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Multipliers on certain topological algebras

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Abstract. There are many well-known results on multipliers (particularly multipliers with closed range) defined on a commutative semisimple Banach algebra. After stating some of them, similar results will be given for commutative strongly semisimple normed algebras and commutative (not necessariliy semisimple) Banach algebras.

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1 Introduction

If A is a Banach algebra then a mapping $T : A \to A$ is a multiplier of A, if $x(Ty) = (Tx)y$ for all $x, y \in A$. For arbitrary Banach algebras, basically nothing is known concerning multipliers, but for algebras without order, a significant number of results are easily deduced. A Banach algebra A is without order, if for all $x \in A$, $xA = 0$ implies $x = 0$, or, for all $x \in A$, $Ax = 0$ implies $x = 0$. Clearly, if A has an identity or bounded approximate identity, it is without order. Due to its wide theoretical and practical applications, the theory of multipliers has been considered by many researchers, see for example [2, 5, 11, 12].

When dealing with a C^*-algebra A, an equivalent definition is available. It is often desirable to embed A in a larger C^*-algebra with unit. The smallest such is the algebra A obtained by adjoining a unit to A, and the largest is A^{**}, the second dual of A. The embedding algebra is the multiplier algebra $M(A)$ of A. In order to define $M(A)$ we consider A as embedded in A^{**}. In this case, $M(A)$ is defined as the idealizer of A in A^{**} (i.e., the largest C^*-subalgebra of A^{**} in which A is an ideal). If A is commutative, say $A = C_0(X)$ with X a locally compact Hausdorff space, then $M(A)$ is isomorphic to the algebra $C_b(X)$ of all bounded continuous functions on X. If $\beta(X)$ denotes the Stone-Čech compactification of X we therefore have $M(A) = C(\beta(X))$.

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The multiplier algebra can thus be regarded as a noncommutative algebraic counterpart of the Stone–Čech compactification, see [1].

Let A be a commutative semisimple Banach algebra with multiplier algebra $M(A)$ and Gelfand spectrum $\triangle(A)$. There are some well known conditions under which a multiplier on A will have closed range. For instance, it has been shown in [9] that if T is a multiplier on A with $dist(0, \sigma(T)\ \{0\}) > 0$, then $T(A)$ is closed in A. Furthermore, this is equivalent to factorization of T as $T = BP = PB$, where B and P are invertible, and idempotent multipliers on A, resp. [9, Theorem 3 and Lemma 1].

For $T \in M(A)$, let $\hat{T}$ be the Gelfand transform of T in the Banach algebra $M(A)$, and $\triangle(\hat{T}) = \{f \in \triangle(A) : \hat{T}(f) \neq 0\}$ and $\delta(T) = \inf\{|\hat{T}(f)| : f \in \triangle(\hat{T})\}$. Ulger investigated conditions under which $\delta(T)$ is positive and $T(A)$ is closedin, [12]. For example, if $\triangle(A)$ is closed in (A^*, γ), where γ is weak-star topology on A^*, then $\delta(T) > 0$, if and only if the set $T^*(\triangle(\hat{T}))$ is closed in (A^*, γ). In general, γ can be norm topology, weak topology, weak-star topology or any locally convex topology. Particularly, if $A = A(G)$, the Fourier algebra on a locally compact group G, $a \in B(G) \cap C_0(G)$, and the multiplier T defined on $A(G)$ by $T(x) = ax$, then $\delta(T) > 0$, if and only if the set $\triangle(\hat{a}) = \{g \in G : a(g) \neq 0\}$ is compact.

Each of the following two results from [12] gives necessary and sufficient conditions for a multiplier on a commutative semisimple Banach algebra A to have a closed range.

Theorem 1. *[12] Suppose that A has a bounded approximate identity. Then $T(A)$ is closed in A and has a bounded approximate identity if and only if, T is spectrally invertible.*

Here a multipliers T on A is called spectrally invertible, if there is a multipliers $S \in M(A)$, such that $\hat{T}\hat{S} = 1$ on $\triangle(\hat{T})$.

For certain BSE-algebras the above condition is equivalent to $\delta(T) > 0$. We recall that a BSE-algebra is commutative semisimple Banach algebra B such that for any $u \in B^{**}$ for which $u|_{\triangle(A)}$ is continuous in the Gelfand topology, the first Arens product of u and each $b \in B$ is an element of B.

Theorem 2. *[12] Let A be a BSE-algebra with bounded approximate identity for which closed linear span of $\triangle(\hat{T})$ in A^* is isomorphic to $\ell^1(\triangle(\hat{T}))$. Then a multiplier $T \in M(A)$ is spectrally invertible, if and only if $\delta(T) > 0$.*

Main Results

For a commutative topological algebra (A, τ) we denote the set of all nonzero continuous complex homomorphisms on A by $\triangle(A)$ and the algebra of all continuous multipliers on A by $M(A)$ or $M(A, \tau)$ for emphasis. We note that contrary to the Banach algebra case, here multipliers and complex homomorphisms are not necessarily continuous. However, if A is assumed to be a complete, metrizable LMC-algebra with a bounded approximate identity then each multiplier on A is a continuous linear map (see [7]). Similar to the Banach algebra case, in a complete metrizable LMC-algebra $A, M(A)$ is complete with respect to the strong operator topology (s-topology) and if A has a bounded approximate identity then A is s-dense in $M(A)$ [7].

Now let $(A, ||.||)$ be a commutative normed algebra and $\bar{A}$ be its completion. Then $\bar{A}$ is a Banach algebra and $\triangle(A)$ can be identified by $\triangle(\bar{A})$. For an ideal I in A, let $h(I)$ be the set

of all closed maximal modular ideals of A containing I. Since in each LMC-algebra A there is a one to one correspondence between closed maximal modular ideals and continuous complex homomorphisms on A, we have $h(I) = \{\varphi \in \triangle(A) : \varphi|_I = 0\}$ [3]. We say that A is strongly semisimple if the intersection of all closed maximal modular ideals in A is the trivial set $\{0\}$. We note that if A has an approximate identity or if A is strongly semisimple then it is a without order algebra, that is $ax = ay$ for all $a \in A$ implies that $x = y$. Clearly, in these cases the elements of $M(A)$ are continuous linear operators on A, and $M(A)$ is a normed algebra containing A as an ideal.
Next theorem shows that some classical results on multipliers defined on a Banach algebra are vaild in normed algebra case, whereas the others may not be valid.

Theorem 3. *Let $(A, ||.||)$ be a commutative normed algebra, whose completion $\bar{A}$ is without order Banach algebra. Then*

(i) $M(A) = \{T \in M(\bar{A}) : TA \subseteq A\}$.

(ii) $\triangle(M(A)) = \triangle(A) \cup h(A)$, *where $h(A)$ is the hull of the ideal A in $M(A)$. In particular, if A is strongly semisimple then so is $M(A)$.*

(iii) If A has a bounded approximate identity then A is dense in $M(A)$ with respect to the s-topology (strong operator topology) on $M(A)$, $M(A)$ is s-dense in $M(\bar{A})$ but in general $M(A)$ is not s-complete.

(iv) $M(A) = M(\bar{A})$, *if and only if A is an ideal in $M(\bar{A})$.*

(v) For each $f \in \triangle(A)$, $T^*(f) = \hat{T}(f) \cdot f$.

Proof. (i) Let $T \in M(A)$. Since T is assumed to be continuous it can be easily extended to a multiplier on $\bar{A}$. Conversely each multiplier on $\bar{A}$ is continuous, so that if $TA \subseteq A$ then $T \in M(A)$. It is easy to see that inclusion map from $M(A)$ into $M(\bar{A})$ is an isometry.
(ii) As in the Banach algebra case (see [8]), for each $\mu \in \triangle(A)$ and $x \in A$ with $\mu(x) \neq 0$, $\bar{\mu}(T) = \mu(Tx)/\mu(x)$ is a well defined (non-zero) complex homomorphism on $M(A)$. We need only to note that since μ and T are continuous, $\bar{\mu} \in \triangle(M(A))$.
Conversely if $\mu \in \triangle(M(A))$ and $\mu|_A \neq 0$ then μ is a continuous complex homomorphism on A, that is $\mu|_A \in \triangle(A)$. The second part is now immediate.
(iii) The proof of the first part is as in the Banach algebra case. For the second one let $T \in M(\bar{A}), x_1, \ldots, x_n \in \bar{A}$ and $\epsilon > 0$. Since A and hence $\bar{A}$ have bounded approximate identities, $\bar{A}$ is dense in $M(\bar{A})$ under the s-topology on $M(\bar{A})$ [8]. Thus, there exists $x \in \bar{A}$ with $||xx_i - Tx_i|| < \epsilon/n_1$ for $i = 1, \ldots, n$. Now it is enough to choose an element $a \in A$ sufficiently near to x such that $||ax_i - xx_i|| < \epsilon, 1 \leq i \leq n$. Then $a \in A \subseteq M(A)$ belongs to the neighborhood $\{T \in M(\bar{A}) : ||Tx_i - T_0x_i|| < \epsilon\}$ in the s-topology of $M(\bar{A})$.
Since the relative s-topology on $M(A)$, inherited from $M(\bar{A})$ is the same s-topology on $M(A)$, it follows that $M(A)$ is s-complete, if and only if $M(A) = M(\bar{A})$. An example of a normed algebra $(A, ||.||)$ with a bounded approximate identity and $M(A, ||.||) \neq M(\bar{A})$ is the Fourier algebra $A(G)$ on a locally compact amenable group G endowed with the sup-norm $||.||_\infty$. As we know $A = A(G)$ is dense in $C_0(G)$ and $M(C_0(G)) = C_b(G)$. Since $A(G)$ is an ideal of $B(G)$ it follows

that $M(A,||.||_\infty) = \{f \in C_b(G) : fA \subseteq A\}$ contains $B(G)$. On the other hand the multiplier algebra of the Banach algebra $(A(G), ||.||)$, where $||.||$ is the usual Banach algebra norm of $B(G)$, can be identified by $B(G)$ [6, 10], so that $M(A,||.||_\infty) \subseteq B(G)$ and hence $M(A,||.||_\infty) = B(G)$ whereas $M(\bar{A}) = M(C_0(G)) = C_b(G)$.
(iv) Clearly, if $M(A) = M(\bar{A})$ then A is an ideal in $M(\bar{A})$. Now assume that A is an ideal in $M(\bar{A})$. Then for each $T \in M(\bar{A})$ and $a \in A, L_{T(a)} = TL_a \in A$, and since $\bar{A}$ is without order, it follows that $T(a) \in A$, that is, $M(A) = M(\bar{A})$.
(v) This is an easy consequence of the proof of (i). □

Example 1. (a) Let X be a locally compact Hausdorff space and $A = C_c(X)$, be equipped with sup-norm $||.||_\infty$. Then since A is dense in $C_0(x), M(C_0(X) = C_b(X)$, and $C_c(X)$ is an ideal of $C_b(x)$, part (iv) of the preceding theorem shows that $M(C_c(X), ||.||_\infty) = C_b(X)$. Note that by [4] $M(C_c(X), \tau) = C(X)$ where τ is the compact open topology on $C(X)$.
(b) Let $1 \leq p \leq \infty$ and consider $X = c_{00}$ under the l_p norm. Then by Theorem 3 (iv), $M(c_{00}) = l_\infty$.

Remark 1. *(i) Using (ii) and (iv) one can see easily that the above argument can be applied for a regular Tauberian Banach function algebra.* $(A, ||.||)$ *to conclude that if* A_{00} *is the algebra of all elements* $x \in A$ *whose Gelfand transform has compact support, then* $M(A_{00}, ||.||) = M(A)$.
(ii) The above example shows that for a non-complete normed algebra $A, M(A)$ *may be a Banach algebra (note that the inclusion map form* $M(A)$ *into* $M(\bar{A})$ *is an isometry).*
(iii) In a normed algebra A *if* $T \in M(A)$ *is one-to-one and onto, then its inverse map in not necessarily in* $M(A)$*. For example consider the multiplier* L_g *on* $C_c(\mathbb{R})$*, where* $g \in C_b(\mathbb{R})$ *has no zero on* $\mathbb{R}$ *and* $1/g$ *is not bounded.*

Now we are in a position that enables us to state a similar result to [12] for commutative strongly semisimple normed algebras and commutative (not necessarily semisimple) Banach algebras.

Theorem 4. *Let* A *be either a commutative strongly semisimple normed algebra or a commutative without order Banach algebra such that* $\triangle(A)$ *is* w^**-closed in* A^**. Then for a multiplier* T *whose range is not contained in* $rad(A), \delta(T) > 0$ *if and only if* $T^*(\triangle(\hat{T}))$ *is* w^**-closed in* A^*.

Proof. Let A be a commutative without order Banach algebra and $rad(A)$ be its radical. Then clearly $\triangle(A/rad(A))$ can be identified by $h(rad(A)) = \triangle(A)$. For each $f \in \triangle(A)$ we may denote the corresponding element in $\triangle(A/rad(A))$ again by f.

Now for each $T \in M(A)$ let the operator $\tilde{T}$ be defined on $A/rad(A)$ by $\tilde{T}(x + rad(A)) = T(x) + rad(A)$. One can see easily that $\tilde{T}$ is well defined and, in fact, $\tilde{T} \in M(A/rad(A))$. On the other hand, for $f \in \triangle(A/rad(A)) = \triangle(A)$ and $x \in A$ with $f(x + rad(A)) \neq 0$ we have

$$\hat{\tilde{T}}(f) = \frac{f(\tilde{T}(x + rad(A))))}{f(x + rad(A))} = \frac{f(Tx + rad(A))}{f(x + rad(A))} = \frac{f(Tx)}{f(x)} = \hat{T}(f).$$

This clearly shows that $\triangle(\hat{T}) = \triangle(\hat{\tilde{T}})$ and $\delta(T) = \delta(\tilde{T})$. We note that since $T(A)$ is not comtained in $rad(A)$ and $A/rad(A)$ is semisimple it follows that $\triangle(\hat{\tilde{T}})$ is not empty. Now by [12], if $\triangle(A/rad(A))$ is w^*-closed in the dual of $A/rad(A)$ then $\delta(\tilde{T}) > 0$ if and only if $\tilde{T}^*(\triangle(\hat{\tilde{T}}))$ is w^*-closed in the dual of $A/rad(A)$. So we need only to show that if $\triangle(A)$ is w^*-closed in A^* then $\triangle(A/rad(A)) = \triangle(A)$ is w^*-closed in the dual of $A/rad(A)$ and $T^*(\triangle(\hat{T}))$ is w^*-closed in A^* if and only if $\tilde{T}^*(\triangle(\hat{\tilde{T}}))$ is w^*-closed in the dual of $A/rad(A)$. Since the dual of $A/rad(A)$ is isometrically isomorphic to $\{a^* \in A^* : a^* = 0 \text{ on } rad(A)\}$ it is clear that $\triangle(A/rad(A))$ is w^*-closed in the dual of $A/rad(A)$.

As in the proof of [[10], Lemma 2.1] $T^*(\triangle(\hat{T}))$ is w^*-closed in A^* if and only if zero is not in the w^*-closure of $T^*(\triangle(\hat{T}))$ (here we do not require A to be semisimple). Now, let $\{f_a\}$ be a net in $\triangle(\hat{T})$ such that $T^*(f_a)$ converges in w^*-topology to zero functional, then $f_a(Tx) \longrightarrow 0$, for each $x \in A$. That is $f_a(Tx + rad(A)) = f_a(\tilde{T}(x + rad(A)) \longrightarrow 0$ or equivalently $\tilde{T}^*(f_a) \longrightarrow 0$ in w^*-topology of the dual of $A/rad(A)$.

Now, let A be a strongly semisimple normed algebra. Using Theorem 3, in this case the proof is similar to the Banach algebra case with this notification that for a net (f_a) in $\triangle(A)$ again $(\hat{T}(f_a))$ is a bounded net of complex numbers. This is because the restriction of the Gelfand transform of $T \in M(A)$ to $\triangle(A) = \triangle(\bar{A})$ equals to the restriction of the Gelfand transform of T as an element of $M(\bar{A})$ to $\triangle(A)$. Hence $(\hat{T}(f_a))$ is bounded by the sup-norm of $\hat{T} \in C_0(\triangle(M(\bar{A})))$. □

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