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Classification of 3-GNDB graphs

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Abstract. A nonempty graph Γ is called generalized 3-distance-balanced, (3-GDB) whenever for every edge ab, $|W_{ab}| = 3|W_{ba}|$ or conversely. As well as a graph Γ is called generalized 3-nicely distance-balanced (3-$GNDB$) whenever for every edge ab of Γ, there exists a positive integer γ_Γ, such that: $|W_{ba}| = \gamma_\Gamma$.In this paper, we classify 3-$GNDB$ graphs with, $\gamma_\Gamma \in \{1, 2\}$.

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1 Introduction

Throughout this paper, let Γ be a finite, undirected and connected graph with diameter d, and $V(\Gamma)$ and $E(\Gamma)$ denote the vertex and edge set of Γ, respectively. The distance $d(a, b)$ between vertices $a, b \in V(\Gamma)$ is the length of a shortest path between $a, b \in V(\Gamma)$. For an edge ab of a graph Γ, let W_{ab} be the set of vertices closer to a than to b , that is $W_{ab} = \{x \in \Gamma | d(x, a) < d(x, b)\}$. We call a graph Γ, distance-balanced (DB), if $|W_{ab}| = |W_{ba}|$ for every edge $ab \in E(\Gamma)$. These graphs were studied by Handa [5] who considered DB. For recent results on DB and EDB see [1, 3, 4, 6–8, 10, 11]. A graph Γ is called nicely distance-balanced, whenever there exists a positive integer γ_Γ, such that for two adjacent vertices a, b of Γ; $|W_{ab}| = |W_{ba}| = \gamma_\Gamma$. These graphs were studied by Kutnur and Miklavič in [9].
A graph Γ is called generalized 3-distance-balanced (3-GDB) if for every edge $ab \in E(\Gamma)$; $|W_{ab}| = 3|W_{ba}|$ or conversely. Throughout of this paper, we assume that $|W_{ab}| = 3|W_{ba}|$. A graph Γ is called generalized 3-nicely distance-balanced (3-$GNDB$), if for every edge ab of Γ, there exists a positive integer γ_Γ, such that: $|W_{ba}| = \gamma_\Gamma$. For example we can show that $K_{1,3}$ and $K_{2,6}$ are $3-GNDB$. The aim of this paper is classifying 3-$GNDB$ graphs with $\gamma_\Gamma \in \{1, 2\}$.

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2 Classification

In order to express the problem, it is better to start with parameter 3. In this section, we classify 3-$GNDB$ graphs with $\gamma_\Gamma \in \{1, 2\}$.
For every two non-negative integers i, j, we denote:

$$D^i_j(a, b) = \{x \in V(\Gamma) | d(x, a) = i \quad \text{and} \quad d(x, b) = j\}. \tag{1}$$

We now suppose that Γ is a 3-$GNDB$ graph with diameter d. Since $|W_{ab}| = 3|W_{ba}|$ for every two adjacent vertices a, b and by (1), we have

$$|\{a\} \bigcup_{i=1}^{d-1} D^i_{i+1}(a, b)| = 3|\{b\} \bigcup_{i=1}^{d-1} D^{i+1}_i(a, b)|.$$

Therefore,

$$\sum_{i=1}^{d-1} |D^i_{i+1}(a, b)| = 3 \sum_{i=1}^{d-1} |D^{i+1}_i(a, b)| + 2. \tag{2}$$

Theorem 1. *If Γ be a connected $k - GNDB$ graph, then Γ is a bipartite graph.*

Proof. Inspired by the proof of Theorem 1.1 in [2], let Γ be a $k - GNDB$ graph with diameter d, and the vertex set $\{v_1, v_2, \ldots, v_{2l+1}\}$ form an odd cycle with length $2l+1$ such that $v_i v_{i+1} \in E(\Gamma)$. Set

$$A_{ij} = \{v \in V(\Gamma) | d(v, v_{i+2l}) = m_j k,$$
$$m_j k = \{1, 2, \ldots, d\}, k = 0, 1, \ldots, 2l, 2 \leqslant j \leqslant r\},$$

and

$$W^\Gamma_{v_i, v_{i+l}} = (\bigcup_{j=1}^r A_{ij}) \bigcup \{v_i, v_{i+2l}\},$$
$$W^\Gamma_{v_{i+1}, vi} = (\bigcup_{j=1}^r A_{(i+1)j}) \bigcup \{v_{i+1}, v_{i+2}\},$$

where the calculation in indexes i are performed modulo $2l + 1$ and some $r \in N$. Taking $|A_{ij}| = a_{ij}$ for $i = 0, 1, \ldots, 2l$ and $j = 1, 2, \ldots, r$, by definition $k - GNDB$ graphs, there exists $e_i \in \{\pm 1\}, i = 0, 1, \ldots, 2l$ such that

$$\sum_{j=1}^r a_{0j} + 2 = k^{e_0} (\sum_{j=1}^r a_{1j} + 2),$$
$$\sum_{j=1}^r a_{1j} + 2 = k^{e_1} (\sum_{j=1}^r a_{2j} + 2),$$
$$\vdots$$
$$\sum_{j=1}^r a_{(2l-1)j} + 2 = k^{e_{2l-1}} (\sum_{j=1}^r a_{2l} + 2),$$

$$\sum_{j=1}^{r} a_{(2l)j} + 2 = k^{e_{2l}} (\sum_{j=1}^{r} a_{0j} + 2).$$

Now, multiplying all $(2l+1)$ equations above implies that $k^{\sum_{i=0}^{2i} e_i} = 1$, that is, $\sum_{i=0}^{2i} ei = 0$. On the other hand, $e_i \in \{\pm 1\} \implies 1 \leqslant |\sum_{i}^{2i} = 0 e_i|$, which is a contradiction and henes Γ has no odd cycle. This completes the proof. □

Theorem 2. *If Γ be a 3-GNDB grpah with $d = 2$, then $deg(a) = 3\, deg(b)$ for every edge ab of Γ.*

Proof. It follows from (1) that for a $3-GNDB$ graph with diameter 2, $|D_2^1(a,b)| = 3|D_1^2(a,b)| + 2$, for every edge ab of Γ. If $|D_1^2(a,b)| = t$, then $|D_2^1(a,b)| = 3t+2$. Therefore, $deg(b) = t+1$ and $deg(a) = 3t+3$. Thus, $deg(a)$ is always $3\, deg(b)$. □

Lemma 1. *Let Γ be a $3-GNDB$ graph with diameter 2. Then Γ is only $K_{n,3n}$.*

Proof. Let Γ be a $3-GNDB$ graph with diameter 2. We claim that Γ is a complete bipartite graph. Otherwise, it does not have diameter 2. It follows from Theorem 2 that $deg(a) = 3\, deg(b)$. Since Γ is complete bipartite graph, Γ must be $K_{n,3n}$. □

Lemma 2. *Let Γ be a connected $k-GNDB$ graph with diameter d. Then $d \leqslant k\gamma_\Gamma$*

Proof. Pick vertices x_0 and x_d of Γ such that $d(x_0, x_d) = d$ and a shortest path $x_0, x_1, x_2, \ldots, x_d$ between x_0 and x_d. We may assume without loss of generality that $|W_{x_0,x_1}| = k|W_{x_0,x_1}|$. Then $\{x_1, x_2, \ldots, x_d\} \in W_{x_1,x_0}$. Hence, $|\{x_1, x_2, \ldots, x_d\}| \leqslant |W_{x_1,x_0}| = k|W_{x_0,x_1}|$. This shows that $d \leqslant k\gamma_\Gamma$.

□

We now classify $3-GNDB$ graphs Γ with, $\gamma_\Gamma \in \{1, 2\}$.
First we consider when $\gamma_\Gamma = 1$. By the Lemma 2, $d \leqslant 3$.
If $d = 1$, then Γ is complete graph.
If $d = 2$, by the Lemma 1, Γ is only $K_{1,3}$.
If $d = 3$, then we have only a path of length 2, that it would not be $3-GNDB$.

Now we consider the case $\gamma_\Gamma = 2$.

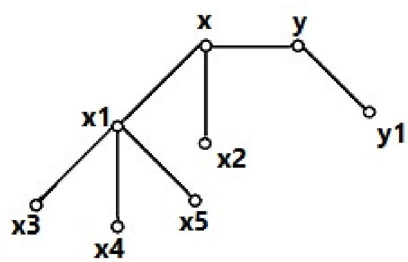
Theorem 3. *A graph Γ is $3-GNDB$ with $\gamma_\Gamma = 2$ if and only if it is $K_{2,6}$.*

Proof. For adjacent vertices a, b of Γ, we say that the edge ab is consistent if $|W_{ab}| = 3|W_{ba}|$. Let d be the diameter of Γ. By the Lemma 2, $d \leqslant 6$. If $d = 1$, then Γ is a complete graph. Therefore, $d \in \{2,3,4,5,6\}$. Pick an edge $xy \in E(\Gamma)$ and for non-negative integers i, j set $D_j^i = D_j^i(x,y)$. Note that, by triangle inequality, $D_j^i = \phi$ whenever $|i-j| > 1$. If $d = 2$, then by Lemma 1, Γ is only $K_{2,6}$.
Note that $|V(\Gamma)| = 8$. Consider that $xy \in E(\Gamma)$ and $d \in \{3,4,5,6\}$. Therefore, $|V(\Gamma)| \setminus \{x,y\} = 6$. Since for every D_j^i, in which $i, j \neq 0$, then there must be at least a neighbour for either vertex x or vertex y. Suppose that $|D_1^2| = 1$ for all cases. We now consider all different cases of $|D_j^i|$, where $i, j \neq 0$ for the 6 remaining vertices in Γ and edge xy. Now we show that,

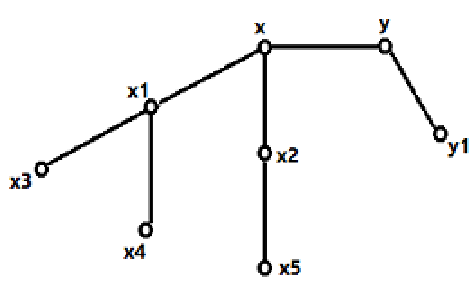
there is no graph for $3 \leqslant d \leqslant 6$.
If $d = 3$, then we split our proof into the following subcases.

Subcase 1: $|D_2^1| = 1, |D_3^2| = 4$ and $|D_1^2| = 1$.
We will show that this case cannot occur. Denote the vertex in D_2^1 by x_1, the vertices in D_3^2 by x_2, x_3, x_4 and x_5, and also the vertex in D_1^2 by y_1. The vertices x_2 up to x_5 cannot be adjacent with y_1, because which is created an odd cycle. Therefore, the vertices x_2 up to x_5 must be adjacent to each other. In this case we have an odd cycle.

Subcase 2: $|D_2^1| = 2, |D_3^2| = 3$ and $|D_1^2| = 1$.
Denote the vertices in D_2^1 by x_1 and x_2, the vertices in D_3^2 by x_3, x_4 and x_5, and also the vertex in D_1^2 by y_1. The vertices x_3, x_4 and x_5 cannot be adjacent with y_1. The vertices x_3, x_4 and x_5 can only be adjacent with x_2. Since the diameter of graph is 3, the vertex y_1 must be adjacent with x_1 or x_2 or both. In each case, the edges xx_1 or yy_1 are not consistent.

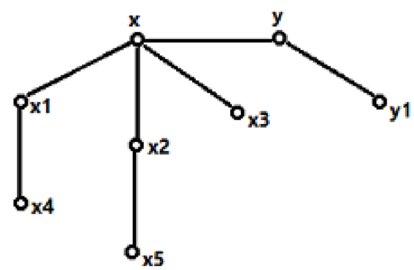


Subcase 3: $|D_2^1| = 2, |D_3^2| = 3$ and $|D_1^2| = 1$.
Denote the vertices in D_2^1 by x_1 and x_2, the vertices in D_3^2 by x_3, x_4 and x_5, and also the vertex in D_1^2 by y_1. The vertices x_3, x_4 and x_5 cannot be adjacent with y_1. The vertices x_3, x_4 and y_1 cannot be adjacent with x_5. The vertices x_3 and x_4 must be adjacent with x_2, and vertex x_5 must be adjacent with x_1. The vertex y_1 can be adjacent with x_1 or x_2 or both. In each case the edge xx_1 is not consistent.

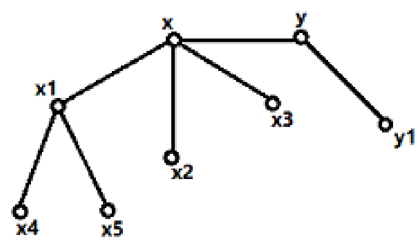


Subcase 4: $|D_2^1| = 3, |D_3^2| = 2, |D_1^2| = 1$.
Denote the vertices in D_2^1 by x_1, x_2 and x_3, the vertices in D_3^2 by x_4 and x_5, and also the vertex

in D_1^2 by y_1. The vertex y_1 cannot be adjacent with x_4 and x_5. The vertex x_3 can be adjacent with x_4, x_5 and y_1, and also The vertex x_4 can be adjacent with x_2 and x_3. Since the diameter of graph is 3, the vertices y_1 and x_4 must be adjacent with x_2 or the vertices y_1 and x_5 must be adjacent with x_1 or the vertex y_1 must be adjacent with x_1 and x_2. In each case the edges xx_2 and x_2x_5 are not consistent.



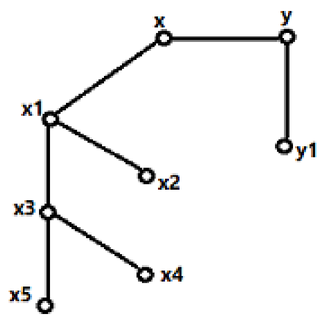
Subcase 5: $|D_2^1| = 3, |D_3^2| = 2, |D_1^2| = 1$.
Denote the vertices in D_2^1 by x_1, x_2 and x_3, the vertices in D_3^2 by x_4 and x_5, and also the vertex in D_1^2 by y_1. The vertex y_1 cannot be adjacent with x_4 and x_5. The vertices x_4 and x_5 can only be adjacent with x_2 and x_3. The vertices x_2 and x_3 can be adjacent with x_4, x_5 and y_1. Since the diameter of graph is 3, the vertices y_1 must be adjacent with x_1. In each case the edges xx_1 and yy_1 are not consistent.



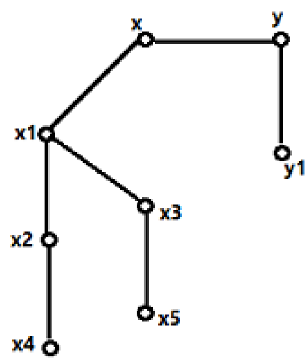
Subcase 6: $|D_2^1| = 4, |D_3^2| = 1, |D_1^2| = 1$.
Denote the vertices in D_2^1 by x_1, x_2, x_3 and x_4, and the vertex in D_3^2 and D_1^2 by x_5 and y_1 respectively. The vertex y_1 cannot be adjacent with x_5. The vertex y_1 can be adjacent with x_1, x_2, x_3 and x_4, and also the vertex x_5 can be adjacent with x_1, x_2 and x_3. In each case the edges x_4x_5 and yy_1 are not consistent.
If $d = 4$ we split our proof into the following subcases.

Subcase 1: $|D_2^1| = 1, |D_3^2| = 2, |D_4^3| = 2, |D_1^2| = 1$.
Denote the vertex in D_2^1 by x_1, the vertices in D_3^2 by x_2 and x_3, the vertices in D_4^3 by x_4 and x_5, and the vertex in D_1^2 by y_1. The vertex y_1 cannot be adjacent with x_2 and x_3. The vertex y_1 can be adjacent with x_1, x_4 and x_5. The vertex x_2 can be adjacent with x_4 and x_5, and also the vertices x_4 and x_5 can only be adjacent with x_2 and y_1. In each case the edge x_1x_3 is not

consistent.



Subcase 2: $|D_2^1| = 1, |D_3^2| = 2, |D_4^3| = 2, |D_1^2| = 1$.
Denote the vertex in D_2^1 by x_1, the vertices in D_3^2 by x_2 and x_3, the vertices in D_4^3 by x_4 and x_5, and the vertex in D_1^2 by y_1. The vertex y_1 cannot be adjacent with x_2 and x_3. The vertex y_1 can be adjacent with x_1, x_4 and x_5. The vertex x_4 can be adjacent with x_3 and y_1, and also the vertex x_5 can be adjacent with x_2 and y_1. In each case the edge x_2x_4 is not consistent.



Subcase 3: $|D_2^1| = 1, |D_3^2| = 1, |D_4^3| = 3, |D_1^2| = 1$.
Denote the vertex in D_2^1 by x_1, the vertex in D_3^2 by x_2 and x_3, the vertices in D_4^3 by x_3, x_4 and x_5, and the vertex in D_1^2 by y_1. The vertex y_1 cannot be adjacent with x_2. The vertices x_3, x_4 and x_5 must be adjacent with y_1. The vertex y_1 can be adjacent with x_1, x_3, x_4 and x_5. In each case the edge x_1x_2 is not consistent.

Subcase 4: $|D_2^1| = 1, |D_3^2| = 3, |D_4^3| = 1, |D_1^2| = 1$.
Denote the vertex in D_2^1 by x_1, the vertices in D_3^2 by x_2, x_3 and x_4, the vertex in D_4^3 by x_5, and the vertex in D_1^2 by y_1. The vertex y_1 cannot be adjacent with x_2, x_3 and x_4. The vertices x_3 and x_4 most be adjacent with x_5. The vertex x_5 can be adjacent with x_3, x_4 and y_1, and also the vertex y_1 can be adjacent with x_1 and x_5. In each case the edge x_4x_5 is not consistent.

Subcase 5: $|D_2^1| = 2, |D_3^2| = 1, |D_4^3| = 2, |D_1^2| = 1$.

Denote the vertices in D_2^1 by x_1 and x_2, the vertex in D_3^2 by x_3, the vertices in D_4^3 by x_4 and x_5, and the vertex in D_1^2 by y_1. The vertex y_1 cannot be adjacent with x_3. The vertices x_4 and x_5 must be adjacent with y_1. The vertex y_1 can be adjacent with x_2, x_3, x_4 and x_5. The vertex x_2 can only be adjacent with x_3 and y_1. In each case the edge x_1x_3 is not consistent.

Subcase 6: $|D_2^1| = 2, |D_3^2| = 2, |D_4^3| = 1, |D_1^2| = 1$.
Denote the vertices in D_2^1 by x_1 and x_2, the vertices in D_3^2 by x_3 and x_4, the vertex in D_4^3 by x_5, and the vertex in D_2^1 by y_1. The vertex y_1 cannot be adjacent with x_3 and x_4. The vertex x_4 can be adjacent with x_1 and x_5. The vertex y_1 can be adjacent with x_1, x_2 and x_5. The vertex x_2 can be adjacent with x_3 and y_1. The vertex x_3 can only be adjacent with x_2. The vertex x_5 can be adjacent with x_4 and y_1. In each case the edge xx_1 is not consistent.

Subcase 7: $|D_2^1| = 3, |D_3^2| = 1, |D_4^3| = 1, |D_1^2| = 1$.
Denote the vertices in D_2^1 by x_1, x_2 and x_3, the vertex in D_3^2 by x_4, the vertex in D_4^3 by x_5 and the vertex in D_1^2 by y_1. The vertex y_1 cannot be adjacent with x_4. The vertices x_2 and x_3 can be adjacent with y_1 and x_4. The vertex y_1 can be adjacent with x_1, x_2, x_3 and x_5. The vertex x_4 can be adjacent with x_2 and x_3. The vertex x_5 must be adjacent with y_1. In each case the edge xx_1 is not consistent.
If $d = 5$ we split our proof into the following subcases.

Subcase 1: $|D_2^1| = |D_3^2| = |D_4^3| = 1, |D_5^4| = 2$ and $|D_1^2| = 1$.
Denote the vertex in D_2^1, D_3^2 and D_4^3 by x_1, x_2 and x_3 respectively, the vertices in D_5^4 by x_4 and x_5, and also the vertex in D_1^2 by y_1. The vertex y_1 cannot be adjacent with x_2, x_4 and x_5. In this case the vertices x_4 and x_5 cannot be adjacent with other vertices.

Subcase 2: $|D_2^1| = |D_3^2| = 1, |D_4^3| = 2, |D_5^4| = 1$ and $|D_1^2| = 1$.
Denote the vertex in D_2^1, D_3^2 by x_1 and x_2 respectively, the vertex in D_4^3 by x_3 and x_4, the vertex in D_5^4 by x_5 and also the vertex in D_1^2 by y_1. The vertices y_1 cannot be adjacent with x_2 and x_5. The vertex x_4 must be adjacent with x_5. The vertex y_1 can be adjacent with x_3 and x_4. In each case the edge x_1x_2 is not consistent.

Subcase 3: $|D_2^1| = 1, |D_3^2| = 2, |D_4^3| = 1, |D_5^4| = 1$ and $|D_1^2| = 1$.
Denote the vertex in D_2^1 by x_1, the vertices in D_3^2 by x_2 and x_3, the vertex in D_4^3 by x_4, the vertex in D_5^4 by x_5, and also the vertex in D_1^2 by y_1. The vertex y_1 cannot be adjacent with x_2, x_3 and x_5. In this case the vertex x_5 cannot be adjacent with other vertices.
If $d = 6$, we have $|D_2^1| = |D_3^2| = |D_4^3| = |D_5^4| = |D_6^5| = |D_1^2| = 1$.
Denote the vertex in $D_2^1, D_3^2, D_4^3, D_5^4, D_6^5$ and D_1^2 by x_1, x_2, x_3, x_4, x_5 and y_1, respectively. The vertex y_1 cannot be adjacent with x_2 and x_4, and also the vertex y_1 can be adjacent with x_1, x_3 and x_5. In each case the edge x_1x_2 is not consistent. □

Acknowledgments

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