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Analytical investigation of fractional SEIRVQD measles mathematical model

Milad Fahimi, Kazem Nouri*, Leila Torkzadeh

†*Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, Semnan, Iran*
Email(s): M_fahimi@semnan.ac.ir, knouri@semnan.ac.ir, Torkzadeh@semnan.ac.ir

Abstract. In this paper, we construct and formulate a new mathematical model for the spread of epidemic diseases with vaccination especially for Chinese measles. This model including susceptible (S), exposed (E), infected (I), recovered (R), vaccinated (V), quarantined (Q) and died individuals (D) is been studied by applying Caputo fractional derivatives (CFD). We introduce the feasibility region and prove positively invariant property for this region. Then we prove the existence of a unique solution of our fractional measles model. Furthermore, the equilibrium points of the model are presented and the stability analysis of the model is proved based on Lyapunov and Ulam-Hyer criteria. The basic reproduction number (R_0) is calculated by the next generation matrix method in order to demonstrate the level of measles virus invasion. Moreover, numerical simulations including data fitting are performed for different fractional orders to illustrate and validate the efficiency of the proposed model.

Keywords: Epidemic model, fractional SEIRVQD model, transmission simulation, differential equation, Ulam-Hyer stability, measles, Lyapunov stability.

AMS Subject Classification 2020: 92C60, 92-10, 60H10

1 Introduction

Infectious diseases continue to debilitate and cause inconvenience in humans and animals originating from the invasion and growth of germs in the body. In the global complex biological situation, more and more attention is being paid overtime to fundamental specialized studies about infectious diseases such as HIV [5, 30], HBV [6, 38], Ebola [24, 25], Measles [11, 33], COVID-19 [26, 39] and Zika [29, 32].

Mathematical models are essential tools for comprehending the spread of disease transmission. They are also advantageous to control and forecast the severity and potential scope of epidemic diseases. Many mathematical models have been performed to study the dynamics of epidemic disease. To study the

*Corresponding author
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research background overtime, we can refer to the Kermack–McKendrick epidemic model as a pioneer in this field [8]. The authors in [5, 26] analyzed the spread of infectious diseases by employing differential equations in the process of simulation. Some models ((SIR) [7], (SEIRQ) [39], (SIS) [9, 40], (SEIR) [18, 31] and (SIRD) [5, 35]) are based on the idea of categorizing individuals in the category of infected, deceased, susceptible, etc.

We can portray numerous biological problems related to real processes with a higher degree of realism as can be seen in the research [3–5, 14, 26, 34]. It should be mentioned that the application of fractional differential equations creates special properties which do not appear in integer-order differential equations. Articles in which the simulation of the disease outbreak process is carried out applying classical integer order derivatives have some limitations. One of the reasons for using fractional derivatives is the possibility of adapting the order of the fractional differential equation to the real data of the epidemic disease. Therefore, according to the hidden carrying of the measles virus by individuals, the number of infected individuals at time t will affect the number of infected individuals in the future days, and the number of infected people at time t is affected by the number of infected individuals in previous days. Thus, we need to investigate the memory effect in the study of epidemic diseases by Caputo fractional-order differential equations.

In this paper, we construct our model based on seven variables fractional differential equation (FDE) with a network abbreviated name (SEIRVQD). We have divided the proposed model into several mutually exclusive epidemiological compartments: susceptible individuals, exposed individuals, infectious individuals with the infectious agents, recovered individuals, vaccinated individuals against the infectious agents, quarantined individuals for limiting means of transmission and dead individuals due to infection. This paper aims to create a new innovative epidemic model for the transmission of measles virus, taking into account the vaccination and quarantine process which was not studied by the mentioned researchers above. Furthermore, using fractional-order differential equations is the main motivation behind the development of our epidemic model.

The current manuscript is organized and outlined as follows. In Section 2, we present some fundamental definitions and theorems related to the propoed model based on fractional differential equations. In Section 3, we construct and formulate a new mathematical model for the spread of epidemic diseases with vaccination in fractional approach and also the modified form of FDE. Moreover, in Section 4, the theoretical approach of the model and dynamical analysis of solution including positively invariant property of the solution and existence and uniqueness of the solution is presented. Section 5 is devoted to present equilibrium points. Additionally, Section 6 is devoted to present stability analysis based on the Lyapunov and Ulam-Hyer criteria. Finally, in Section 7, numerical simulations including data fitting are performed to validate the theoretical results.

2 Fundamentals and preliminaries

In this section, we express some fundamental definitions and theorems related to the measles disease modeling based on FDE.

Definition 1 ([22]). *The Caputo fractional derivatives of order τ is given by*

$$ {}_0^c\mathscr{D}_t^{\tau}\mathscr{H}(t) = \frac{1}{\Gamma(m-\tau)}\int_0^t (t-x)^{m-\tau-1}\mathscr{H}^{(m)}(x)dx, $$

where $\mathscr{H}$ is defined as $\mathscr{H}:[0,T]\to\mathbb{R}$, $m-1<\tau\leq m$ is the order of fractional derivative operator and $m\in\mathbb{N}$.

Definition 2 ([22]). *The Caputo fractional integral of order τ is given by*

$$ {}_0^c\mathscr{I}_t^{\tau}\mathscr{H}(t)=\frac{1}{\Gamma(\tau)}\int_0^t(t-x)^{\tau-1}\mathscr{H}(x)dx $$

for the fractional integral order τ with $Re(\tau)>0$.

Definition 3. *If $\mathscr{G}(j)$ is the Laplace transform of $\mathscr{H}(t)$, then*

$$ \mathscr{L}[{}_0^c\mathscr{D}_t^{\tau}\mathscr{H}(t),j]=j^{\tau}\mathscr{G}(j)-\sum_{i=0}^{m-1}j^{\tau-i-1}\mathscr{G}^{(i)}(0),\ \ m-1<\tau\leq m,\ m\in\mathbb{N}. $$

Definition 4 ([22]). *The generalized Mittag-Leffler functions are defined as*

$$ E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ (\alpha,\beta>0). $$

Mittag-Leffler functions satisfy the following relation

$$ E_{\alpha,\beta}(z)=zE_{\alpha,\alpha+\beta}(z)+\frac{1}{\Gamma(\beta)}. $$

Theorem 1 ([22]). *Suppose that f is a continuous function such that ${}_0^c\mathscr{D}_t^{\tau}\mathscr{H}$ is also continuous for $\tau\in(0,1]$. Then there exists some $K\in(0,t)$ such that*

$$ \mathscr{H}(t)=\mathscr{H}(0)+\frac{1}{\Gamma(\tau+1)}{}_0^c\mathscr{D}_t^{\tau}\mathscr{H}(K)t^{\tau}. $$

Theorem 2 ([20]). *Assume that g is a locally Lipschitz function defined over domain $\mathscr{B}\subset\mathbb{R}^n$, $f(0)=0$ and $\varpi\subset\mathscr{B}$ is a compact set that is positively invariant. Also, let $V(x)$ be a $\mathbb{C}^1$ fuction defined over $\mathscr{B}$ such that $V'(x)\leq 0$ in ϖ. Then the origin is a globally asymptotically stable equilibrium point of $x'=g(x)$.*

Lemma 1 ([22]). *The Caputo FDE, ${}_{t_0}^c\mathscr{D}_t^{\tau}F(t)=\mathscr{X}(t,F(t))$ with initial value $F(t_0)=F_0$ and fractional order $0<\tau\leq 1$ has equilibrium point at x^* if $\mathscr{X}(t,x^*)=0$.*

Lemma 2 ([22]). *Let $0<a\leq 1$, $y(t)\in C[a,b]$, and ${}_a^C\mathscr{D}_t^{\tau}\in C(a,b]$. Then $y(t)$ is non-increasing for $t\in[a,b]$, if ${}_a^C\mathscr{D}_t^{\tau}\leq 0,\forall t\in(a,b)$. Similarly, $y(t)$ is non-decreasing $\forall t\in(a,b)$, if ${}_a^C\mathscr{D}_t^{\tau}\geq 0$, $a<t<b$.*

3 Epidemic disease model

In this work, we formulate and construct a mathematical epidemic disease model. This model consists of seven variables. In other words, in this model, the population is investigated in seven compartments. Assume S and E refer to the number of susceptible and exposed people and I refers to the symptomatic infected people. Additionally, R denotes recovered people, V denotes vaccinated people, Q refers to

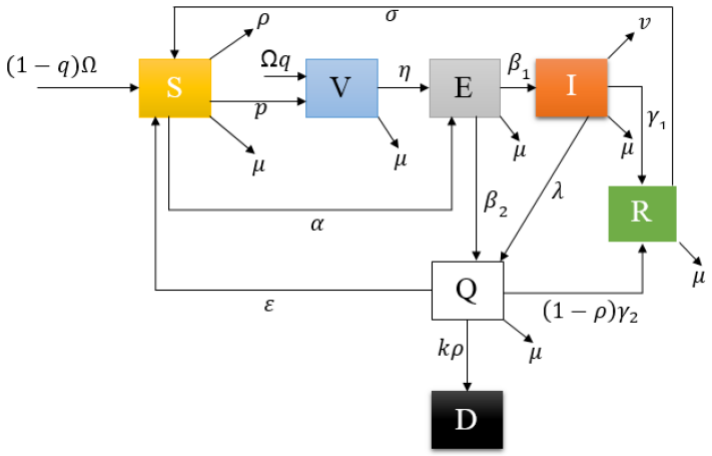

Figure 1: A schematic representation of the disease dynamics.

quarantined people and finally, D refers to dead people. The unit of the variables is according to the number of cases that are considered at time t. Therefore, the resulting epidemic disease transmission model comprising of seven dimensions ordinary differential equation is achieved by:

$$
\begin{aligned}
{}_0^C\mathscr{D}_t^\tau S &= (1-q)\Omega + \delta R + \varepsilon Q - (p+\rho+\mu+\alpha I)S,\\
{}_0^C\mathscr{D}_t^\tau E &= \eta V + \alpha I S - (\beta_1+\beta_2+\mu)E,\\
{}_0^C\mathscr{D}_t^\tau I &= \beta_1 E - (\nu+\gamma_1+\mu+\lambda)I,\\
{}_0^C\mathscr{D}_t^\tau R &= (1-\rho)\gamma_2 Q + \gamma_1 I - (\delta+\mu)R,\\
{}_0^C\mathscr{D}_t^\tau V &= (\Omega q + pS - (\mu+\eta)V),\\
{}_0^C\mathscr{D}_t^\tau Q &= \lambda I + \beta_2 E - (\varepsilon + k\rho + \mu + (1-\rho)\gamma_2)Q,\\
{}_0^C\mathscr{D}_t^\tau D &= k\rho Q,
\end{aligned}
\tag{1}
$$

where ε is the rate of exiting individuals from quarantine situations to the class of susceptible people, k is the average days until death, ρ is the mortality rate emanating from disease invasion, Ω is the recruitment rate which means a constant input of new members into the population per unit of time, q is the fraction of vaccinated individuals who are recruited into the population, p is the fraction of vaccinated susceptible individuals, δ is the rate of change of position from the recovered class to the susceptible class, δ is the rate of reduced immunity in recovered individuals, μ is the natural death rate, ν is the rate of losing immunity for vaccinated individuals, η is vaccine inefficacy, β_1 is the rate of the intensification of the clinical symptoms during quarantine, β_2 is the quarantine rate of exposed individuals, α is the effective contact rate, γ_1 is the recovery rate in symptomatic individuals, γ_2 is the average days until recovery, λ is the quarantine rate of symptomatic individuals. These parameters are all positive. A schematic representation of the disease dynamics can be comprehended in Fig. 1.

3.1 Modified Caputo derivatives for the SEIRVQD measles model

Due to the mismatch of the dimensions of the equations, we modify the system through adding the auxiliary parameter θ. To verify that the right and left sides of the FDE have the same dimensions (time^{-1}), we define the following time consistency transform

$$ {}_0^C\mathscr{D}_t^\tau \to \frac{1}{\theta^{\tau-1}} {}_0^C\mathscr{D}_t^\tau,\ 0<\tau\leq 1. \tag{2} $$

The auxiliary parameter θ demonstrates the fractional time components in the system. This non-local time is called cosmic time [15]. This auxiliary parameter modifies our FDE as follows

$$ \begin{aligned}
\theta^{\tau-1}{}_0^C\mathscr{D}_t^\tau S &= (1-q)\Omega+\delta R+\varepsilon Q-(p+\rho+\mu+\alpha I)S,\\
\theta^{\tau-1}{}_0^C\mathscr{D}_t^\tau E &= \eta V+\alpha IS-(\beta_1+\beta_2+\mu)E,\\
\theta^{\tau-1}{}_0^C\mathscr{D}_t^\tau I &= \beta_1 E-(\nu+\gamma_1+\mu+\lambda)I,\\
\theta^{\tau-1}{}_0^C\mathscr{D}_t^\tau R &= (1-\rho)\gamma_2 Q+\gamma_1 I-(\delta+\mu)R,\\
\theta^{\tau-1}{}_0^C\mathscr{D}_t^\tau V &= (\Omega q+pS-(\mu+\eta)V),\\
\theta^{\tau-1}{}_0^C\mathscr{D}_t^\tau Q &= \lambda I+\beta_2 E-(\varepsilon+k\rho+\mu+(1-\rho)\gamma_2)Q,\\
\theta^{\tau-1}{}_0^C\mathscr{D}_t^\tau D &= k\rho Q,
\end{aligned} \tag{3} $$

where $0<\tau\leq 1$ is non-integer order of FDE (3) and $t\in[0,T]$. As can be seen, the fractional operator of our model has been taken in the sense of Caputo as a great tool that can be implemented to describe real-life biological phenomena with the so-called memory effect.

4 Theoretical results and dynamical analysis of solution

In this section, we will introduce the feasibility region and study the existence and uniqueness of the solution of the proposed model (3).

4.1 Positively invariant solution

Theorem 3. *The solution of fractional model* (3) *with non-negative initial conditions* $(S(0),E(0),I(0),R(0),V(0),Q(0),D(0))$ *will be positively invariant for every* $t>0$ *in the closed set*

$$ \Pi=\{(S,E,I,R,V,Q,D)\in\mathscr{R}_+^7 \mid N=S+E+I+R+V+Q+D\leq\frac{(1-q)\Omega}{\mu+P+\rho}\}. $$

Proof. Assume $(S(0),E(0),I(0),R(0),V(0),Q(0),D(0))$ is the initial vector with positive components. According to the idea of [23], we compute ${}_0^cD_t^\tau S$ by setting $S=0$ in the first equation of model (3). Similarly, we compute ${}_0^cD_t^\tau E\mid_{E=0}$, ${}_0^cD_t^\tau I\mid_{I=0}$, ${}_0^cD_t^\tau R\mid_{R=0}$, ${}_0^cD_t^\tau V\mid_{V=0}$, ${}_0^cD_t^\tau Q\mid_{Q=0}$ and ${}_0^cD_t^\tau D\mid_{D=0}$ based on the model (3). So, for all $t\geq 0$, we obtain

$$ \begin{aligned}
\theta^{\tau-1}{}_0^c\mathscr{D}_t^\tau S\mid_{S=0} &= (1-q)\Omega+\delta R+\varepsilon Q\geq 0,\\
\theta^{\tau-1}{}_0^c\mathscr{D}_t^\tau E\mid_{E=0} &= \eta V+\tau IS\geq 0,
\end{aligned} $$

$$\theta^{\tau-1}{}_0^c\mathscr{D}_t^\tau I\,|_{I=0}=\beta_1 E\geq 0,$$
$$\theta^{\tau-1}{}_0^c\mathscr{D}_t^\tau R\,|_{R=0}=(1-\rho)\gamma_2 Q+\gamma_1 I\geq 0,$$
$$\theta^{\tau-1}{}_0^c\mathscr{D}_t^\tau V\,|_{V=0}=\Omega q+PS\geq 0,$$
$$\theta^{\tau-1}{}_0^c\mathscr{D}_t^\tau Q\,|_{Q=0}=\lambda I+\beta_2 E\geq 0$$
$$\theta^{\tau-1}{}_0^c\mathscr{D}_t^\tau D\,|_{D=0}=K\rho Q\geq 0.$$

Now we consider the following auxiliary FDE:

$$\begin{aligned}
{}_0^c\mathscr{D}_t^\tau S(t)&=\theta^{1-\tau}\left[(1-q)\Omega+\delta R+\varepsilon Q-(P+\rho+\mu+\alpha I)S\right]+\frac{1}{e},\\
{}_0^c\mathscr{D}_t^\tau E&=\theta^{1-\tau}\left[\eta V+\alpha IS-(\beta_1+\beta_2+\mu)E\right]+\frac{1}{e},\\
{}_0^c\mathscr{D}_t^\tau I&=\theta^{1-\tau}\left[\beta_1 E-(\nu+\gamma_1+\mu+\lambda)I\right]+\frac{1}{e},\\
{}_0^c\mathscr{D}_t^\tau R&=\theta^{1-\tau}\left[(1-\rho)\gamma_2 Q+\gamma_1 I-(\delta+\mu)R\right]+\frac{1}{e},\\
{}_0^C\mathscr{D}_t^\tau V&=\theta^{1-\tau}\left[(\Omega q+pS-(\mu+\eta)V)\right]+\frac{1}{e},\\
{}_0^C\mathscr{D}_t^\tau Q&=\theta^{1-\tau}\left[\lambda I+\beta_2 E-(\varepsilon+k\rho+\mu+(1-\rho)\gamma_2)Q\right]+\frac{1}{e},\\
{}_0^C\mathscr{D}_t^\tau D&=\theta^{1-\tau}\left[k\rho Q\right]+\frac{1}{e},
\end{aligned}$$

where $e\in N$. We assume that there is an achievable time sample such that the solutions are negative and we consider

$$t_k=\inf\{t>0\mid(S_1(t),E_1(t),I_1(t),R_1(t),V_1(t),Q_1(t),D_1(t))\notin(\mathbb{R}_0^+)^4\}.$$

Then there exists $(S_1(t_k),E_1(t_k),I_1(t_k),R_1(t_k),V_1(t_k),Q_1(t_k),D_1(t_k))\notin(\mathbb{R}_0^+)^4$, and one of the components is zero. So, we assume $S_1(t_k)=0$. On the other hand, ${}_0^c\mathscr{D}_t^\tau S_1(t_k)$ is positive and continuous. According to the Theorem 1 and Lemma 2 we obtain

$$\exists\,\varepsilon'>0:S_1\left([t_k,t_k+\varepsilon)\right)\subseteq\mathbb{R}_0^+,$$

which demonstrate that $S_1\geq 0$, therefore this statement (assertion) conducts to the contradiction. This procedure can be implemented for E_1,I_1,R_1,V_1,Q_1 and D_1. Thus for all $t_e\geq 0$ as $e\to\infty$; the solution will be nonnegative. Now we consider the sum of the population components as follows

$$\begin{aligned}
\theta^{\tau-1}\mathscr{D}_t^\tau(S+E+I+R+V+Q)&=\Omega-\mu(S+E+I+R+V+Q)-\rho(S+\kappa Q)-\nu I\\
&\leq\Omega-\mu(S+E+I+R+V+Q).
\end{aligned}$$

Using the Laplace transform (3) and using $\mathbb{E}_\tau$ and $\mathbb{E}_{\tau\tau}$ as Mittag-Leffler functions mentioned in Definition 4, we achieve

$$(S+E+I+R+V+Q)(t)=(S+E+I+R+V+Q)(0)E_\tau\left(-\mu\theta^{1-\tau}t^\tau\right)$$

$$+\int_0^t \Omega\theta^{1-\tau}x^{\tau-1}\mathbb{E}_{\tau,\tau}\left(-\mu\theta^{1-\tau}x^{\tau}\right)dx.$$

By simplifying the above equation, we obtain

$$\begin{aligned}(S+E+I+R+V+Q)(t) &= (S+E+I+R+V+Q)(0)\mathbb{E}_{\tau}\left(-\mu\theta^{1-\tau}t^{\tau}\right)\\ &\quad+\int_0^t \Omega\theta^{1-\tau}x^{\tau-1}\sum_{i=0}^{\infty}\frac{(-1)^i\mu^i\theta^{i(1-\tau)}x^{i\tau}}{\Gamma(i\tau+\tau)}dx\\ &= \frac{\Omega}{\mu}+\mathbb{E}_{\tau}\left(-\mu\theta^{1-\tau}t^{\tau}\right)\left((S+E+I+R+V+Q)(0)-\frac{\Omega}{\mu}\right).\end{aligned}$$

Thus if $(S+E+I+R+V+Q)(0)\leq\frac{\Omega}{\mu}$, then for $t>0$ we conclude $(S+E+I+R+V+Q)(t)\leq\frac{\Omega}{\mu}$. By considering the feasible region

$$\Pi=\{(S,I,E,R,V,Q,D)\in R_+^7 \mid N=S+E+I,R+V+Q+D\leq\frac{(1-q)\Omega}{\mu+p+\rho}\leq\frac{\Omega}{\mu}\},\tag{4}$$

and according to Lemma 2 we can conclude that the set Π is positively invariant with respect the model (3). □

4.2 Existence and uniqueness of the solution

The proposed epidemic disease model (3) is a dynamical system on the biologically feasible region (Π), introduced in (4). In this section, the existence of a unique solution of model (3) will be proved. First, we define the following kernels

$$\begin{aligned}&\theta^{\tau-1}{}_0^C\mathscr{D}_t^{\tau}S(t)=\phi_1(t,S(t)),\ \theta^{\tau-1}{}_0^C\mathscr{D}_t^{\tau}V(t)=\phi_2(t,V(t)),\\ &\theta^{\tau-1}{}_0^C\mathscr{D}_t^{\tau}E(t)=\phi_3(t,E(t)),\ \theta^{\tau-1}{}_0^C\mathscr{D}_t^{\tau}I(t)=\phi_4(t,I(t)),\\ &\theta^{\tau-1}{}_0^C\mathscr{D}_t^{\tau}R(t)=\phi_5(t,R(t)),\ \theta^{\tau-1}{}_0^C\mathscr{D}_t^{\tau}Q(t)=\phi_6(t,Q(t)),\\ &\theta^{\tau-1}{}_0^C\mathscr{D}_t^{\tau}D(t)=\phi_7(t,D(t)).\end{aligned}$$

By applying fractional integral operators to the both sides of the above equation, we obtain

$$G(t)-G(0)=\frac{\theta^{1-\tau}}{\Gamma(\tau)}\int_0^t\phi_1(\zeta,G(\zeta))(t-\zeta)^{\tau-1}d\zeta,$$

where G is a delegate of the seven variables of the proposed model. It is necessary to mention that kernels $\phi_1,\phi_2,...,\phi_7$ can be defined for variables S,V,E,I,R,Q and D, respectively. Now, in the following theorem, we will consider the Lipshitz property and contraction condition for the kernels $\phi_1,\phi_2,...,\phi_7$.

Theorem 4. *The Lipschitz and contraction property are established for the kernels $\phi_1,\phi_2,...,\phi_7$, if $0\leq\mu_i<1$ for $i=1,2,...,7$, where $\mu_1=\alpha b_1+p+\rho+\mu$, $\mu_2=\mu+n$, $\mu_3=\beta_1$, $\mu_4\nu+\gamma_1+\mu+\lambda$, $\mu_5=\delta+\mu$ and $\mu_6=\varepsilon+k\rho+\mu+(1-\rho)\gamma_2$ are the Lipschitz-contraction coefficients.*

Proof. First, we have

$$\begin{aligned}\|\phi_1(t,S(t))-\phi_1(t,S_1(t))\| &= \|-(p+\rho+\mu+\alpha I)(S(t)-S_1(t))\| \\ &\leq (\alpha\|I\|+P+\rho+\mu)\|S(t)-S_1(t)\|.\end{aligned}$$

By considering $\mu_1=\alpha b_1+p+\rho+\mu$, where $\|I(t)\|\leq b_1$ is a bounded function, we obtain $\|\phi_1(t,S(t))-\phi_1(t,S_1(t))\|\leq\mu_1\|S(t)-S_1(t)\|$. Similarly, it can be written for the other kernels, as follows

$$\|\phi_2(t,V(t))-\phi_2(t,V_1(t))\| = \|-(\mu+n)(V(t)-V_1(t))\| \leq (\mu+\eta)\|V(t)-V_1(t)\|.$$

Having in mind that $\mu_2=\mu+n$, we obtain $\|\phi_2(t,S(t))-\phi_2(t,S_1(t))\|\leq\mu_2\|V(t)-V_1(t)\|$. In the same way, we obtain

$$\|\phi_3(t,E(t))-\phi_3(t,E_1(t))\| = \|\beta_1(E(t)-E_1(t))\| \leq \beta_1\|E(t)-E_1(t)\|.$$

On the other hand, from $\mu_3=\beta$, we obtain $\|\phi_3(t,E(t))-\phi_3(t,E_1(t))\|\leq\mu_3\|E(t)-E_1(t)\|$. Similarly, we obtain

$$\begin{aligned}\|\phi_4(t,I(t))-\phi_4(t,I_1(t))\| &= \|-(\nu+\gamma_1+\mu+\lambda)(I(t)-I_1(t))\| \\ &\leq (\nu+\gamma_1+\mu+\lambda)\|I(t)-I_1(t)\|.\end{aligned}$$

It follows from $\mu_4=\nu+\gamma_1+\mu+\lambda$, that $\|\phi_4(t,I(t))-\phi_4(t,I_1(t))\|\leq\mu_4\|I(t)-I_1(t)\|$. Similarly, we see that

$$\|\phi_5(t,R(t))-\phi_5(t,R_1(t))\| = \|-(\delta+\mu)(R(t)-R_1(t))\| \leq (\delta+\mu)\|R(t)-R_1(t)\|.$$

By considering $\mu_5=\delta+\mu$, we obtain $\|\phi_5(t,R(t))-\phi_5(t,R_1(t))\|\leq\mu_5\|R(t)-R_1(t)\|$. Similarly, we get

$$\begin{aligned}\|\phi_6(t,Q(t))-\phi_6(t,Q_1(t))\| &= \|-(\varepsilon+k\rho+\mu+(1-\rho)\gamma_2)(Q(t)-Q_1(t))\| \\ &\leq (\varepsilon+k\rho+\mu+(1-\rho)\gamma_2)\|Q(t)-Q_1(t)\|.\end{aligned}$$

By considering $\mu_6=\varepsilon+k\rho+\mu+(1-\rho)\gamma_2$, we obtain $\|\phi_6(t,Q(t))-\phi_6(t,Q_1(t))\|\leq\mu_6\|Q(t)-Q_1(t)\|$. Similarly, we deduce that

$$\|\phi_7(t,D(t))-\phi_7(t,D_1(t))\| = \|k\rho Q-k\rho Q\| = 0.$$

By considering $\mu_7=0$, the condition for establishing Lipshitz-contraction property will be provided. Thus the kernels $\phi_2,\phi_3,...,\phi_7$ are also contraction because $0\leq\mu_i<1$ for $i=2,3,...,7$. □

To present the existence analysis, we construct the following recursive formulas. Therefore we suppose the following recursive formulas concerning the system (3):

$$\psi_{1,i}=S_i(t)-S_{i-1}(t)=\frac{\theta^{1-\tau}}{\Gamma(\tau)}\int_0^t\Big(\phi_1(\chi,S_{n-1}(\chi))-\phi_1(\chi,S_{n-2}(\chi))\Big)d\chi.$$

By taking the norm of $\psi_{1,i}(t)$ we obtain

$$\|\psi_{1,i}(t)\|=\|S_i(t)-S_{i-1}(t)\|=\|\frac{\theta^{1-\tau}}{\Gamma(\tau)}\int_0^t(\phi_1(\chi,S_{n-1}(\chi))-\phi_1(\chi,S_{n-2}(\chi)))(t-\chi)^{\tau-1}d\chi\|$$

$$\leq \frac{\theta^{1-\tau}}{\Gamma(\tau)} \int_0^t \|(\phi_1(\chi, S_{n-1}(\chi)) - \phi_1(\chi, S_{n-2}(\chi))(t-\chi)^{\tau-1}\| d\chi. \tag{5}$$

Similarly, the norms of $\psi_{2,i}(t), \psi_{3,i}(t), ..., \psi_{7,i}(t)$ can be defined for variables E, I, R, V, Q and D respectively.

Theorem 5. *Model (3) has a solution if* $0 \leq \frac{\theta^{1-\tau}}{\Gamma(\tau)} \mathscr{T} \mu_i < 1$ *holds, where* $\mathscr{T}$ *is a time instant such that* $\mathscr{T} > 0$.

Proof. By the Lipschitz condition proved in Theorem 4 and recursive formulas (5), we get

$$\|\psi_{1,i}(t)\| \leq \frac{\theta^{1-\tau}}{\Gamma(\tau)} \mu_1 \int_0^1 \|\psi_{1,i-1}(\chi)\| d\chi.$$

This property can be similarly extended to to $\psi_{2,i}(t), \psi_{3,i}(t), \psi_{4,i}(t), \psi_{5,i}(t), \psi_{6,i}(t)$ and $\psi_{7,i}(t)$. According to the condition $0 \leq \frac{\theta^{1-\tau}}{\Gamma(\tau)} \mathscr{T} \mu_i < 1$, we conclude

$$\begin{aligned} \|\psi_{1,i}(t)\| &\leq \frac{\theta^{1-v}}{\Gamma(\tau)} \mu_1 \mathscr{T} \mu_1 \int_0^t \|\psi_{1,i-1}(\chi)\| d\chi \leq \Big(\frac{\theta^{1-\tau}}{\Gamma(\tau)} \mu_1 \mathscr{T}\Big)^2 \|S_i(0)\| \int_0^t \|\phi_{1,i-2}\| \\ &\leq \cdots \leq \|S_i(0)\| \Big(\frac{\theta^{1-v}}{\Gamma(\tau)} \mu_1 \mathscr{T}\Big)^i, \end{aligned}$$

where $S_n, V_n, E_n, I_n, R_n, V_n, Q_n$ and D_n can be defined as $S_n(t) = \sum_{i=1}^{n} \psi_{1,i}(t)$, $V_n(t) = \sum_{i=1}^{n} \psi_{2,i}(t)$, $E_n(t) = \sum_{i=1}^{n} \psi_{3,i}(t)$, $I_n(t) = \sum_{i=1}^{n} \psi_{4,i}(t)$, $R_n(t) = \sum_{i=1}^{n} \psi_{5,i}(t)$, $Q_n(t) = \sum_{i=1}^{n} \psi_{6,i}(t)$, $D_n(t) = \sum_{i=1}^{n} \psi_{7,i}(t)$. This property can be similarly extended to $\psi_{2,i}(t), \psi_{3,i}(t), \psi_{4,i}(t), \psi_{5,i}(t), \psi_{6,i}(t)$ and $\psi_{7,i}(t)$. Therefore the system has a continuous solution. Now we define the function

$$U_{1,j}(t) = \frac{\theta^{1-\tau}}{\Gamma(\tau)} \int_0^t \Big(\phi_1(\eta, S(\eta)) - \phi_1(\eta, S_{j-1}(\eta))\Big) d\eta.$$

So, we obtain

$$\begin{aligned} &S(t) - S(0) = S_j(t) - U_{1,j}(t), \quad V(t) - V(0) = V_j(t) - U_{2,j}(t), \\ &E(t) - E(0) = E_j(t) - U_{3,j}(t), \quad I(t) - I(0) = I_j(t) - U_{4,j}(t), \\ &R(t) - R(0) = R_j(t) - U_{5,j}(t), \quad Q(t) - Q(0) = Q_j(t) - U_{6,j}(t), \\ &D(t) - D(0) = D_j(t) - U_{7,j}(t). \end{aligned}$$

As a result, we achieve

$$\begin{aligned} \|U_{1,j}(t)\| &= \|\frac{\theta^{1-\tau}}{\Gamma(\tau)} \int_0^t \Big(\phi_1(\eta, S(\eta)) - \phi_2(\eta, S_{j-1}(\eta))\Big) d\eta\| \\ &\leq \frac{\theta^{1-\tau}}{\Gamma(\tau)} \int_0^t \|\phi_1(\eta, S(\eta)) - \phi_2(\eta, S_{j-1}(\eta))\| d\eta \leq \frac{\theta^{1-\tau}}{\Gamma(\tau)} \mu_1 \|S(t) - S_{j-1}(t)\|. \end{aligned}$$

We repeat the method, so we achieve

$$\|U_{1,j}(t)\| \leq \left(\frac{\theta^{1-\tau}}{\Gamma(\tau)}t\right)^{j+1}\mu_1^{j+1}.$$

By applying $\mathscr{T}$ we have

$$\|U_{1,j}(t)\| \leq \left(\frac{\theta^{1-\tau}}{\Gamma(\tau)}\mathscr{T}\right)^{j+1}{\mu_1}^{j+1}.$$

So, $\lim_{j\to\infty} U_{1,j}(t) = 0$. Similarly, we have $\lim_{j\to\infty} U_{i,j}(t) = 0$ for $i = 2,3,\ldots,7$. □

Theorem 6. *If the inequality*

$$1 - \frac{\theta^{1-\tau}}{\Gamma(\tau)}\mathscr{T}\mu_i > 0$$

holds, then the solution of model (3) is unique.

Proof. We assume that another solutions $S_2, V_2, E_2, I_2, R_2, Q_2$ and D_2 exist for system (3). So, we can write $S_2(t) - S_1(t) = \frac{\theta^{1-\tau}}{\Gamma(\tau)}\int_0^t \Big(\phi_2(\eta, S_2(\eta)) - \phi_1(\eta, S_1(\eta))\Big)d\eta$. According to the Lipschitz property, we take the norm and achieve

$$\|S_2(t) - S_1(t)\| \leq \frac{\theta^{1-\tau}}{\Gamma(\tau)}\mu_1\mathscr{T}\|S_2(t) - S_1(t)\|,$$

then $\|S_2(t) - S_1(t)\|\Big(1 - \frac{\theta^{1-\tau}}{\Gamma(\tau)}\mu_1\mathscr{T}\Big) \leq 0$, due to the condition $1 - \frac{\theta^{1-v}}{\Gamma(\tau)}\mu_1\mathscr{T} > 0$, we conclude that

$$\|S_2(t) - S_1(t)\|\Big(1 - \frac{\theta^{1-\tau}}{\Gamma(\tau)}\mu_1\mathscr{T}\Big) > 0.$$

Additionally, we can conclude $\|S_2(t) - S_1(t)\| = 0$. Thus $S_1(t) = S_2(t)$. Similarly we can conclude that $E_1 = E_2, I_1 = I_2, R_1 = R_2, Q_1 = Q_2, V_1 = V_2$ and $D_1 = D_2$. □

5 Equilibrium points

To obtain equilibrium points, the derivatives must be set to zero. Due to the existence of fractional derivatives operators and according to Lemma 1, we set the fractional derivatives to zero and we achieve equilibrium points of our proposed biological model by establishing the following system:

$${}_0^C\mathscr{D}_t^\tau S(t) = {}_0^C\mathscr{D}_t^\tau E(t) = {}_0^C\mathscr{D}_t^\tau I(t) = {}_0^C\mathscr{D}_t^\tau R(t) = {}_0^C\mathscr{D}_t^\tau V(t) = {}_0^C\mathscr{D}_t^\tau Q(t) = {}_0^C\mathscr{D}_t^\tau D(t) = 0.$$

In the first step, we investigate disease-free equilibrium (DFE) in which $I = E = R = V = Q = 0$. Therefore, the situation of the studied population sample will be immune and disease-free at this point. By simplifying equations we achieve the point $E_0 = (S_0^*, 0, 0, 0, 0, 0, D_0^*)$. The second equilibrium point is $E_1 = (S_1^*, E_1^*, I_1^*, R_1^*, V_1^*, 0, 0)$ such that

$$S_1^* = \frac{(1-q)\Omega + \delta R}{p + \rho + \mu + \alpha I},\quad E_1^* = \frac{\eta V + \alpha IS}{\beta_1 + \beta_2 + \mu},\quad I_1^* = \frac{\beta_1 E}{V + \gamma_1 + \mu + \lambda},\quad R_1^* = \frac{\gamma_1 I}{\delta + \mu},\quad V_1^* = \frac{\Omega q + PS}{\mu + \eta}.$$

At this point, the process of vaccination and immunity continuity of the population is complete and mortality and quarantine rates originating from infection are zero. The third equilibrium point is $E_2 = (0, E_2^*, I_2^*, 0, V_2^*, 0, 0)$ where

$$E_2^* = \frac{\eta V}{\beta_1 + \beta_2 + \mu}, \quad I_2^* = \frac{\beta_1 E}{V + \gamma_1 + \mu + \lambda}, \quad V_2^* = \frac{\Omega q}{\mu + \eta}.$$

At this point, the susceptible population is considered to be zero.

6 Stability analysis of the model

In this section, we investigate the stability analysis of the fractional model (3) based on Ulam-Hyers and Lyapunov criteria.

6.1 Ulam-Hyer stability

We define $g(t, y(t))$ such that the following inequality holds

$$\|{}_0^c D_t^\tau y(t) - g(t, y(t))\| \leq \varepsilon, \quad \forall t \in [o, T]. \tag{6}$$

Now $\bar{y} \in \Pi$ is solution of (3) if and only if there is $h \in \Pi$ such that [19, 36]:

$$|h(t)| \leq \varepsilon, \tag{7}$$

$${}_0^c D_t^\tau \bar{y}(t) = g(t, \bar{y}(t)) + h(t), \quad \forall t \in [0, T]. \tag{8}$$

Definition 5. *If there exists a constant $\xi_g > 0$ such that for any $\varepsilon > 0$, and any $\overline{\mathscr{J}}(t)$ satisfying (6), then the SEIRVQD model (3) comprises a solution $\mathscr{J}(t)$ satisfying*

$$\|\overline{\mathscr{J}}(t) - \mathscr{J}(t)\| \leq \varepsilon \cdot \xi_g, \quad \forall t \in [0, T].$$

Theorem 7. *Assume $|h(t)| \leq \varepsilon$ and ${}_o^c D_t^\tau = g(t, \mathscr{J}(t)) + h(t)$ holds. Then the SEIRVQD model has Hyers-Ulam stability on $[0, T]$, if $\Gamma(\tau + 1) > T^\tau h \theta^{1-\tau}$ holds.*

Proof. Due to the Theorem 6, we assume that $\overline{\mathscr{J}}(t)$ is a unique solution of the proposed fractional model (3). Furthermore we let $\overline{\mathscr{J}}(t)$ satisfies (6). Now, apply the fractional Caputo integral to both sides of (8), that gives

$$\overline{\mathscr{J}}(t) = \overline{\mathscr{J}}(0) + \frac{\theta^{1-\tau}}{\Gamma(\tau)} \int_0^t (t - \zeta)^{\tau - 1} g(\zeta, \overline{\mathscr{J}}(\zeta)) d\zeta + \frac{\theta^{1-\tau}}{\Gamma(\tau)} \int_0^t (t - \zeta)^{\tau - 1} h(\zeta) d\zeta, \quad \forall t \in [0, T].$$

By mentioned condition (7), we get

$$\|\overline{\mathscr{J}}(t) - \overline{\mathscr{J}}(0) - \frac{\theta^{1-\tau}}{\Gamma(\tau)} \int_0^t (t - \zeta)^{\tau - 1} g(\zeta, \overline{\mathscr{J}}(\zeta)) d\zeta\| \leq \frac{\varepsilon \theta^{1-\tau}}{\Gamma(v)} \int_0^t (t - \zeta)^{\tau - 1} d\zeta.$$

So, we have

$$\|\overline{\mathscr{J}}(t) - \overline{\mathscr{J}}(0) - \frac{\theta^{1-\tau}}{\Gamma(\tau)} \int_0^t (t - \zeta)^{\tau - 1} g(\zeta, \overline{\mathscr{J}}(\zeta)) d\zeta\| \leq \frac{\varepsilon T^v \theta^{1-\tau}}{v\Gamma(\tau)} = \frac{\varepsilon T^\tau \theta^{1-\tau}}{\Gamma(\tau + 1)}, \quad \forall t \in [0, T].$$

So, we have

$$\begin{aligned}\|\overline{\mathscr{J}}(t)-\mathscr{J}(t)\| &= \|\overline{\mathscr{J}}(t)-\mathscr{J}(0)-\frac{\theta^{1-\tau}}{\Gamma(v)}\int_0^t(t-\zeta)^{\tau-1}g(\zeta,\mathscr{J}(\zeta))d\zeta\|\\ &\leq \|\overline{\mathscr{J}}(t)-\mathscr{J}(0)-\frac{\theta^{1-\tau}}{\Gamma(\tau)}\int_0^t(t-\zeta)^{\tau-1}g(\zeta,\overline{\mathscr{J}}(\zeta))d\zeta\|\\ &\quad+\|\frac{\theta^{1-\tau}}{\Gamma(\tau)}\int_0^t(t-\zeta)^{\tau-1}\left(g(\zeta,\overline{\mathscr{J}}(\zeta))-g(\zeta,\mathscr{J}(\zeta))\right)d\zeta\|.\end{aligned}$$

According to the Lipzchits property, we achieve

$$\|\overline{\mathscr{J}}(t)-\mathscr{J}(t)\| \leq \frac{\varepsilon T^{\tau}\theta^{1-\tau}}{\Gamma(\tau+1)}+\frac{T^{\tau}h\theta^{1-\tau}}{v\Gamma(\tau+1)}\|\overline{\mathscr{J}}(t)-\mathscr{J}(t)\|.$$

So, we obtain

$$\|\overline{\mathscr{J}}(t)-\mathscr{J}(t)\|\left(1-\frac{T^{\tau}h\theta^{1-\tau}}{\Gamma(\tau+1)}\right)\leq\frac{\varepsilon T^{\tau}h\theta^{1-\tau}}{\Gamma(\tau+1)}.$$

So, we get

$$\|\overline{\mathscr{J}}(t)-\mathscr{J}(t)\|\leq\frac{\varepsilon T^{\tau}h\theta^{1-\tau}}{\Gamma(\tau+1)-T^{\tau}h\theta^{1-\tau}}.$$

By assuming

$$\xi_g=\frac{T^{\tau}h\theta^{1-\tau}}{\Gamma(\tau+1)-T^{\tau}h\theta^{1-\tau}},$$

and using definition 7 we obtain $\|\overline{\mathscr{J}}(t)-\mathscr{J}(t)\|\leq\xi_g\varepsilon$. Therefore, we conclude that model (3) has Ulam-Hyers stability. □

6.2 Lyapunov stability of fractional model

The DFE of system (3) is $E_0=(S_0^*,0,0,0,0,0,D_0^*)$ at which the population remains in the absence of disease. In this point, S_0^* and D_0^* are defined as follows

$$S_0^*=\frac{(1-q)\Omega}{p+\rho+\mu},\quad D_0^*=N-\frac{(1-q)\Omega}{p+\rho+\mu}.$$

Now, we will compute the basic reproduction number (R_0) using the next generation matrix method (NGM) [12] and computing the matrix FV^{-1} as follows:

$$FV^{-1}=\frac{1}{\mu+p+\rho}\begin{bmatrix}-\alpha I & 0 & \frac{-\alpha S(\mu+p+\rho)}{\lambda+\mu+\delta_1+V}-\frac{\alpha I\delta\gamma_1}{(\mu+\delta)(\mu+\delta_1+V+\lambda)} & \frac{-\alpha I\delta}{(\mu+\delta)}\\ -\alpha I & 0 & \frac{-\alpha S(\mu+p+\rho)}{\lambda+\mu+\delta_1+V}-\frac{\alpha I\delta\gamma_1}{(\mu+\delta)(\mu+\delta_1+V+\lambda)} & \frac{-\alpha I\delta}{(\mu+\delta)}\\ 0 & \frac{\beta_1(\mu+p+\rho)}{\mu+\beta_1+\beta_2} & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

where F is a non-negative matrix and V is a non-singular matrix, defined in [12]. Considering the spectral radius definition of R_0 as $R_0 = \rho'(FV^{-1})$ and implementing DFE point $E_0 = (S_0^*, 0, 0, 0, 0, 0, D_0^*)$, we achieve

$$R_0 = \frac{\sqrt{-\alpha S_0^* \beta_1 \big((p^2+\rho^2+2\mu p+2\mu\rho+2p\rho)(\mu+\delta)+\mu(\delta+\mu^2)\big)}}{\mu+p+\rho}. \tag{9}$$

Theorem 8. *Assume that $h \in C([t_0,+\infty)\times\mathbb{R},\mathbb{R})$ is a function and $h(t,0)\equiv 0$. There exists a constant $\iota > 0$, a function $\varpi \in \iota([t_0,+\infty),S_\iota)$ with $\varpi(1,0)=0$ and for any initial value $x_0 \in S_\iota$, $x(t)=x(t;t_0,x_0)\in S_\iota$ for $t \geq t_0$. Furthermore, assume that $g(||x||) \leq \varpi(t,x) \leq f(||x||)$ for $t > t_0$, $x \in S_\iota$ where $f,g \in \mathscr{K}$ and the generalized proportional Caputo fractional derivative [1] has the property*

$$^{C}_{0}\mathscr{D}^{\tau}_{t}(\cdot,x(\cdot))(t) \leq h(t,\varpi(t,x(t))),\ t \geq t_0.$$

Then, the solution of FDE has Lyapunov stability, where Lyapunov functions are satisfied on the ball S_ι, x is strictly increasing and the sets $\mathscr{K}$ and S_ι are defined as follows

$$\mathscr{K} = \{x \in C([0,+\infty),[0,+\infty))\},\ \ S_\iota = \{x \in \mathbb{R}^n : ||x|| \leq \iota\}. \tag{10}$$

Proof. We use the following Lyapunov function to prove the global stability

$$L = f_1 S + f_2 V + f_3 E + f_4 I + f_5 R + f_6 Q + f_7 D,$$

where $f_i > 0$ for $i = 1,2,\ldots,7$. Additionally, $f_i; i = 1,2,\ldots,7$ are constant functions that will be introduced later. By applying Caputo fractional differential operator, we obtain

$$^{c}_{0}\mathscr{D}_t L = {}^{c}_{0}\mathscr{D}^{\tau}_{t} S f_1 + {}^{c}_{0}\mathscr{D}^{\tau}_{t} V f_2 + {}^{c}_{0}\mathscr{D}^{\tau}_{t} E f_3 + {}^{c}_{0}\mathscr{D}^{\tau}_{t} I f_4 + {}^{c}_{0}\mathscr{D}^{\tau}_{t} R f_5 + {}^{c}_{0}\mathscr{D}^{\tau}_{t} Q f_6 + {}^{c}_{0}\mathscr{D}^{\tau}_{t} D f_7.$$

We derive the Lyapunov function and substitute them into the equations of the main model (3). Therefore, by simplification and classification, we obtain

$$\begin{aligned}
^{c}_{0}\mathscr{D}_t L =& \Big(f_1\delta - f_5(\delta+\mu)\Big)R + \Big(f_1\varepsilon + f_5(1-\rho)\delta_2 - f_6(\varepsilon + k\rho + \mu + (1-\rho)\gamma_2) + f_7 k\rho\Big)Q \\
&+ \Big(f_2 p - f_1(p+\rho+\mu+\alpha I) + f_3\alpha I\Big)S + \Big(-f_3(\beta_1+\beta_2+\mu) + f_4\beta_1 + f_6\beta_2\Big)E \\
&+ \Big(-f_2(\mu+\eta) + f_3\eta\Big)V + \Big(-f_4(V+\gamma_1+\mu+\lambda) + f_5\gamma_1 + f_6\lambda\Big)I + \Big(f_1(1-q)\Omega \\
&+ f_2\Omega q\Big).
\end{aligned}$$

Now we choose $f_1 = f_2 = f_3 = f_4 = \delta+\mu, f_5 = f_6 = f_7 = \delta$. So, we achieve

$$\begin{aligned}
^{c}_{0}\mathscr{D}_t L =& \mu(\varepsilon-\delta)Q - (\delta+\mu)(\rho+\mu)S - (\delta+\mu)\mu E - (\delta+\mu)\mu V \\
&- \Big(\delta(V+\mu) + \mu(V+\mu+\gamma_1+\lambda)\Big)I + (\delta+\mu)\Omega.
\end{aligned}$$

In the special case, if we set $\gamma_1 = -\lambda$, then we conclude $^{c}_{0}\mathscr{D}_t L \geq 0$. It is impossible due to the positiveness of all the model parameters. Furthermore $^{c}_{0}\mathscr{D}_t L \leq 0$ if and only if $Q = E = I = V = 0$. As long as in

model (3), (S,E,I,R,V,Q,D) tends to $E_0 = (S_0^*,0,0,0,0,0,D_0^*)$ for $t \to +\infty$. Now the essential condition for establishing $R_0 < 1$ is

$$R_0 = \frac{\sqrt{\alpha S_0^* B}}{\mu+p+\rho} < 1 \Longrightarrow \sqrt{\alpha S_0^* B} < \mu+p+\rho \Longrightarrow \alpha S_0^* B < (\mu+p+\rho)^2, \tag{11}$$

where B is defined as

$$\begin{aligned} B = &-\mu^3\beta_1 - \mu^2\beta_1\delta - \mu\beta_1 p^2 - 2\mu^2\beta_1 p - \mu\beta_1\rho^2 - 2\mu^2\beta_1\rho - \beta_1\delta\rho^2 \\ &- \beta_1\delta\rho^2 - 2\mu\beta_1\delta\rho - 2\mu\beta_1 + \delta\rho - 2\mu\beta_1 p\rho - 2\beta_1\delta p\rho. \end{aligned} \tag{12}$$

By substituting (12) into (11), and considering the equilibrium point E_0 and substituting into ${}_0^c\mathscr{D}_t L$, we obtain

$${}_0^c\mathscr{D}_t L = -(\delta+\mu)(\rho+\mu)S_0^* + (\delta+\mu)\Omega \leq \frac{-(\mu+p+\rho)^2(\rho+\mu)}{\alpha\beta_1(p^2+\rho^2+2\mu p+2\mu\rho+2p\rho)} + (\delta+\mu)\Omega \leq 0.$$

Therefore we can conlude ${}_0^c\mathscr{D}_t L \leq 0$. Hence E_0 will be the largest invariant set. Thus according to Barbashin-Krasovskii-Lasalle's invariance principle [20] mentioned in Theorem 2, we conclude that the system (3) is global asymptotically stable. □

7 Numerical simulation and data fitting

In this section, we will present numerical simulations of the solution of our fractional epidemic model. To underpin and evaluate the analysis of the model's dynamical behavior, we solve our proposed model using Caputo fractional order derivatives. We investigate numerical simulations employing the mathematical software MATLAB (R2020 version). To apply the fractional Euler method (FEM), we reconstruct the interval $[0,T]$ to partition it into n subintervals $[(k-1)\mathbb{M}, k\mathbb{M}]$ where $\mathbb{M} = T/n$ and $k = 1,2,...,n$. Therefore, the approximated solution can be calculated from the discretized equations below

$$\begin{aligned}
S(t_k) =& S(t_0) + \frac{\mathbb{M}^\tau}{\Gamma(\tau+1)}\sum_{j=0}^{k-1} C_{j,k}[(1-q)\Omega + \delta R + \varepsilon Q - (p+\rho+\mu+\alpha I)S(t_j)],\\
E(t_k) =& S(t_0) + \frac{\mathbb{M}^\tau}{\Gamma(\tau+1)}\sum_{j=0}^{k-1} C_{j,k}[\eta V + \alpha IS - (\beta_1+\beta_2+\mu)E(t_j)],\\
I(t_k) =& I(t_0) + \frac{\mathbb{M}^\tau}{\Gamma(\tau+1)}\sum_{j=0}^{k-1} C_{j,k}[\beta_1 E - (\nu+\gamma_1+\mu+\lambda)I(t_j)],\\
R(t_k) =& V(t_0) + \frac{\mathbb{M}^\tau}{\Gamma(\tau+1)}\sum_{j=0}^{k-1} C_{j,k}[(1-\rho)\gamma_2 Q + \gamma_1 I - (\delta+\mu)R(t_j)],\\
V(t_k) =& V(t_0) + \frac{\mathbb{M}^\tau}{\Gamma(\tau+1)}\sum_{j=0}^{k-1} C_{j,k}[(\Omega q + pS - (\mu+\eta))V(t_j)],\\
Q(t_k) =& V(t_0) + \frac{\mathbb{M}^\tau}{\Gamma(\tau+1)}\sum_{j=0}^{k-1} C_{j,k}[\lambda I + \beta_2 E - (\varepsilon + k\rho + \mu + (1-\rho)\gamma_2)Q(t_j)],
\end{aligned}$$

Table 1: The parameter values used in Example 1.

Parameters	Value	Source
α	0.09	[11]
β_1	0.025	[11, 17]
β_2	0.1	[11, 17]
δ	0.02	[17, 28]
μ	0.0003	[2]
ν	0.02	[11, 28]
η	0.05	[11]
ε	0.03	[11, 17]
k	8	[10]
ρ	0.6	[11]
Ω	397338	[2]
q	0.02	[2]
p	0.175	[11]
λ	0.8	[11, 17]
γ_1	0.5	[2]
γ_2	12	[10]

$$D(t_k) = D(t_0) + \frac{\mathbb{M}^{\tau}}{\Gamma(\tau+1)} \sum_{j=0}^{k-1} C_{j,k}[k\rho Q],$$

for $k = 1, \ldots, N$. In these equations, $C_{j,k}$ are defined as the weights illustrated below

$$C_{j,k} = (k-j)^{\tau} - (k-1-j)^{\tau}. \tag{13}$$

Example 1. In this numerical example, the parameter values taken from the literature and estimated are used to perform the solutions of the proposed model. The values and the sources of these parameters are presented in Table 1. This paper studies the variables of the human population model. Thus, the parameters and initial values of the model variables are presented in Table 1, on a scale of 10^{-4}.

Although global evolution in measles control has been achieved, measles extinction has not been achieved. However, China still faces challenges in measles control, especially in children under eight years old who were not licensed for vaccination. According to the reported data [16], we set the average life expectancy of Chinese people (both sexes) equal to 77.47 years. Thus the recruitment rate of this region is $\Omega = 397338$ considered and estimated in [2, 10]. In Fig. 2, we present a graphical simulation of the process of measles transmission under the proposed model (SEIRVQD) in China. This figure depicts that overtime, the number of susceptible individuals increases in initial observation, and after reaching the peak of the disease outbreak and the increase in immunity due to vaccination and infection, the number of susceptible individuals starts a descending trend until it reaches the equilibrium point.

As conceived, we have observed that the more time taken, the number of the exposed individuals has decreased. Furthermore, the number of infected and recovered and quarantined individuals increases in initial observation, and after reaching the peak of the disease outbreak and the increase in immunity

due to vaccination and infection, the number of susceptible individuals starts a descending trend until it reaches the equilibrium point.

According to the applied vaccination rates (p,q), the number of vaccinated individuals has increased and finally reached an equilibrium level. Finally, death cases have increased steeply at first, and after reaching the peak of the epidemic wave, it has taken a downward trend. It is also very important to mention this calculation matter that according to the relation (9), we find the basic reproduction number $(R_0 = 1.159)$ for Chinese measles.

By assuming $(S_0, E_0, I_0, R_0, V_0, Q_0, D_0) = (100, 9000, 1000, 600, 100, 800, 0)$ as an initial point, we can calculate the basic reproduction number $R_0 = 1.159$. We find the equilibrium point as $(S_1, E_1, I_1, R_1, V_1, Q_1, D_1) = (3251, 892, 292, 182, 13245, 98, 41)$ which tends to E_2 presented as third equilibrium point in Section 5 and can be clearly recognized in Fig. 2 as the level of reaching equilibrium and stability of values.

The stability of the system can be illustrated in Fig. 3. This explanatory numerical simulation depicts the stable behavior of the model for various orders of fractional derivatives. In order to show the validity, accuracy and efficiency of the presented model, Fig. 3 obtained by FEM method is presented. Fig. 3 demonstrate that the fractional model constructed with the Caputo fractional operator is more accurate in approximating and simulating the real number of Chinese measles patients. This real data is extracted from the World Health Organization (WHO) and China CDC weekly reports [13, 21, 37].

Furthermore, by analyzing the numerical results of other related models, such as models *SEIR* [26] and *SVEIR* [27], we can understand the accuracy, efficiency, and predictability of the proposed model in Fig. 3.

8 Conclusion and remarks

In this paper, we construct and formulate a new mathematical model for the spread of epidemic diseases namely SEIRVQD model using fractional calculus. The Caputo fractional operator is implemented in the model, which has memory according to the inherent property of its definition. Also, we have presented an analytical approach to this model. For the stability analysis, the Lyapunov and Ulam-Hyer criteria are applied. The theoretical results of the model and dynamical analysis of solution including positively invariant property and existence and uniqueness of the solution. The basic reproduction number (R_0) is calculated by the next generation matrix method in order to demonstrate the level of measles virus invasion. Moreover, numerical simulations including data fitting are performed for different fractional orders. Thereupon, we conclude that the fractional model constructed with the Caputo fractional operator is more accurate in approximating and simulating the real number of Chinese measles patients. This real data is extracted from the World Health Organization (WHO) and China CDC weekly reports. Additionally, the stability of the system can be illustrated based on the graphical output of the FEM numerical method. This explanatory numerical simulation depicts the stable behavior of the model for various orders of fractional derivatives.

As an innovation of this paper, various components of epidemiology are considered such that the factors vaccination, quarantine, and treatment of infection are applied together in the simulation of the proposed model using fractional Caputo derivatives. Due to the mismatch of the dimensions of the equations, we modify the system through adding the auxiliary parameter θ. This auxiliary parameter modifies our FDE for measles transmission modeling. This idea yields a comprehensive model with

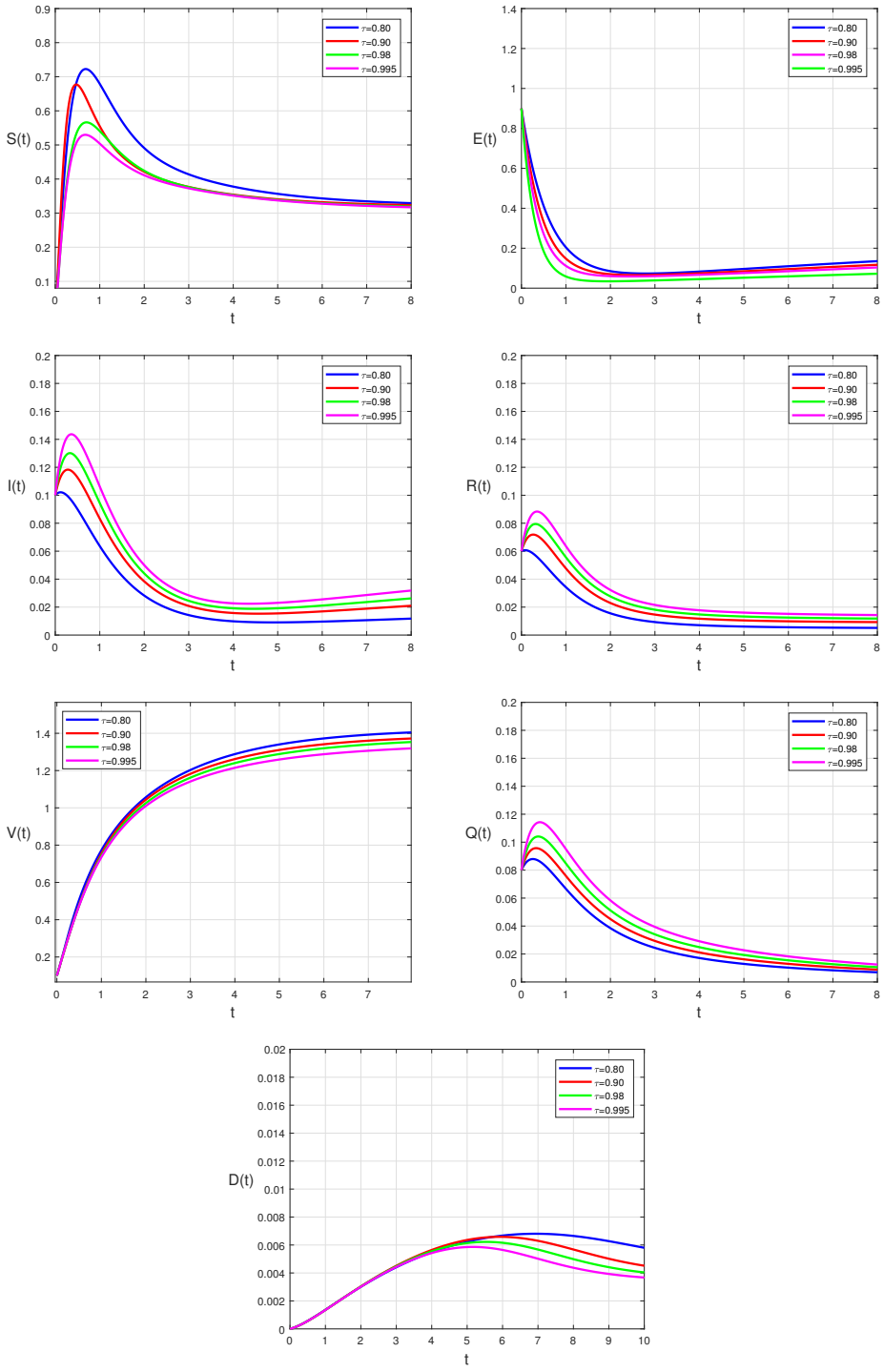

Figure 2: Graphical representation of the susceptible, exposed, infected, recovered, vaccinated, quarantined and died cases in China (measles) based on SEIRVQD model (Number of individuals$\times 10^{-4}$, t-Year).

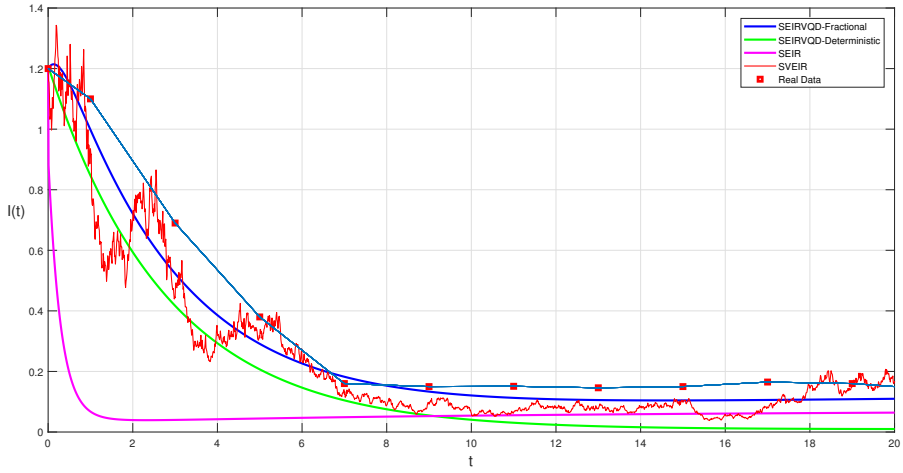

Figure 3: Measles infected cases in China vs. model fitting (Number of individuals $\times 10^{-4}$, t-Year).

realistic and useful results for the prediction of the level of invasion and the spread of the virus. On the other hand, it is necessary to devote further research to extend this idea to other epidemic systems, especially in stochastic or hybrid approaches.

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