X$, let $\theta : L \longrightarrow \frac{X}{N}$ be a $\mathcal{LA}$-projective cover of $\frac{X}{N}$. As X is projective, there is a homomorphism $\beta : X \longrightarrow L$ with $\theta\beta = \pi$, where $\pi : X \longrightarrow \frac{X}{N}$ is the canonical projection. Then we have $L = \beta(X) \oplus ker(\theta)$ and so $L = \beta(X)$. So we have $\beta : P \longrightarrow L$ is an epimorphism. As X is projective, X possesses the decomposition $X = \mu(L) \oplus ker(\theta)$, where $\mu : L \longrightarrow X$ is a homomorphism with $\beta\mu = id_X$. It can be achived that $\mu(L)$ is a locally artinian supplement of N in X where $\mu(L)$ is a direct summand of X.

(2) $\Rightarrow$ (1) Clear. □

Since every submodule of a locally artinian module is locally artinian, we have the following result of Theorem 2.

Corollary 4. *Let X be a projective S-module of which radical is locally artinian. Then X is $\mathcal{LA}$-semiperfect if and only if it is semiperfect.*

Recall from [9] that a module X is named *ss-semilocal* if each submodule L of X possesses a weak supplement K in X provided that $L \cap K$ is semisimple. For a ring S, ${}_SS$ is an ss-semilocal module if and only if S is semilocal and $Rad(S) \subseteq Soc({}_SS)$.

The following example shows that each locally artinian semiperfect module need not be ss-semilocal.

Example 1. Consider the ring $S = \mathbb{Z}_{q^l}$, where $q = 2$ and $l = 4$. By [9, Example 2.16], ${}_SS$ is semiperfect but not ss-semilocal. It is clear that $Rad(S) = \langle \overline{2} \rangle$ is locally artinian. So ${}_SS$ is $\mathcal{LA}$-semiperfect by Corollary 4.

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