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Commutativity of prime rings involving multiplicative b-generalized derivation

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Abstract. Let $\mathfrak{Q}_{mr}$ be a maximal right ring of quotients of $\mathfrak{A}$, where $\mathfrak{A}$ is a prime ring. A map $\mathfrak{F}:\mathfrak{A}\to\mathfrak{Q}_{mr}$ associated with derivation $d:\mathfrak{A}\to\mathfrak{A}$ is called a multiplicative b-generalized derivation (need not necessarily additive) if $\mathfrak{F}(lm)=\mathfrak{F}(l)m+bld(m)$ holds for all $l,m\in\mathfrak{A}$ and for some $b\in\mathfrak{Q}_{mr}$. In this article, we study the commutativity of prime rings when the map b-generalized derivation satisfies the strong commutativity preserving condition and some central identities.

Keywords: Derivation, Prime ring, Multiplicative generalized derivation, Multiplicative b-generalized derivation.
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1 Motivation and Preliminaries

Throughout, unless otherwise mentioned, $\mathfrak{A}$ always denotes an associative prime ring with center $\mathfrak{Z}$ but not necessarily with an identity element. $\mathfrak{C}=\mathfrak{Z}(\mathfrak{Q}_{mr})$ is the extended centroid of $\mathfrak{A}$ and is also known as the center of $\mathfrak{Q}_{mr}$. It is known that $\mathfrak{A}\subseteq\mathfrak{Q}_{mr}$, and the overrings $\mathfrak{Q}_{mr}$ is prime if $\mathfrak{A}$ is prime. Also, $\mathfrak{C}$ is a field if and only if $\mathfrak{A}$ is a prime ring. We refer the reader to the book [4] for details. "A ring $\mathfrak{A}$ is prime if $a\mathfrak{A}b=(0)$, specifies that either $a=0$ or $b=0$ for any $a,b\in\mathfrak{A}$, and is considered as a semi-prime if $a\mathfrak{A}a=(0)$, implies $a=0$ for any $a\in\mathfrak{A}$". The commutator (anti-commutator) is represented by $[l,m]=lm-ml$ ($l\circ m=lm+ml$), for every $l,m\in\mathfrak{A}$. A map $f:\mathfrak{A}\to\mathfrak{A}$ is called centralizing on a non-empty subset $\mathfrak{S}$ of $\mathfrak{A}$ if $[f(l),l]\in\mathfrak{Z}$ for all $l\in\mathfrak{S}$, and is

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called commuting on $\mathfrak{S}$ if $[f(l), l] = 0$, for all $l \in \mathfrak{S}$. An additive mapping $d : \mathfrak{A} \to \mathfrak{A}$ is known as a derivation if $d(lm) = d(l)m + ld(m)$ holds, for all $l, m \in \mathfrak{A}$. A derivation $\mathfrak{I}_a$ is called $\mathfrak{Q}_{mr}$-inner if there exists $a \in \mathfrak{Q}_{mr}$ such that $\mathfrak{I}_a(l) = [a, l]$, for all $l \in \mathfrak{A}$. Otherwise, it is called $\mathfrak{Q}_{mr}$-outer. Let $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ be a map associated with a derivation d such that $\mathfrak{F}(lm) = \mathfrak{F}(l)m + ld(m)$ holds, for all $l, m \in \mathfrak{A}$. If $\mathfrak{F}$ is additive, then $\mathfrak{F}$ is known as a generalized derivation. However, if $\mathfrak{F}$ is not necessarily additive, then $\mathfrak{F}$ is said to be a multiplicative generalized derivation.

The algebra of derivation and generalized derivation play a crucial role in studying functional identities and their applications. There are many generalizations of derivation, viz., generalized derivation, multiplicative generalized derivation, skew generalized derivation, b-generalized derivation, etc. Koşan and Lee introduced the notion of b-generalized derivation. The most important and systematic research on the b-generalized derivations has been accomplished in [7,12] and references therein.

In 2016, Gölbaşi [8] studied certain identities having multiplicative generalized derivation $\mathfrak{F}$ on a non-zero ideal $\mathfrak{I}$ of a semi-prime ring $\mathfrak{A}$ and showed that $\mathfrak{A}$ contains a non-zero central ideal. In the same study, it was also reported that a prime ring $\mathfrak{A}$ must be commutative if $\mathfrak{F}([l, m]) = 0$, for all $l, m \in \mathfrak{I}$. In 2018, Koç and Gölbaşi [10] described the study of strong commutativity preserving (SCP) maps having multiplicative generalized derivations $\mathfrak{F}$ associated with a non-zero additive map d and they established that, for a semi-prime ring $\mathfrak{A}$ it contains a non-zero central ideal if $\mathfrak{F}$ is SCP on $\mathfrak{I}$, where $\mathfrak{I}$ a non-zero ideal of $\mathfrak{A}$. Similar studies of derivation/generalized derivation/multiplicative generalized derivation can be seen in [1–3,5,14,16] and references therein.

In this article, we have presented a map $\mathfrak{F} : \mathfrak{A} \to \mathfrak{Q}_{mr}$ associated with derivation (need not be additive) $d : \mathfrak{A} \to \mathfrak{A}$ such that $\mathfrak{F}(lm) = \mathfrak{F}(l)m + bld(m)$ holds for all $l, m \in \mathfrak{A}$ for some $b \in \mathfrak{Q}_{mr}$. If $\mathfrak{F}$ is additive (not necessarily additive), then $\mathfrak{F}$ is called b-generalized derivation (multiplicative b-generalized derivation). Also, if b is unity, then we see that the map $\mathfrak{F}$ from $\mathfrak{A}$ to $\mathfrak{Q}_{mr}$ is given by $\mathfrak{F}(lm) = \mathfrak{F}(l)m + ld(m)$, for all $l, m \in \mathfrak{A}$ is considered as a 1-generalized derivation (multiplicative 1-generalized derivation) provided that $\mathfrak{F}$ is additive (non-additive). So we can say that b-generalized derivation (multiplicative b-generalized derivation) is a generalization of generalized derivation (multiplicative generalized derivation) provided that $\mathfrak{F}$ is additive (non-additive). Here, we present some related examples

Example 1. Let $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ is a map defined by $\mathfrak{F}(l) = bd(l)$, where $d : \mathfrak{A} \to \mathfrak{A}$ is a derivation and $b \in \mathfrak{A}$ and for all $l \in \mathfrak{A}$, then clearly we can observe that the map $\mathfrak{F}(lm) = \mathfrak{F}(l)m + bld(m)$, for all $l, m \in \mathfrak{A}$, is a multiplicative b-generalized derivation associated with a derivation d.

Example 2. Let $\mathfrak{A} = \left\{ \begin{bmatrix} 0 & l & m \\ 0 & 0 & n \\ 0 & 0 & 0 \end{bmatrix} \Big| l, m, n \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of integers and $\mathfrak{F}$ and d are maps from $\mathfrak{A} \to \mathfrak{A}$ such that $\mathfrak{F}\left(\begin{bmatrix} 0 & l & m \\ 0 & 0 & n \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & mn \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $d\left(\begin{bmatrix} 0 & l & m \\ 0 & 0 & n \\ 0 & 0 & 0 \end{bmatrix}\right) =$

$\begin{bmatrix} 0 & 0 & l \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then it is verified that $\mathfrak{F}$ is a multiplicative b-generalized derivation associated with derivation d and for any fixed $b \in \mathfrak{A}$.

2 Preliminaries

Lemma 1 (Lemma 3 [15]). *If a prime ring $\mathfrak{A}$ contains a commutative non-zero right ideal $\mathfrak{I}$, then $\mathfrak{A}$ is commutative.*

Lemma 2. *Let $\mathfrak{A}$ be a prime ring. If $l \in \mathfrak{Z}$ and $m \in \mathfrak{A}$ such that $lm \in \mathfrak{Z}$, then $m \in \mathfrak{Z}$ or $l = 0$.*

Lemma 3. *Let $\mathfrak{A}$ be a prime ring. If $[[l, m], p] = 0$, $\forall\, l, m, p \in \mathfrak{A}$, then $[l, m] = 0$.*

Proof. We have given

$$[[l, m], p] = 0, \ \forall\ l, m, p \in \mathfrak{A}. \tag{1}$$

Replacing l by nl in (1), $\forall\ n \in \mathfrak{A}$, and using the identity of commutator, we have

$$[n[l, m], p] + [[n, m]l, p] = 0, \ \forall\ l, m, p, n \in \mathfrak{A}. \tag{2}$$

Again, we use the identity of commutator in (2), we find that

$$n[[l, m], p] + [n, p][l, m] + [n, m][l, p] + [[n, m], p]l = 0, \ \forall\ l, m, p, n \in \mathfrak{A}. \tag{3}$$

Using our hypothesis in (3), we obtain

$$[n, p][l, m] + [n, m[l, p] = 0, \ \forall\ l, m, p, n \in \mathfrak{A}. \tag{4}$$

In particular $p = l$, the above equation yields

$$[n, l][l, m] = 0, \ \forall\ l, m, n \in \mathfrak{A}. \tag{5}$$

Substituting n by mn in (5), we get

$$m[n, l][l, m] + [m, l]n[l, m] = 0, \ \forall\ l, m, n \in \mathfrak{A}. \tag{6}$$

Using (5) in (6), we have

$$[m, l]n[l, m] = 0, \ \forall\ l, m, n \in \mathfrak{A}. \tag{7}$$

This implies that,

$$[m, l]\mathfrak{A}[l, m] = (0), \ \forall\ l, m \in \mathfrak{A}. \tag{8}$$

Since $\mathfrak{A}$ is a prime ring, then we get either $[m, l] = 0$ or $[l, m] = 0$. Consequently, in both cases, it follows that $[l, m] = 0$, $\forall\ l, m \in \mathfrak{A}$. □

3 Main results

Theorem 1. *Let $\mathfrak{A}$ be a prime ring with char$(\mathfrak{A}) \neq 6$ and let $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ be a multiplicative b-generalized derivation associated with derivation $d : \mathfrak{A} \to \mathfrak{A}$. If $\mathfrak{F}$ is centralizing on $\mathfrak{A}$, then either $\mathfrak{A}$ is commutative or $\mathfrak{F}(l) = \lambda l$, $\forall\, l \in \mathfrak{A}$ and some $\lambda \in \mathfrak{C}$.*

Proof. In our hypothesis, we substitute l by l^2 and using the definition of multiplicative b-generalized derivation, we get

$$[\mathfrak{F}(l)l + bld(l), l^2] \in \mathfrak{Z},\ \forall\, l, m \in \mathfrak{A}. \tag{9}$$

Which implies that

$$[\mathfrak{F}(l), l^2]l + [bld(l), l^2] \in \mathfrak{Z},\ \forall\, l, m \in \mathfrak{A}. \tag{10}$$

On simplifying above relation and using the hypothesis in (10), we obtain

$$2[\mathfrak{F}(l), l]l^2 + [bld(l), l^2] \in \mathfrak{Z},\ \forall\, l, m \in \mathfrak{A}. \tag{11}$$

Commuting (11) by l, we find that

$$[[bld(l), l^2], l] = 0,\ \forall\, l, m \in \mathfrak{A}. \tag{12}$$

We divided this into two cases: (i) when d is an outer derivation and (ii) when d is an inner derivation.

Case(I) Let us suppose that d is an outer derivation, by using *[9, Lemma 2]*, (12) reduces to

$$[[blm, l^2], l] = 0,\ \forall\, l, m \in \mathfrak{A}.$$

Now, we can extend this result for $\mathfrak{Q}_{mr}$.

$$[[blm, l^2], l] = 0,\ \forall\, l, m \in \mathfrak{Q}_{mr}. \tag{13}$$

Now, we substitute l by $l + 1$ in previous relation, we found that

$$[[blm + by, l^2 + 2l + 1], l + 1] = 0,\ \forall\, l, m \in \mathfrak{Q}_{mr}. \tag{14}$$

On solving (14), we see that

$$[[blm, l^2], l] + [[blm, 2l], l] + [[by, l^2], l] + [[by, 2l], l] = 0,\ \forall\, l, m \in \mathfrak{Q}_{mr}. \tag{15}$$

Using (13) in (15), we have

$$2[[blm, l], l] + [[by, l^2], l] + 2[[by, l], l] = 0,\ \forall\, l, m \in \mathfrak{Q}_{mr}. \tag{16}$$

Again we substitute l by $l + 1$ in (16), we obtain

$$\begin{aligned} &2[[blm + by, l + 1], l + 1] + [[by, l^2 + 2l + 1], l + 1] \\ &+ 2[[by, l + 1], l + 1] = 0,\ \forall\, l, m \in \mathfrak{Q}_{mr}. \end{aligned} \tag{17}$$

After simplifying we obtain

$$2[[blm, l], l] + 2[[by, l], l] + [[by, l^2], l] + 2[[by, l], l] + 2[[by, l], l] = 0, \ \forall\ l, m \in \mathfrak{Q}_{mr}. \tag{18}$$

Using (16) in (18) and char($\mathfrak{A}$) $\neq 2$, we get

$$[[by, l], l] = 0, \ \forall\ l, m \in \mathfrak{Q}_{mr}. \tag{19}$$

Substituting m by $[l, a]$ for fixed $a \in \mathfrak{Q}_{mr}$ in (19), we have

$$[[b[l, a], l], l] = 0, \ \forall\ l \in \mathfrak{Q}_{mr}. \tag{20}$$

Since $b[l, a]$ is a b-generalized derivation, so we replace $b[l, a]$ by $\mathfrak{G}(l)$. Equation (20) becomes $[[\mathfrak{G}(l), l], l] = 0$, $\forall\ l \in \mathfrak{Q}_{mr}$. By using *[13, Theorem 3.2]* we get $\mathfrak{G}(l) = \lambda l$, $\forall\ l \in \mathfrak{Q}_{mr}$ and some $\lambda \in \mathfrak{C}$. Thus, we have

$$b[l, a] - \lambda l = 0, \ \forall\ l \in \mathfrak{Q}_{mr}. \tag{21}$$

Replacing l by lp, $\forall\ r \in \mathfrak{Q}_{mr}$ in (21), we find that

$$bl[r, a] + b[l, a]r - \lambda lp = 0, \ \forall\ l, r \in \mathfrak{Q}_{mr}. \tag{22}$$

Multiplying (21) by r from right, we see that

$$b[l, a]r - \lambda lp = 0, \ \forall\ l, r \in \mathfrak{Q}_{mr}. \tag{23}$$

From (22) and (23), we obtain

$$bl[r, a] = 0, \ \forall\ l, r \in \mathfrak{Q}_{mr}. \tag{24}$$

Using the primeness of $\mathfrak{Q}_{mr}$, we get either $b = 0$ or $[r, a] = 0$, $\forall\ r \in \mathfrak{Q}_{mr}$ and for fixed $a \in \mathfrak{Q}_{mr}$. If $[r, a] = 0$ implies $a \in \mathfrak{C}$, then $\mathfrak{Q}_{mr}$ and so $\mathfrak{A}$ is commutative. Next, if $b = 0$, then by the definition of multiplicative b-generalized derivation we get

$$\mathfrak{F}(lm) = \mathfrak{F}(l)m, \ \forall\ l, m \in \mathfrak{A}. \tag{25}$$

We substitute l by lm in hypothesis, we get

$$[\mathfrak{F}(lm), lm] = [\mathfrak{F}(l)m, lm] \in \mathfrak{Z}, \ \forall\ l, m \in \mathfrak{A}.$$

We can extend the above identity for $m \in \mathfrak{Q}_{mr}$, we get

$$[\mathfrak{F}(lm), lm] = [\mathfrak{F}(l)m, lm] \in \mathfrak{Z}, \ \forall\ l \in \mathfrak{A}, m \in \mathfrak{Q}_{mr}.$$

Thus

$$[\mathfrak{F}(l), l]m^2 + l[\mathfrak{F}(l), m]m + \mathfrak{F}(l)[m, l]m \in \mathfrak{Z}, \ \forall\ l \in \mathfrak{A}, m \in \mathfrak{Q}_{mr}. \tag{26}$$

Substituting m by $m + 1$ in (26) and using it, we find that

$$[\mathfrak{F}(l), l](2m + 1) + l[\mathfrak{F}(l), m] + \mathfrak{F}(l)[m, l] \in \mathfrak{Z}, \ \forall\ l \in \mathfrak{A}, m \in \mathfrak{Q}_{mr}. \tag{27}$$

Since $\mathfrak{F}$ is centralizer, then above relation reduces to

$$2[\mathfrak{F}(l), l]m + l[\mathfrak{F}(l), m] + \mathfrak{F}(l)[m, l] \in \mathfrak{Z},\ \forall\ l \in \mathfrak{A}, m \in \mathfrak{Q}_{mr}. \tag{28}$$

Replacing m by $m + l$ in (28) and using it, we see that

$$2[\mathfrak{F}(l), l]l + l[\mathfrak{F}(l), l] \in \mathfrak{Z},\ \forall\ l \in \mathfrak{A}. \tag{29}$$

This implies that

$$3l[\mathfrak{F}(l), l] \in \mathfrak{Z},\ \forall\ l \in \mathfrak{A}. \tag{30}$$

Since char$(\mathfrak{A}) \neq 6$, (30) reduces to

$$l[\mathfrak{F}(l), l] \in \mathfrak{Z},\ \forall\ l \in \mathfrak{A}. \tag{31}$$

For all $p \in \mathfrak{A}$, (31) becomes

$$[l[\mathfrak{F}(l), l], p] = 0,\ \forall\ l, p \in \mathfrak{A}. \tag{32}$$

The above equation can be re-written as

$$[l, p][\mathfrak{F}(l), l] = 0,\ \forall\ l, p \in \mathfrak{A}. \tag{33}$$

Putting rp in place of p in (33) and using it, we get

$$[l, r]p[\mathfrak{F}(l), l] = 0,\ \forall\ l, p \in \mathfrak{A}. \tag{34}$$

In particular, for $r = \mathfrak{F}(l)$ we have $[l, \mathfrak{F}(l)]p[\mathfrak{F}(l), l] = 0$, for all $l, p \in \mathfrak{A}$. Since $\mathfrak{A}$ is prime ring then we get $[\mathfrak{F}(l), l] = 0$. Then we substitute these value in (26), we find that

$$l[\mathfrak{F}(l), m]m + \mathfrak{F}(l)[m, l]m \in \mathfrak{Z},\ \forall\ l, m \in \mathfrak{A}. \tag{35}$$

Linearizing (35) for m, we obtain

$$\begin{aligned} &l[\mathfrak{F}(l), m]r + l[\mathfrak{F}(l), r]m + \mathfrak{F}(l)[m, l]r \\ &+ \mathfrak{F}(l)[r, l]m \in \mathfrak{Z},\ \forall\ l, m, r \in \mathfrak{A}. \end{aligned} \tag{36}$$

Substituting r by l and using $[\mathfrak{F}(l), l] = 0$ in (36), we see that

$$(l[\mathfrak{F}(l), m] + \mathfrak{F}(l)[m, l])l \in \mathfrak{Z},\ \forall\ l, m, r \in \mathfrak{A}. \tag{37}$$

Substituting m by $m + 1$ in (35), we have

$$l[\mathfrak{F}(l), m + 1](m + 1) + \mathfrak{F}(l)[m + 1, l](m + 1) \in \mathfrak{Z},\ \forall\ l, m \in \mathfrak{A}. \tag{38}$$

Simplifying above relation and using (35), we found that

$$l[\mathfrak{F}(l), m] + \mathfrak{F}(l)[m, l] \in \mathfrak{Z},\ \forall\ l, m \in \mathfrak{A}. \tag{39}$$

From (39) and (37) and by using Lemma 2, we get

$$l[\mathfrak{F}(l), m] + \mathfrak{F}(l)[m, l] = 0 \text{ or } l \in \mathfrak{Z},\ \forall\ l, m \in \mathfrak{A}. \tag{40}$$

If $l \in \mathfrak{Z}$, then $\mathfrak{A}$ is commutative. Now, we assume that $l[\mathfrak{F}(l), m] + \mathfrak{F}(l)[m, l] = 0$ i.e., $l\mathfrak{F}(l)m - lm\mathfrak{F}(l) + \mathfrak{F}(l)ml - \mathfrak{F}(l)lm = 0, \forall\ l, m \in \mathfrak{A}$. Using $[\mathfrak{F}(l), l] = 0$ in the previous relation, we obtain $\mathfrak{F}(l)ml - lm\mathfrak{F}(l) = 0, \forall\ l, m \in \mathfrak{A}$. Using *[6, Lemma 3.2]* , we get either $l = 0$ or $\mathfrak{F}(l) = \lambda l$, where $\lambda \in \mathfrak{C}$. Since, we know that $l = 0$ can not possible. So we get either $\mathfrak{F}(l) = \lambda l$.

Case(II) Now, we assume that d is an inner derivation, then for some $a \in \mathfrak{A}$, (12) reduces to

$$[[bl[l, a], l^2], l] = 0,\ \forall\ l \in \mathfrak{A}. \tag{41}$$

Equation (41) can be extended to $\mathfrak{Q}_{mr}$ i.e.,

$$[[bl[l, a], l^2], l] = 0,\ \forall\ l \in \mathfrak{Q}_{mr}. \tag{42}$$

Substituting l by $l + 1$ in (42) and using it, we find that

$$2[[bl[l, a], l], l] + [[b[l, a], l^2], l] + 2[[b[l, a], l], l] = 0,\ \forall\ l \in \mathfrak{Q}_{mr}. \tag{43}$$

again we replace l by $l + 1$ in (43) and using it, we see that

$$[[b[l, a], l], l] = 0,\ \forall\ l \in \mathfrak{Q}_{mr}. \tag{44}$$

Above relation is same as (20), by using previous discussion we get $a \in \mathfrak{Z}(\mathfrak{A})$. In this case , d becomes zero. by the definition of multiplicative b-generalized derivation we get $\mathfrak{F}(lm) = \mathfrak{F}(l)m$ which is same as (25). So, using similar argument we get the conclusion. □

Corollary 1. *Let $\mathfrak{A}$ be a prime ring with char$(\mathfrak{A}) \neq 2$ and let $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ be a multiplicative b-generalized derivation associated with derivation $d : \mathfrak{A} \to \mathfrak{A}$. If $\mathfrak{F}$ is commuting on $\mathfrak{A}$, then either $\mathfrak{A}$ is commutative or $\mathfrak{F}(l) = \lambda l$, $\forall\ l \in \mathfrak{A}$ and some $\lambda \in \mathfrak{C}$.*

Corollary 2. *Let $\mathfrak{A}$ be a prime ring with char$(\mathfrak{A}) \neq 2$ and let $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ be a multiplicative b-generalized derivation associated with derivation $d : \mathfrak{A} \to \mathfrak{A}$. Then either $\mathfrak{A}$ is commutative or $\mathfrak{F}(l) = \lambda l$, $\forall\ l \in \mathfrak{A}$ and some $\lambda \in \mathfrak{C}$, if $\mathfrak{F}$ satisfies any one of the following conditions:*

1. $[\mathfrak{F}(l), m] \pm [l, m] \in \mathfrak{Z}$, $\forall\ l, m \in \mathfrak{A}$.
2. $[\mathfrak{F}(l), m] \pm [d(l), d(m)] \in \mathfrak{Z}$, $\forall\ l, m \in \mathfrak{A}$.
3. $[\mathfrak{F}(l), m] \pm \mathfrak{F}([l, m]) \in \mathfrak{Z}$, $\forall\ l, m \in \mathfrak{A}$.
4. $[\mathfrak{F}(l), m] \pm d([l, m]) \in \mathfrak{Z}$, $\forall\ l, m \in \mathfrak{A}$.

Theorem 2. *Let $\mathfrak{A}$ be a prime ring and $\mathcal{K}$ be a non-zero two-sided ideal of $\mathfrak{A}$. Next, let $\mathfrak{F} : \mathfrak{A} \to \mathfrak{Q}_{mr}$ be a multiplicative b-generalized derivation associated with derivation $d : \mathfrak{A} \to \mathfrak{A}$ such that $\mathfrak{F}([l, m]) = 0$, $\forall\ l, m \in \mathcal{K}$ and any $b \in \mathfrak{Q}_{mr}$. Then $\mathfrak{A}$ is commutative.*

Proof. We have given that $\mathfrak{F}([l, m]) = 0$, $\forall\ l, h \in \mathcal{K}$. Substituting ml for m in our hypothesis, we get

$$\mathfrak{F}([l, m]l) = 0,\ \forall\ l, m \in \mathcal{K}. \tag{45}$$

This can be rewritten as

$$\mathfrak{F}([l,m])l + b[l,m]d(l) = 0,\ \forall\ l,m \in \mathcal{K}. \tag{46}$$

By using our hypothesis in (46), we get

$$b[l,m]d(l) = 0,\ \forall\ l,m \in \mathcal{K}.$$

Now, we can extend this result for $\mathfrak{Q}_{mr}$.

$$b[l,m]d(l) = 0,\ \forall\ l,m \in \mathfrak{Q}_{mr}. \tag{47}$$

We substitute m by mbk, $\forall\ k \in \mathfrak{Q}_{mr}$ in (47) and then using it, we obtain

$$b[l,mb]kd(l) = 0,\ \forall\ l,m \in \mathfrak{Q}_{mr}. \tag{48}$$

Since, $\mathfrak{Q}_{mr}$ is prime, from (48) we get either $b[l,mb]k = 0$ or $d(l) = 0$, $\forall\ l,m \in \mathfrak{Q}_{mr}$. We divide this into two cases:

Case(I) Suppose that $d(l) = 0$, Substituting m by lm in hypothesis and using $d(l) = 0$, we get

$$0 = \mathfrak{F}(l[l,m]) = \mathfrak{F}(l)[l,m],\ \forall\ l,m \in \mathfrak{Q}_{mr}. \tag{49}$$

Replacing m by rm in (49), $\forall\ r \in \mathfrak{Q}_{mr}$, we obtain $0 = \mathfrak{F}(l)r[l,m]$ $\forall\ l,m,r \in \mathfrak{Q}_{mr}$. By the primeness of $\mathfrak{A}$ we get either $[l,m] = 0$ or $\mathfrak{F}(l) = 0$. If $[l,m] = 0$ then $\mathfrak{Q}_{mr}$ is commutative. So, $\mathfrak{A}$ is commutative and the second case is not possible.

Case(II) If $b[l,mb] = 0$, $\forall\ l,m \in \mathfrak{Q}_{mr}$. Now we replace l by lm in $b[l,hb] = 0$, $\forall\ m \in \mathcal{K}$ and using it, we have

$$bl[m,hb] = 0,\ \forall\ l,m,h \in \mathcal{K}. \tag{50}$$

Replacing l by lr, $\forall\ r \in \mathfrak{A}$ in (50) and using the primeness of $\mathfrak{A}$, then we get either $bl = 0$, which is a contradiction (by using the property of maximal right symmetric ring of quotients) or $[m,hb] = 0$, $\forall\ m,h \in \mathcal{K}$. Now replacing h by lh, $\forall\ l \in \mathcal{K}$ in $[m,hb] = 0$ and using it, we obtain

$$[m,l]hb = 0,\ \forall\ l,m,h \in \mathcal{K}. \tag{51}$$

Again we substitute h by rh, $\forall\ r \in \mathfrak{A}$ in (51), then we get either $hb = 0$ (not possible by above argument) or $[m,l] = 0$, $\forall\ l,m \in \mathcal{K}$. So, $\mathcal{K}$ is commutative, then by Lemma (1), we get $\mathfrak{A}$ is commutative. □

Theorem 3. *Let $\mathfrak{A}$ be a prime ring with char$(\mathfrak{A}) \neq 2$ and $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ be a multiplicative b-generalized derivation associated with non-zero derivation $d : \mathfrak{A} \to \mathfrak{A}$ such that $[\mathfrak{F}(l), \mathfrak{F}(m)] - [l,m] \in \mathfrak{Z}$, $\forall\ l,m \in \mathfrak{A}$, then either $\mathfrak{A}$ is commutative or $\mathfrak{F}(l) = \lambda l$, $\forall\ l \in \mathfrak{A}$ and some $\lambda = \pm 1$.*

Proof. By using *[11, Theorem 1.1]*, we know that multiplicative b-generalized derivation can be written as

$$\mathfrak{F}(l) = \lambda l + \mu(l),\ \forall\ l,m \in \mathfrak{A}, \tag{52}$$

where $\mu : \mathfrak{A} \to \mathfrak{C}$. Substituting l by lm in (52), we get

$$\mathfrak{F}(lm) = \lambda lm + \mu(lm), \ \forall \ l, m \in \mathfrak{A}. \tag{53}$$

Using the definition of multiplicative b-generalized derivation and (52), we have

$$\mathfrak{F}(lm) = \lambda lm + \mu(l)m + bld(m), \ \forall \ l, m \in \mathfrak{A}. \tag{54}$$

Equating (53) and (54), we obtain $\mu(lm) = \mu(l)m + bld(m)$. Since, we know that $\mu(lm) \in \mathfrak{C}$, then we find that

$$\mu(l)m + bld(m) \in \mathfrak{C}, \ \forall \ l, m \in \mathfrak{A}. \tag{55}$$

Commuting the above relation with m, we see that

$$[bld(m), m] = 0, \ \forall \ l, m \in \mathfrak{A}. \tag{56}$$

Since, can extend this result for $\mathfrak{Q}_{mr}$ i.e., $[bld(m), m] = 0$, $\forall \ l \in \mathfrak{Q}_{mr}$ and $m \in \mathfrak{A}$. Now, substituting l by bl and using it, we have

$$[b, m]bld(m) = 0, \ \forall \ l \in \mathfrak{Q}_{mr}, m \in \mathfrak{A}. \tag{57}$$

Using the primeness of $\mathfrak{A}$, we get either $d(m) = 0$ or $[b, m]b = 0$, $\forall \ m \in \mathfrak{A}$. If $d(m) = 0$, substituting these value in (55), we get $\mu(l)m \in \mathfrak{C}$. By Lemma 2 we have either $\mu(l) = 0$ or $m \in \mathfrak{C}$ i.e., $\mathfrak{A}$ is commutative, and if $\mu(l) = 0$ then we have $\mathfrak{F}(l) = \lambda l$. Next, we assume that $[b, m]b = 0$, $\forall \ m \in \mathfrak{A}$, which implies that $[b, m]b = 0$, $\forall \ m \in \mathfrak{Q}_{mr}$. Substituting m by lm and using it, we obtain

$$[b, l]mb = 0, \ \forall \ l, m \in \mathfrak{Q}_{mr}. \tag{58}$$

By the primeness of $\mathfrak{A}$, we get either $b = 0$ or $[b, l] = 0$, $\forall \ l \in \mathfrak{Q}_{mr}$. If $b = 0$, then from (55), we have $\mu(l)m \in \mathfrak{C}$. Using previous argument we get $\mathfrak{F}(l) = \lambda l$ or $\mathfrak{A}$ is commutative. Now, we assume that $[b, l] = 0$, $\forall \ l \in \mathfrak{Q}_{mr}$. From (56), we get

$$b[ld(m), m] + [b, m]ld(m) = 0, \ \forall \ l, m \in \mathfrak{A}. \tag{59}$$

Using $[b, l] = 0$ in (59), we observe that

$$b[ld(m), m] = 0, \ \forall \ l, m \in \mathfrak{A}. \tag{60}$$

Left multiplying (60) from $r \in \mathfrak{A}$ and $[b, l] = 0$ $i.e., bl = lb$, we find that

$$br[ld(m), m] = 0, \ \forall \ l, m, r \in \mathfrak{A}. \tag{61}$$

Using primeness of $\mathfrak{A}$, we get either $b = 0$ which is already discuss or $[ld(m), m] = 0$, $\forall \ l, m \in \mathfrak{A}$. Replacing l by rl, $\forall \ r \in \mathfrak{A}$ and using it, we get

$$[r, m]ld(m) = 0, \ \forall \ l, m, r \in \mathfrak{A}. \tag{62}$$

Since, $\mathfrak{A}$ is prime ring, we get either $[r, m] = 0$ i.e., $\mathfrak{A}$ is commutative or $d(m) = 0$ then we get $\mathfrak{A}$ is commutative or $\mathfrak{F}(l) = \lambda l$. □

Corollary 3. *Let $\mathfrak{A}$ be a prime ring with char$(\mathfrak{A}) \neq 2$ and $\mathfrak{F} : \mathfrak{A} \to \mathfrak{A}$ be a multiplicative b-generalized derivation associated with non-zero derivation $d : \mathfrak{A} \to \mathfrak{A}$ such that $[\mathfrak{F}(l), \mathfrak{F}(m)] - [l, m] = 0$, $\forall\ l, m \in \mathfrak{A}$. Then either $\mathfrak{A}$ is commutative or $\mathfrak{F}(l) = \lambda l$, $\forall\ l \in \mathfrak{A}$ and some $\lambda = \pm 1$.*

Example 3. Let $\mathfrak{A} = \left\{ \begin{bmatrix} 0 & l & m \\ 0 & 0 & n \\ 0 & 0 & 0 \end{bmatrix} \Big| l, m, n \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of integers and $\mathfrak{F}$ and d are maps from $\mathfrak{A} \to \mathfrak{A}$ such that $\mathfrak{F}\left(\begin{bmatrix} 0 & l & m \\ 0 & 0 & n \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & mn \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $d\left(\begin{bmatrix} 0 & l & m \\ 0 & 0 & n \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & l \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then it is easy to verify that $\mathfrak{F}$ is a multiplicative b-generalized derivation associated with derivation d and for any fixed $0 \neq b \in \mathfrak{A}$. Let $\mathcal{K} = \left\{ \begin{bmatrix} 0 & 0 & m \\ 0 & 0 & n \\ 0 & 0 & 0 \end{bmatrix} \Big| m, n \in \mathbb{Z} \right\}$. Here we see that $\mathcal{K}$ is a two-sided ideal of $\mathfrak{A}$ and satisfies all the hypothesis of Theorems 1, 2 and Corollary 1 but $\mathfrak{A}$ is non-commutative. Hence, the primeness of hypothesis is essential in Theorems 1, 2 and Corollary 1.

Conjecture 1. *In Theorems 1, 2 and 3, If we consider semi-prime rings instead of prime rings, then what can we say about the validity of these results?*

Possibility of generalizing the results : The generalization of these results would aim to broaden the types of rings and derivations under consideration, while still obtaining similar structural conclusions about commutativity or linearity. This could involve expanding the class of rings (e.g., from prime to semiprime or from rings with certain commutativity conditions to non-commutative rings with particular symmetries), considering different types of derivations (e.g., higher-order or generalized in more complex ways), or exploring different algebraic identities and conditions that lead to analogous outcomes. Such generalizations would deepen the understanding of how derivations interact with the structure of various algebraic systems, potentially uncovering new patterns and relationships in the broader context of ring theory and its applications.

Applications: These results contribute to the broader understanding of the structure of prime rings and the role of derivations in determining commutativity. They are particularly significant in the context of algebraic studies related to ring theory, where such conditions can simplify the classification and analysis of algebraic systems. Applications might extend to mathematical physics, coding theory, and other areas where algebraic structures are used to model complex systems.

Conclusion

In this article, we characterize the multiplicative b-generalized derivation map on prime rings involving certain differential identities. More, precisely we have seen the commutativity of prime ring $\mathfrak{A}$ in the presence of strong commutative preserving, commuting and centralizing maps. And also, we have given example to show that our condition is not unnecessary.

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