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On the solutions of the Diophantine equation $F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} = 11^a$

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Abstract. Let F_n denote the nth Fibonacci number. In this paper, we solve the Diophantine equation $F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} = y^a$ in integers n_1, n_2, n_3, n_4, a for $y = 11$. In doing so, we disprove a recent conjecture made by Diouf and Tiebekabe in [3].

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1 Introduction

Let α and β be the roots of the polynomial $x^2 - x - 1$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. We define the Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$ by

$$F_n := \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

which is known as Binet's formula. Equivalently, each term F_n in the sequence can be defined by the recurrence

$$F_n = F_{n-1} + F_{n-2},$$

with initial terms $F_0 = 0$ and $F_1 = 1$.

Recently, the Diophantine equations

$$F_n \pm F_m = y^a, \tag{1}$$

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with $n \geq m \geq 0, y, a \geq 2$ have been studied. The equation $F_n + F_m = y^a$ has been solved in the case $y = 2$ in [1], and for general y under the condition $n \equiv m \pmod 2$ in [7]. The equation $F_n - F_m = y^a$ has been solved when $y = 2, 3$ and 5 in [9], [2], and [5] respectively. Most recently, it was found in [6] that for any fixed y, the equations (1) have finitely many solutions, and all solutions were found for $y \in [2, 1000]$.

A recent paper [3] has investigated the equation

$$F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} = y^a, \tag{2}$$

with $n_1 \geq n_2 \geq n_3 \geq n_4$ and $a \geq 1$. All solutions were found in the case when $y = 2$, and it was conjectured that if y is any prime greater than 7, no solutions exist with $a \geq 2$.

Our main result is as follows.

Theorem 1. *The solutions to equation (2) with $y = 11$, $n_1 \geq n_2 \geq n_3 \geq n_4$ and $a \geq 1$ are*

$$\begin{aligned}(n_1, n_2, n_3, n_4, a) = \{&(6,1,1,1,1), (6,2,1,1,1), (6,2,2,1,1),\\ &(6,2,2,2,1), (5,4,3,1,1), (5,4,3,2,1),\\ &(5,3,3,3,1), (4,4,4,3,1), (11,8,6,4,2),\\ &(10,10,6,4,2), (26,23,21,10,5), (25,25,21,10,5)\}.\end{aligned}$$

The latter four of the listed solutions can be seen to be counterexamples to the aforementioned conjecture. A brief check reveals that counterexamples may be found for many different primes. The largest found by the present authors was $p = 4999$, $a = 2$, for which there are three solutions, one being $F_{12} + F_{30} + F_{35} + F_{36} = (4999)^2$.

2 Preliminaries

We will use Baker's method for bounding linear forms in logarithms of algebraic numbers to obtain our results. The main tool will be the following theorem of Matveev.

Theorem 2 (Matveev [8])**.** *Let $n \in \mathbb{Z}^+$. Let $\mathbb{L}$ be a number field of degree D and let $\eta_1, \ldots, \eta_l$ be non-zero elements of $\mathbb{L}$. Let $b_1, b_2, \ldots, b_l$ be integers and define*

$$B := \max\{|b_1|, \ldots, |b_l|\},$$

and

$$\Lambda := \eta_1^{b_1} \cdots \eta_l^{b_l} - 1 = \left(\prod_{i=1}^{l} \eta_i^{b_i}\right) - 1.$$

Let $A_1, \ldots, A_l$ be real numbers such that

$$A_j \geq \max\{Dh(\eta_j), |\log(\eta_j)|, 0.16\}, 1 \leq j \leq l.$$

Assume that $\Lambda \neq 0$ and $\mathbb{L}$ is real. Then we have

$$\log|\Lambda| > -1.4 \times 30^{l+3} \times l^{4.5} \times d^2 \times A_1 \cdots A_l(1 + \log D)(1 + \log B).$$

We will use the following version of Baker-Davenport reduction in order to reduce our bounds.

Lemma 1 (Dujella-Pethő [4])**.** *Let M be a positive integer, let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let A, B, μ be real numbers such that $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\kappa q\|$.*
If $\epsilon > 0$, then there is no solution of the inequality

$$0 < m\kappa - n + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m \leq M.$$

3 Main Results

In this section we are concerned with the resolution of the following equation

$$F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} = 11^a \tag{3}$$

in positive integers n_1, n_2, n_3, n_4 and a with

$$n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4 \quad \textit{and} \quad n_1 \geqslant 1.$$

Assume that (n_1, n_2, n_3, n_4, a) is a solution to (3). Using the well-known inequality $F_n \leq \alpha^{n-1}$,

$$\begin{aligned}
11^a = F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} &\leqslant \alpha^{n_1-1} + \alpha^{n_2-1} + \alpha^{n_3-1} + \alpha^{n_4-1} \\
&< \alpha^{n_1-1}\left(1 + \alpha^{n_2-n_1} + \alpha^{n_3-n_1} + \alpha^{n_4-n_1}\right) \\
&< 11^{n_1-1}(1+1+1+1) \\
&\leqslant 11^{n_1-1} \times 4 \\
&= 11^{n_1-1} \times 11 \\
&< 11^{n_1}.
\end{aligned}$$

Hence, $11^a < 11^{n_1}$, and it follows that $a < n_1$.

3.1 A Bound on $n_1 - n_2$

Using Binet's formula, we may rewrite equation (3) as:

$$\frac{\alpha^{n_1}}{\sqrt{5}} - 11^a = \frac{\beta^{n_1}}{\sqrt{5}} - (F_{n_2} + F_{n_3} + F_{n_4})$$

Taking absolute values, we obtain
$\left|\frac{\alpha^{n_1}}{\sqrt{5}} - 11^a\right| \leqslant \left|\frac{\beta^{n_1}}{\sqrt{5}}\right| + (F_{n_2} + F_{n_3} + F_{n_4}) < \frac{|\beta|^{n_1}}{\sqrt{5}} + (\alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4})$, and

$$\left|\frac{\alpha^{n_1}}{\sqrt{5}} - 11^a\right| < \frac{1}{2\sqrt{5}} + (\alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4}),$$

where we have used the fact that $1 \leq a < n_1$. After dividing by $\alpha^{n_1}/\sqrt{5}$, we get

$$\begin{aligned}
\left|1 - 11^a \cdot \alpha^{-n_1} \cdot \sqrt{5}\right| &< \frac{\sqrt{5}}{\alpha^{n_1}}\left(\frac{1}{2\sqrt{5}} + \alpha^{n_2-1} + \alpha^{n_3-1} + \alpha^{n_4-1}\right) \\
&< \frac{1}{2\alpha^{n_1}} + \frac{\sqrt{5}}{\alpha^{n_1-n_2}} + \frac{\sqrt{5}}{\alpha^{n_1-n_3}} + \frac{\sqrt{5}}{\alpha^{n_1-n_4}} \\
&\leq \frac{1}{2\alpha^{n_1}} + \sqrt{5}\left(\alpha^{n_2-n_1} + \alpha^{n_3-n_1} + \alpha^{n_4-n_1}\right) \\
&\leq \frac{1}{2}\alpha^{n_2-n_1} + 3\sqrt{5}\alpha^{n_2-n_1} \\
&< \frac{10}{\alpha^{n_1-n_2}},
\end{aligned}$$

and so,

$$\left|1 - 11^a \cdot \alpha^{-n_1} \cdot \sqrt{5}\right| < \frac{10}{\alpha^{n_1-n_2}}. \tag{4}$$

Now let us put

$$\Lambda_1 = 11^a \cdot \alpha^{-n_1} \cdot \sqrt{5} - 1.$$

If $\Lambda_1 = 0$, then $11^a = \dfrac{\alpha^{n_1}}{\sqrt{5}}$. Taking the conjugate in $\mathbb{Q}(\sqrt{5})$ gives $-11^a = \dfrac{\beta^{n_1}}{\sqrt{5}}$. But $|\beta|^{n_1} < |\alpha|^{n_1}$ for $n_1 > 1$, and so $\Lambda_1 \neq 0$. We now apply Matveev's theorem with $\gamma_1 := 11$, $\gamma_2 := \alpha$, $\gamma_3 := \sqrt{5}$, $b_1 := a$, $b_2 := -n$, and $b_3 := 1$. Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K} := \mathbb{Q}(\sqrt{5})$, we can take $D := 2$. We compute the logarithmic height of each γ_i and choose an appropriate corresponding A_i as follows:
$h(\gamma_1) = \log 11 = 2.3978\ldots$, so we can choose $A_1 := 5$.
$h(\gamma_2) = \frac{1}{2}\log\alpha = 0.2406\ldots$, so we can choose $A_2 := 0.5$.
$h(\gamma_3) = \log\sqrt{5} = 0.8047\ldots$, it follows that we can choose $A_3 := 1.7$.
Since $a < n_1, B := \max\{|b_1|, |b_2|, |b_3|\} = n_1$.
Matveev's result informs us that

$$\left|1 - 11^a \cdot \alpha^{n_1} \cdot \sqrt{5}\right| > \exp\left(-c_1 \cdot (1 + \log n_1) \cdot 2.5 \cdot 0.5 \cdot 1.7\right),$$

where $c_1 := 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2) < 9.7 \times 10^{11}$. Taking logarithms in the last inequality, we get

$$\log|\Lambda_1| > -2.42 \times 10^{12} \log n_1.$$

Taking log in inequality (4), we get

$$\log|\Lambda_1| < \log 10 - (n_1 - n_2)\log\alpha.$$

Comparing the previous two inequalities, we get $(n_1 - n_2)\log\alpha < 2.5 \times 10^{12}\log n_1 + \log 10$, and so it follows that

$$(n_1 - n_2)\log\alpha < 2.5 \times 10^{12}\log n_1. \tag{5}$$

3.2 A Bound on $n_1 - n_3$

Now we write our equation differently to get another linear form in logarithms

$$\frac{\alpha^{n_1}}{\sqrt{5}} + \frac{\alpha^{n_2}}{\sqrt{5}} - 11^a = \frac{\beta^{n_1}}{\sqrt{5}} + \frac{\beta^{n_2}}{\sqrt{5}} - (F_{n_3} + F_{n_4}).$$

Taking absolute values on the above equation, we get

$$\begin{aligned}\left|\frac{\alpha^{n_1}}{\sqrt{5}}\left(1+\alpha^{n_2-n_1}\right) - 11^a\right| &\leqslant \frac{|\beta|^{n_1} + |\beta|^{n_2}}{\sqrt{5}} + |\, F_{n_3} + F_{n_4}| \\ &\leq \frac{1}{\sqrt{5}} + \alpha^{n_3-1} + \alpha^{n_4-1} \\ &< \frac{1}{\sqrt{5}} + \alpha^{n_3} + \alpha^{n_4}\end{aligned}$$

Dividing both sides of the above inequality by $\frac{\alpha^{n_1}}{\sqrt{5}}\left(1+\alpha^{n_2-n_1}\right)$, we obtain that

$$\begin{aligned}\left|1 - 11^a \cdot \alpha^{n_1} \cdot \sqrt{5}\left(1+\alpha^{n_2-n_1}\right)^{-1}\right| &< \left(\frac{1}{\sqrt{5}} + \alpha^{n_3} + \alpha^{n_4}\right) \times \frac{\sqrt{5}}{\alpha^{n_1} + \alpha^{n2}} \\ &< \frac{1}{(\alpha^{n_1} + \alpha^{n_2})} + \sqrt{5}\left(\frac{\alpha^{n_3} + \alpha^{n_4}}{\alpha^{n_1} + \alpha^{n_2}}\right) \\ &< \frac{1}{(\alpha^{n_1} + \alpha^{n_2})} + \sqrt{5}\left(\frac{\alpha^{n_3}}{\alpha^{n_1}} + \frac{\alpha^{n_4}}{\alpha^{n_1}}\right) \\ &< \frac{1}{(\alpha^{n_1} + \alpha^{n_2})} + \frac{2\sqrt{5}}{\alpha^{n_1-n_3}} \\ &< \frac{1}{\alpha^{n_1-n_3}} + \frac{2\sqrt{5}}{\alpha^{n_1-n_3}} \\ &< \frac{6}{\alpha^{n_1-n_3}}.\end{aligned}$$

So,

$$\left|1 - 11^a \cdot \alpha^{n_1} \cdot \sqrt{5}\left(1+\alpha^{n_2-n_1}\right)^{-1}\right| < \frac{6}{\alpha^{n_1-n_3}}. \tag{6}$$

Let us consider

$$\Lambda_2 = 11^a \cdot \alpha^{n_1} \cdot \sqrt{5}\left(1+\alpha^{n_2-n_1}\right)^{-1} - 1.$$

Before applying Matveev's theorem with the parameters:

$$\gamma_1 := 11, \quad \gamma_2 := \alpha, \quad \gamma_3 := \sqrt{5}\left(1+\alpha^{n_2-n_1}\right)^{-1},$$
$$b_1 := a, \quad b_2 := -n_1, \quad \text{and} \quad b_3 := 1,$$

we must ensure that $\Lambda_2 \neq 0$, otherwise, we would get the relation

$$11^a\sqrt{5} = \alpha^{n_1} + \alpha^{n_2}.$$

Conjugating in the field $\mathbb{Q}(\sqrt{5})$, we get

$$-11^a\sqrt{5} = \beta^{n_1} + \beta^{n_2}.$$

Combining the two expressions, we get

$$\alpha^{n_1} < \alpha^{n_1} + \alpha^{n_2} = |\beta^{n_1} + \beta^{n_2}| \leqslant |\beta|^{n_1} + |\beta|^{n_2} < 1,$$

which is impossible. Hence $\Lambda_2 \neq 0$.
Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K} := \mathbb{Q}(\sqrt{5})$, we can take $D := 2$. We know that, $h(\gamma_1) = \log 11$ and $h(\gamma_2) = \frac{1}{2}\log\alpha$. We estimate $h(\gamma_3)$ using the fact that

$$\gamma_3 = \frac{\sqrt{5}}{1+\alpha^{n_2-n_1}} < \sqrt{5} \text{ and } \gamma_3^{-1} = \frac{1+\alpha^{n_2-n_1}}{\sqrt{5}} < \frac{2}{\sqrt{5}},$$

so $|\log\gamma_3| < 1$.
Using proprieties of logarithmic height , we have

$$\begin{aligned} h(\gamma_3) &= h\left(\frac{\sqrt{5}}{1+\alpha^{n_2-n_1}}\right) \\ &\leq h\left(\sqrt{5}\right) + h\left(1+\alpha^{n_2-n_1}\right) \\ &\leqslant \log\sqrt{5} + |n_2-n_1|\left(\frac{\log\alpha}{2}\right) + \log 2 \\ &= \log(2\sqrt{5}) + (n_1-n_2)\left(\frac{\log\alpha}{2}\right). \end{aligned}$$

Hence, we can take

$$A_3 := 2 + (n_1-n_2)\log\alpha > \max\{2h(\gamma_3), |\log\gamma_3|, 0.16\}.$$

Matveev's theorem implies that

$$|\Lambda_2| > \exp\left(-c_1(1+\log n_1)\cdot 2.5\cdot 0.5\cdot(2+(n_1-n_2)\log\alpha)\right)$$

where $c_1 := 1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2) < 9.7\times 10^{11}$.
Since $(1+\log n_1) < 2\log n_1$ hold for $n_1 \geqslant 3$, from the first inequality, we have

$$(n_1-n_3)\log\alpha < 1.4\times 10^{12}\log n_1\left(3+(n_1-n_2)\log\alpha\right).$$

combining this with (5), we get

$$(n_1-n_3)\log\alpha < 3.6\times 10^{24}\log^2 n_1. \tag{7}$$

3.3 A Bound on $n_1 - n_4$

Let us consider a third linear form in logarithms. Rewriting equation (3) yields

$$\frac{\alpha^{n_1}+\alpha^{n_2}+\alpha^{n_3}}{\sqrt{5}} - 11^a = \frac{\beta^{n_1}+\beta^{n_2}+\beta^{n_3}}{\sqrt{5}} - F_{n_4}.$$

In a similar manner to our earlier two bounds, we obtain

$$\begin{aligned}\left|\frac{\alpha^{n_1}}{\sqrt{5}}\left(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}\right) - 11^a\right| &\leqslant \frac{|\beta|^{n_1}+|\beta|^{n_2}+|\beta|^{n_3}}{\sqrt{5}} + F_{n_4} \\ &< \frac{\alpha}{\sqrt{5}} + \alpha^{n_4}.\end{aligned}$$

Thus we have

$$\begin{aligned}\left|1 - \frac{11^a \cdot \alpha^{-n_1} \cdot \sqrt{5}}{(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1})}\right| &< \left(\frac{\alpha}{\sqrt{5}} + \alpha^{n_4}\right)\left(\frac{\sqrt{5}}{\alpha^{n_1}+\alpha^{n_2}+\alpha^{n_3}}\right) \\ &< \frac{4}{\alpha^{n_1-n_4}}.\end{aligned}$$

We apply Matveev's theorem a third time with

$$\Lambda_3 = 11^a \cdot \alpha^{-n_1} \cdot \sqrt{5}\left(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}\right)^{-1} - 1,$$

where we can take the parameters

$$\gamma_1 := 11, \quad \gamma_2 := \alpha, \quad \gamma_3 := \sqrt{5}\left(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}\right)^{-1},$$
$$b_1 := a, \quad b_2 := -n_1, \quad \text{and }, \quad b_3 := 1.$$

Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K} := \mathbb{Q}(\sqrt{5})$, we can also in this case take $D := 2$.
Suppose, for a contradiction, that $|\Lambda_3| = 0$. Then

$$11^a\sqrt{5} = \alpha^{n_1}+\alpha^{n_2}+\alpha^{n_3}.$$

Taking the conjugate in the field $\mathbb{Q}(\sqrt{5})$, we get

$$-11^a\sqrt{5} = \beta^{n_1}+\beta^{n_2}+\beta^{n_3},$$

which leads to

$$\alpha^{n_1} < \alpha^{n_1}+\alpha^{n_2}+\alpha^{n_3} = |\beta^{n_1}+\beta^{n_2}+\beta^{n_3}| \leqslant |\beta|^{n_1}+|\beta|^{n_2}+|\beta|^{n_3} < 1$$

and leads to a contradiction since $n_1 > 1$. Hence $\Lambda_3 \neq 0$.

As we did before, we can take $A_1 := 5, A_2 := 0.5$ and $B := n_1$. We can also see that $\gamma_3 = \frac{\sqrt{5}}{1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}} < \sqrt{5}$ and $\gamma_3^{-1} = \frac{1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}}{\sqrt{5}} \leq \frac{3}{\sqrt{5}}$, so $|\log\gamma_3| < 1$. We apply the proprieties of the logarithmic height to estimate $h(\gamma_3)$

$$\begin{aligned}h(\gamma_3) &\leqslant \log\sqrt{5} + |n_2 - n_1|\left(\frac{\log\alpha}{2}\right) + |n_3 - n_1|\left(\frac{\log\alpha}{2}\right) + \log 3 \\ &= \log(3\sqrt{5}) + (n_1 - n_2)\left(\frac{\log\alpha}{2}\right) + (n_1 - n_3)\left(\frac{\log\alpha}{2}\right);\end{aligned}$$

so we can take

$$\begin{aligned} A_3 &= 3 + (n_1 - n_2) \log \alpha + (n_1 - n_3) \log \alpha \\ &> \max \{2h(\gamma_3), |\log \gamma_3|, 0.16\}, \end{aligned}$$

which yields the bound

$$|\Lambda_3| > \exp(-c_1 (1 + \log n_1)(2.5)(0.5)(3 + (2n_1 - n_2 - n_3) \log \alpha))$$

where $c_1 = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2) < 9.7 \times 10^{11}$.
Using the bounds (5) and (7), this leads to the upper bound

$$(n_1 - n_4) \log \alpha < 1.69 \times 10^{37} \times \log^3 n_1. \tag{8}$$

3.4 An absolute bound on n_1

Let us now consider a final linear form in logarithms. Rerwriting equation (3) once again, we get

$$\frac{\alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4}}{\sqrt{5}} - 11^a = \frac{\beta^{n_1} + \beta^{n_2} + \beta^{n_3} + \beta^{n_4}}{\sqrt{5}}.$$

Taking absolute values on both sides, we get

$$\begin{aligned} &\left| \frac{\alpha^{n_1}}{\sqrt{5}} \left(1 + \alpha^{n_2 - n_1} + \alpha^{n_3 - n_1} + \alpha^{n_4 - n_1}\right) - 11^a \right| \\ &\leqslant \frac{|\beta|^{n_1} + |\beta|^{n_2} + |\beta|^{n_3} + |\beta|^{n_4}}{\sqrt{5}} \\ &< \frac{2}{\sqrt{5}}. \end{aligned}$$

Dividing both sides of the above relation by

$$\frac{\alpha^{n_1}}{\sqrt{5}} \left(1 + \alpha^{n_2 - n_1} + \alpha^{n_3 - n_1} + \alpha^{n_4 - n_1}\right),$$

we get

$$\left| 1 - 11^a \cdot \alpha^{-n_1} \cdot \sqrt{5} \left(1 + \alpha^{n_2 - n_1} + \alpha^{n_3 - n_1} + \alpha^{n_4 - n_1}\right)^{-1} \right| < \frac{2}{\alpha^{n_1}}$$

let us take

$$\Lambda_4 = 11^a \cdot \alpha^{-n_1} \cdot \sqrt{5} \left(1 + \alpha^{n_2 - n_1} + \alpha^{n_3 - n_1} + \alpha^{n_4 - n_1}\right)^{-1} - 1$$

In the last application of Matveev's theorem, we have the following parameters:

$$\gamma_1 := 11, \quad \gamma_2 := \alpha, \quad \gamma_3 := \sqrt{5} \left(1 + \alpha^{n_2 - n_1} + \alpha^{n_3 - n_1} + \alpha^{n_4 - n_1}\right)^{-1},$$

and we can also take $b_1 := a, \quad b_2 := -n$ and $b_3 := 1$.
Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K} := \mathbb{Q}(\sqrt{5})$, we can take $D := 2$. To ensure that $\Lambda_4 \neq 0$ suppose, for a contradiction, that $\Lambda_4 = 0$. It follows that

$$11^a \sqrt{5} = \alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4}.$$

by taking the conjugate of the above relation in the field $\mathbb{Q}(\sqrt{5})$, we get

$$-11^a\sqrt{5} = \beta^{n_1} + \beta^{n_2} + \beta^{n_3} + \beta^{n_4}.$$

Combining the above two equations, we get

$$\begin{aligned}\alpha^{n_1} < \alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4} &= |\beta^{n_1} + \beta^{n_2} + \beta^{n_3} + \beta^{n_4}| \\ &\leqslant |\beta|^{n_1} + |\beta|^{n_2} + |\beta|^{n_3} + |\beta|^{n_4} < 1\end{aligned}$$

which leads to a contradiction since $n_1 > 0$. Moving to the logarithmic heights, we can take $A_1 := 5, A_2 := 0.5$ and $B := n_1$. For $h(\gamma_3)$, we can see that,

$$\gamma_3 = \frac{\sqrt{5}}{1 + \alpha^{n_2-n_1} + \alpha^{n_3-n_1} + \alpha^{n_4-n_1}} < \sqrt{5}$$

and $\gamma_3^{-1} = \frac{1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}+\alpha^{n_4-n_1}}{\sqrt{5}} < \frac{4}{\sqrt{5}}$. Hence $|\log\gamma_3| < 1$. Then by using the properties of the logarithmic height we get

$$\begin{aligned}h(\gamma_3) &\leqslant \log(4\sqrt{5}) + (|n_2 - n_1| + |n_3 - n_1| + |n_4 - n_1|)\left(\frac{\log\alpha}{2}\right) \\ &= \log(4\sqrt{5}) + ((n_1 - n_2) + (n_1 - n_3) + (n_1 - n_4))\left(\frac{\log\alpha}{2}\right),\end{aligned}$$

and so we can take

$$A_3 := 4 + (n_1 - n_2)\log\alpha + (n_1 - n_3)\log\alpha + (n_1 - n_4)\log\alpha.$$

Matveev's theorem then yields the bound

$$|\Lambda_4| > \exp\left(-c_1 \cdot (1 + \log n_1)(1.25)(4 + (3n_1 - n_2 - n_3 - n_4)\log\alpha)\right),$$

where $c_1 = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2) < 9.7 \times 10^{11}$.
But since

$$|\Lambda_4| < \frac{2}{\alpha^{n_1}},$$

so by taking log and using the both inequalities we have

$$n_1\log\alpha < 1.25 \times 10^{12}\log n_1 \cdot (4 + (3n_1 - n_2 - n_3 - n_4)\log\alpha)\,. \tag{9}$$

This, in combination with (5), (7) and (8) allows us to get an upper bound M on n_1. We will take

$$M = 1.5 \times 10^{58}.$$

This is still too large to allow us to compute all solutions to (3), so in the next section we will focus on reducing it.

3.5 Reduction of the Bounds

Let
$$z_1 := a \log 11 - n_1 \log \alpha + \log \sqrt{5}.$$
Recall that from (4), we have
$$|1 - e^{z_1}| < \frac{10}{\alpha^{n_1 - n_2}}.$$
By Binet's formula, we have
$$\frac{\alpha^{n_1}}{\sqrt{5}} = F_{n_1} + \frac{\beta^{n_1}}{\sqrt{5}} < F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} = 11^a,$$
hence
$$\frac{\alpha^{n_1}}{\sqrt{5}} < 11^a$$
which yields $z_1 > 0$, and so
$$0 < z_1 < e^{z_1} - 1 < \frac{10}{\alpha^{n_1 - n_2}}.$$
Dividing both sides of the resulting inequality by $\log \alpha$, we get
$$0 < a \left(\frac{\log 11}{\log \alpha} \right) - n_1 + \left(\frac{\log \sqrt{5}}{\log \alpha} \right) < \frac{10}{\log \alpha} \cdot \alpha^{n_1 - n_2} < 21 \cdot \alpha^{n_1 - n_2}. \tag{10}$$
Now, we define
$$\tau := \frac{\log 11}{\log \alpha}, \quad \mu := \frac{\log \sqrt{5}}{\log \alpha}, \quad A := 21, \quad \text{and} \quad B := \alpha.$$
We use the Baker-Davenport reduction method with these parameters. Observe that the inequalities $A > 0, B > 1$ are satisfied. We find that the 123rd convergent of the continued fraction expansion of τ has denominator q satisfying the conditions $q > 6M$ and $\varepsilon = \|\mu q\| - \|\tau q\| N > 0$. As $a \leqslant n_1 < M$ and (10) holds for a, n_1 and $n_1 - n_2$ integers, we must have $n_1 - n_2 < \frac{\log(Aq/\varepsilon)}{\log B}$, from which we may deduce that $n_1 - n_2 \leq 291$.

A similar method allows us to reduce the bound on $n_1 - n_3$. Set
$$z_2 := a \log 11 - n_1 \log \alpha + \log \sqrt{5} \left(1 + \alpha^{-(n_1 - n_2)}\right)^{-1}$$
From (6), we have
$$\left|1 - 11^a \cdot \alpha^{n_1} \cdot \sqrt{5} \left(1 + \alpha^{n_2 - n_1}\right)^{-1}\right| < \frac{6}{\alpha^{n_1 - n_3}},$$
which yields
$$|1 - e^{z_2}| < \frac{6}{\alpha^{n_1 - n_3}}.$$
As
$$\frac{\alpha^{n_1}}{\sqrt{5}} + \frac{\alpha^{n_2}}{\sqrt{5}} = F_{n_1} + F_{n_2} + \frac{\beta^{n_1}}{\sqrt{5}} + \frac{\beta^{n_2}}{\sqrt{5}} < F_{n_1} + F_{n_2} + 1 \leqslant F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4}$$

$$= 11^a.$$

Therefore, $1 < 11^a\sqrt{5}\alpha^{-n_1}\left(1+\alpha^{n_2-n_1}\right)^{-1}$ and so $z_2 > 0$. This gives

$$0 < z_2 \leqslant e^{z_2} - 1 < \frac{6}{\alpha^{n_1-n_3}},$$

and so we obtain

$$0 < a\left(\frac{\log 11}{\log\alpha}\right) - n_1 + \frac{\log\sqrt{5}\left(1+\alpha^{-(n_1-n_2)}\right)^{-1}}{\log\alpha} < 13\cdot\alpha^{-(n_1-n_3)}. \tag{11}$$

We apply the Baker-Davenport reduction with the parameters:

$$\tau := \frac{\log 11}{\log\alpha}, \quad \mu := \frac{\log\sqrt{5}\left(1+\alpha^{-(n_1-n_2)}\right)^{-1}}{\log\alpha},$$
$$A := 20, \quad B := \alpha \quad \textit{and} \quad N = M.$$

We attempt to find $\frac{p}{q}$ a convergent of τ such that $\varepsilon = \|\mu q\| - \|\tau q\|N > 0$, but since μ depends on (n_1-n_2), we must exclude the values of (n_1-n_2) which lead to the linear dependence of τ and μ. Since we have established a reasonable bound on n_1-n_2, these exceptional cases may be found through a direct check for all values of $n_1-n_2\in[0,291]$, which reveals that only the case $n_1-n_2=2$ must be dealt with separately. We use Maple™ to apply the Baker-Davenport reduction method to (11), for all $n_1-n_2\in[0,291]$ not equal to to 2, finding that in all cases, either the 122nd or 123rd convergent suffices, and we obtain the bound $n_1-n_3\leq 303$. When $n_1-n_2=2$, equation (11) becomes

$$0 < a\frac{\log 11}{\log\alpha} - n_1 + 1 < 13\cdot\alpha^{-(n_1-n_3)}.$$

Since a is less than the denominator of the 122nd convergent of $\frac{\log 11}{\log\alpha}$, we check the first 123 partial quotients, finding that the largest of them is 57, and so

$$\left|a\frac{\log 11}{\log\alpha} - (n_1-1)\right| > \frac{1}{59a}.$$

This gives $\alpha^{n_1-n_3} < 13\cdot 59a$, and so

$$n_1 - n_3 < \frac{\log 767a}{\log\alpha},$$

which yields $n_1-n_3\leq 292$ in the case $n_1-n_2=2$. So in either case, $n_1-n_3\leq 303$.

We repeat this process again in order to bound n_1-n_4. In this case, we use the inequality

$$0 < \left|a\frac{\log 11}{\log\alpha} - n_1 + \frac{\log\sqrt{5}(1+\alpha^{-(n_1-n_2)}+\alpha^{-(n_1-n_3)})^{-1}}{\log\alpha}\right| < \frac{18}{\alpha^{n_1-n_4}}, \tag{12}$$

and apply Lemma 1 for all choices of $n_1 - n_2 \leq 291$ and $n_1 - n_3 \leq 303$ apart from the pairs $(n_1-n_2, n_1-n_3) \in \{(1,1),(0,3),(3,4)\}$, which we deal with separately using the same continued fraction method as above. In all cases, we find that $n_1 - n_4 \leq 318$.

We may use the bounds we have obtained for $n_1 - n_2, n_1 - n_3$ and $n_1 - n_4$ to reduce the bound on n_1 immediately using (9). We obtain $n_1 < 4.42 \times 10^{16}$. We apply Lemma 1 once more, using the inequality

$$0 < \left| a\frac{\log 11}{\log \alpha} - n_1 + \frac{\log \sqrt{5}(1+\alpha^{-(n_1-n_2)}+\alpha^{-(n_1-n_3)}+\alpha^{-(n_1-n_4)})^{-1}}{\log \alpha} \right| < \frac{9}{\alpha^{n_1}},$$

along with the bound $a < 4.42 \times 10^{16}$. A check reveals that the cases that must be dealt with separately this time are $(n_1-n_2, n_1-n_3, n_1-n_4) \in \{(0,0,1),(0,4,5),(1,2,3),(3,5,6),(4,4,5)\}$. In all cases, n_1 was found to be no greater than 210, and a search for n_1 up to this bound revealed only the solutions listed in the statement of Theorem 1.

Throughout this paper, calculations were carried out in Maple™, with results rounded to an accuracy of 200 decimal places.

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