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On the continued fraction expansions of some transcendental series in $\mathbb{F}_q((T^{-1}))$

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Abstract. In this paper we describe the continued fraction expansions of certain infinite series over $\mathbb{F}_q(T)$, where $\mathbb{F}_q$ is a finite field with q elements. As the first application, we determine the continued fraction expansion of the sum of rational functions with exponential elements. As the second application, we exhibit the continued fraction expansions of many classes of transcendental series that have bounded partial quotients.

Keywords: Continued fraction, Formal power series, Transcendance.
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1 Introduction

Let p be a prime number, $q = p^s$ with $s \geq 1$, and let $\mathbb{F}_q$ be the finite field with q elements. Given a formal indeterminate T, we consider the ring $\mathbb{F}_q[T]$ of polynomials, the field $\mathbb{F}_q(T)$ of rational functions and $\mathbb{F}_q((T^{-1}))$ the field of power series in $1/T$ over $\mathbb{F}_q$. For $\alpha \in \mathbb{F}_q((T^{-1}))$ we can write

$$\alpha = \sum_{n \geqslant n_0} c_n T^{-n}, \quad \text{where} \quad n_0 \in \mathbb{Z} \quad \text{and} \quad c_n \in \mathbb{F}_q.$$

An ultrametric absolute value is defined over this field by $|0| = 0$ and $|\alpha| = |T|^{n_0}$ where $|T|$ is a fixed real number greater than 1. Note that $\mathbb{F}_q((T^{-1}))$ is the completion of the field $\mathbb{F}_q(T)$ for this absolute value. As in classical continued fraction theory of real numbers, if $\alpha \in \mathbb{F}_q((T^{-1}))$, then we can write

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0, a_1, a_2, \ldots],$$

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where $a_0 = [\alpha]$, $a_i \in \mathbb{F}_q[T]$, with $\deg(a_i) \geqslant 1$ for any $i \geq 1$. The sequence $(a_i)_{i\geq 0}$ is called the partial quotients of α.

Several beautiful result connecting certain types of series with continued fraction expansion having symmetric patterns can be found in the literature: we refer the reader in particular to papers [1, 4–6]. In [10–12] different types of families of explicit continued fractions for transcendental analogues of Euler's "e" and Hurwitz numbers $(ae^{2/n}+b)/(ce^{2/n}+d)$ in the setting of Carlitz-Drinfeld modules for $\mathbb{F}_q[t]$ were produced, where the pattern of the sequence of partial quotients is based on block reversal.

As we know, there are uncountable many continued fractions of formal power series with bounded sequence of partial quotients, most of them are transcendental, but it is not very easy to give examples. One of basic tool to compute continued fraction of a given power series is the following simple lemma, due to Mendès France [8] and which has been rediscovered many times.

Lemma 1. *(Folding Lemma) Let $y \in \mathbb{F}_q((T^{-1}))$ and $\dfrac{P_n}{Q_n} = [a_0, a_1, \ldots, a_n]$. Then*

$$\big[a_0, a_1, \ldots, a_n, y, -a_n, \ldots, -a_1\big] = \big[a_0, a_1, \ldots, a_n, y - \frac{Q_{n-1}}{Q_n}\big] \tag{1}$$

$$= \frac{P_n}{Q_n} + \frac{(-1)^n}{yQ_n^2} \tag{2}$$

We fix now our used notations. The continued fraction $[a_0, a_1, \ldots, a_n]$ can be written as $[a_0, \overrightarrow{W}]$ where $\overrightarrow{W} = a_1, \ldots, a_n$ is the word of length n defining the fractional part and we note by $\overleftarrow{W}$ the reversal of the word $\overrightarrow{W}$.

We will define an operator ψ labeled by a polynomial R by

$$\psi_R\big([a_0, \overrightarrow{W}]\big) = \big[a_0, \overrightarrow{W}, R, -\overleftarrow{W}\big].$$

and if $S \in \mathbb{F}_q((T^{-1}))$,

$$\psi_S\psi_R\big([a_0, \overrightarrow{W}]\big) = \big[a_0, \overrightarrow{W}, R, -\overleftarrow{W}, S, \overleftarrow{W}, -R, -\overrightarrow{W}\big].$$

These operators are analogous to the folding maps employed in [9].

An infinite continued fraction representation for an irrational number is useful because its initial segments provide rational approximations to the number. These rational numbers are the convergents of the continued fraction. In this paper, first, we explore interesting continued fractions of the linear fractional transformation of "e". Second, we will create many families of continued fractions, in an interesting way using the explicit representation of infinite series. This families, are ultimately connected to those studied in [3, 9].

2 Main Results

We begin by the following theorem, whose proof is derived from the successive application of the Folding lemma, in order to calculate the continued fraction expansion of many classes of infinite series. Note that this theorem is an improvement of the main theorem of the paper [6], with fairly closely proof.

Theorem 1. *Let $(A_i)_{i=1}^{\infty}$ be a sequence of nonzero monic polynomials of $\mathbb{F}_q[T]$. Let B be a monic polynomial of $\mathbb{F}_q[T]$ such that $\gcd(B, A_1) = 1$ and $\deg B < \deg A_1$. Let*

$$\alpha(1) = \frac{B}{A_1} \text{ and } \alpha(n) = \frac{B}{A_1} + \sum_{i=2}^{n} \frac{1}{A_1 \cdots A_i} \text{ for all } n \geq 2,$$

and let

$$\alpha(\infty) = \frac{B}{A_1} + \sum_{i=2}^{\infty} \frac{1}{A_1 \cdots A_i}.$$

Assume that $A_1 \cdots A_i | A_{i+1}$ for all $i \geq 1$. Suppose that
$\alpha(1) = [0, b_1, \ldots, b_l] = [0, \overrightarrow{W}]$, $l \geq 1$ and let $z_i = \dfrac{A_{i+1}}{A_1 \cdots A_i}$, $i \geq 1$. Then

$$\alpha(\infty) = \prod_{i=2}^{\infty} \psi_{-z_i}\big(\psi_{(-1)^l z_1}\big([0, \overrightarrow{W}]\big)\big).$$

Proof. We have $\alpha(1) = \dfrac{B}{A_1} = [0, b_1, \ldots, b_l] = \dfrac{U_{k_1}}{V_{k_1}}$. So that $U_{k_1} = B$ and $V_{k_1} = A_1$. By Folding lemma we have

$$\begin{aligned} \alpha(2) &= \frac{B}{A_1} + \frac{1}{A_1 A_2} = \frac{B}{A_1} + \frac{(-1)^l}{(-1)^l \frac{A_2}{A_1} A_1^2} \\ &= [0, b_1, \ldots, b_l, (-1)^l \frac{A_2}{A_1}, -b_l, \ldots, -b_1] = \psi_{z_1}\big([0, b_1, \ldots, b_l]\big) := \frac{U_{k_2}}{V_{k_2}}. \end{aligned}$$

It suffice to prove that for all $n \geq 3$, if $\alpha(n) = [0, a_1, \ldots, a_{k_n}]$, then

$$\alpha(n+1) = \big[0, a_1, \ldots, a_{k_n}, -z_n, -a_{k_n}, \ldots, -a_1\big].$$

Let $[0, a_1, \ldots, a_{k_n}] = U_{k_n}/V_{k_n}$ be the k_n-th convergent to $\alpha(n)$. Observe that k_n is an odd integer. By Folding Lemma, we get

$$\begin{aligned} \psi_{-z_n}\big([0, \underbrace{a_1, \ldots, a_{k_n}}]\big) &= [0, \underbrace{a_1, \ldots, a_{k_n}}, -\frac{A_{n+1}}{A_1 \cdots A_n}, \underbrace{-a_{k_n}, \ldots, -a_1}] \\ &= \frac{U_{k_n}}{V_{k_n}} + \frac{-1}{-z_n (V_{k_n})^2} \\ &= \alpha(n) + \frac{1}{z_n (V_{k_n})^2}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \alpha(n+1) &= \frac{B}{A_1} + \frac{1}{A_1 A_2} + \frac{1}{A_1 A_2 A_3} + \cdots + \frac{1}{A_1 \cdots A_n} + \frac{1}{A_1 \cdots A_n A_{n+1}} \\ &= \frac{B A_2 \cdots A_n + A_3 \cdots A_n + \cdots + A_n + 1}{A_1 \cdots A_n} + \frac{1}{A_1 \cdots A_n A_{n+1}}. \end{aligned}$$

We prove that $\gcd(BA_2\cdots A_n + A_3\cdots A_n + \cdots + A_n + 1, A_1\cdots A_n) = 1$ for $n \geq 2$. Suppose there exists a prime $P \in \mathbb{F}_q[T]$ such that

$$P|(BA_2\cdots A_n + A_3\cdots A_n + \cdots + A_n + 1) \ \ and \ \ P|(A_1\cdots A_n).$$

Since $P|(A_1\cdots A_n)$ then $P|A_k$ for some $1 \leq k \leq n$, then $P|A_j\cdots A_n$ for all $2 \leq j \leq k$. Since $A_1\cdots A_k|A_{k+t}$ for all $1 \leq t \leq n-k$, we have $A_k|A_{k+t}\cdots A_n$ for all $1 \leq t \leq n-k$ and so $P|A_{k+t}\cdots A_n$ for all $1 \leq t \leq n-k$. In particular we also have $P|(BA_2\cdots A_n)$. Since $P|(BA_2\cdots A_n + A_3\cdots A_n + \cdots + A_n + 1)$, then we get $P|1$, which is a contradiction. This gives that $\gcd(B(A_2\cdots A_n) + A_3\cdots A_n + \cdots + A_n + 1, A_1\cdots A_n) = 1$. Since all A_i are monic then $V_{k_n} = A_1\cdots A_n$. Hence

$$\begin{aligned} \alpha(n) + \frac{-1}{-z_n(V_{k_n})^2} &= \alpha(n) + \frac{1}{z_n(V_{k_n})^2} \\ &= \alpha(n) + \frac{1}{A_1\cdots A_nA_{n+1}} = \alpha(n+1). \end{aligned}$$

This gives the desired result.

□

We direct the reader to consult the works of Thakur [10–12] to have a general idea of what we are going to study in this paragraph. The Carlitz exponential $e(z)$ is defined as follows:

Let $[i] := T^{q^i} - T, D_0 := 1 \ and \ D_i = [i]D_{i-1}^q, \ for \ i > 0$. Put

$$e(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i}.$$

The continued fraction expansion of $e(z)$ was described and the simplest pattern is obtained for $z = 1$ that is

$$\begin{aligned} e - 1 &= \sum_{i=1}^{\infty} \frac{1}{D_i} = [0, \underbrace{\underbrace{D_1}, -\frac{D_2}{D_1^2}, \underbrace{-D_1}}, -\frac{D_3}{D_2^2}, \underbrace{D_1, \frac{D_2}{D_1^2}, -D_1}, -\frac{D_4}{D_3^2}, \ldots] \\ &= [0, \underbrace{\underbrace{[1]}, -[2]D_1^{q-2}, \underbrace{-[1]}}, -[3]D_2^{q-2}, \underbrace{[1], [2]D_1^{q-2}, -[1]}, -4D_3^{q-2}, \ldots] \end{aligned}$$

In particular, for $q = 2$ we obtain the continued fraction expansions of the analogue of e in the field of power series.

$$e := e(1) = [1, \underbrace{\underbrace{\underbrace{[1]}, [2], \underbrace{[1]}}, [3], \underbrace{[1], [2], [1]}}, [4], \underbrace{[1], [2], [1], [3], [1], [2], [1]}, [5], \ldots].$$

and also if for $n > 2$, $\sum_{i=0}^{n-2} 1/D_iT^n = [0, \overrightarrow{W_n}]$ then

$$\frac{e}{T^n} = [0, \overrightarrow{W_n}, T^{2^{n-1}-n}, \overrightarrow{W_n}, T^{2^n-n}, \overrightarrow{W_n}, T^{2^{n-1}-n}, \overrightarrow{W_n}, T^{2^{n+1}-n}, \ldots].$$

Corollary 1. *Let C be a nonzero polynomial with coefficient in $\mathbb{F}_q$ such that* $\gcd(C, D_1) = 1$. *Let $C/D_1 = [0, \overrightarrow{W_n}]$ the continued fraction expansion of the rational function C/D_1 of length n. Then the continued fraction expansion of the series*

$$e - 1 + \frac{C-1}{D_1} = \frac{C}{D_1} + \sum_{i=2}^{\infty} \frac{1}{D_i} \tag{3}$$

is given by

$$\begin{aligned} e - 1 + \frac{C-1}{D_1} &= \prod_{i=2}^{\infty} \psi_{-D_{i+1}/D_i^2}\big(\psi_{(-1)^n D_2/D_1^2}([0, \overrightarrow{W_n}])\big) \\ &= [0, \overrightarrow{W_n}, (-1)^n D_2/D_1^2, -\overleftarrow{W_n}, -D_3/D_2^2, \overrightarrow{W_n}, (-1)^{n+1} D_2/D_1^2, \\ &\quad -\overleftarrow{W_n}, -D_4/D_3^2, \dots] \\ &= [0, \overrightarrow{W_n}, (-1)^n [2] D_1^{q-2}, -\overleftarrow{W_n}, -[3] D_2^{q-2}, \overrightarrow{W_n}, (-1)^{n+1} [2] D_1^{q-2}, \\ &\quad -\overleftarrow{W_n}, -[4] D_3^{q-2}, \dots] \end{aligned}$$

Proof. We apply the Theorem 1. Let $A_1 = D_1$ and $A_i = [i] D_{i-1}^{q-1}$ $(i \geq 2)$. Since

$$\begin{aligned} A_1 \cdots A_i &= D_1([2] D_1^{q-1})([3] D_2^{q-1}) \cdots ([i] D_{i-1}^{q-1}) \\ &= D_2([3] D_2^{q-1}) \cdots ([i] D_{i-1}^{q-1}) = D_{i-1}([i] D_{i-1}^{q-1}) = D_i \end{aligned}$$

and $A_{i+1} = [i+1] D_i^{q-1}$ then $A_1 \cdots A_i | A_{i+1}$ for all $i \geq 1$ and we have

$$\frac{A_{i+1}}{A_1 \cdots A_i} = \frac{A_1 \cdots A_i A_{i+1}}{(A_1 \cdots A_i)^2} = \frac{D_{i+1}}{D_i^2}.$$

□

Example 1. Let $q = 3$. Then

$$\begin{aligned} \frac{(T^2-1)e + T}{T^2 - 1} &= [1, T, T, T, -(T^9 - T)(T^3 - T), -T, -T, -T, \\ &\quad -(T^{27} - T)(T^9 - T)(T^9 - T^3), T, T, T, \\ &\quad (T^9 - T)(T^3 - T), \dots] \end{aligned}$$

It suffices to take $C = T^2 + 1$.

Let p be an odd prime number. We consider the series

$$\xi_1 = \sum_{i=0}^{\infty} \frac{1}{T^{2^i}} \quad \textit{and} \quad \xi_2 = T \sum_{i=0}^{\infty} \frac{1}{T^{2^i}}$$

in $\mathbb{F}_q((T^{-1}))$. The continued fraction expansion of this series over $\mathbb{Q}(T)$ was entirely described, see [3, 9], and they have bounded partial quotients:

$$\sum_{i=0}^{\infty} \frac{1}{T^{2^i}} = [0, T-1, T+2, T, T, T-2, T, T+2, T, T-2, T+2, \ldots]$$

$$T\sum_{i=0}^{\infty} \frac{1}{T^{2^i}} = [1, T, -T, -T, -T, T, T, -T, -T, -T, T, T, T, -T, \ldots].$$

This series have similar continued fractions in $\mathbb{F}_q((T^{-1}))$ and they are transcendental, see [2]. Let k an integer with $1 \leq k < p/2$. We consider the polynomials $P_k = (T^2-1)^k$ and $Q_k = \sum_{i=0}^{k-1}(-1)^{k-i-1}C_{k-1}^{i}(2i+1)^{-1}T^{2i+1}$ of $\mathbb{F}_q[T]$. Note that Q_k is simply, up to a constant factor, the remainder in the Euclidean division of T^p by P_k. This pair (P_k, Q_k) of polynomials was introduced in [7]. We recall that there exists a $2k$-tuple $(u_1, u_2, \ldots, u_{2k}) \in (\mathbb{F}_p^*)^{2k}$ such that

$$Q_k/P_k = [0, u_1T, u_2T, \ldots, u_{2k}T],$$

where

$$u_i = (2k-2i+1)\left(\frac{\prod_{1\leq j<i/2}(2j)(2k-2j)}{\prod_{1\leq j<(i+1)/2}(2j-1)(2k-2j+1)}\right)^{(-1)^i}$$

for all $1 \leq i \leq 2k$ and as usual the empty product is equal to 1.

Corollary 2. *The continued fraction of the series*

$$\alpha = \frac{Q_k}{P_k} + \sum_{i=2}^{\infty} \frac{1}{P_k^{2^i-1}} \tag{4}$$

is given by

$$\begin{aligned}\alpha &= \prod_{i=2}^{\infty} \psi_{-P_k}\big(\psi_{P_k}([0, u_1T, u_2T, \ldots, u_{2k}T])\big)\\ &= [0, u_1T, u_2T, \ldots, u_{2k}T, -(T^2-1)^k, -u_{2k}T, \ldots, -u_1T, (T^2-1)^k,\\ &\quad u_1T, \ldots, u_{2k}T, (T^2-1)^k, -u_{2k}T, \ldots, -u_1T, (T^2-1)^k, \ldots]\end{aligned}$$

Proof. We apply the Theorem 1 with $A_i = P_k^{2^{i-1}}$, $i \geq 1$. □

Corollary 3. *The continued fraction of the series*

$$\alpha = \frac{Q_k}{P_k} + \sum_{i=2}^{\infty} \frac{1}{P_k^{2^{i-1}}} \tag{5}$$

is given by

$$\alpha = [0, u_1T, W_\infty],$$

where W_∞ *denote the infinite sequence belong by* W_j *for all* $j \geq 1$ *where*

$$\overrightarrow{W_j} = \overrightarrow{W_{j-1}}, u_1T+1, u_1T-1, \overleftarrow{W_{j-1}} \quad (j \geq 2)$$

with

$$\overrightarrow{W_1} = \overrightarrow{W_0}, u_{2k}T+1, u_{2k}T-1, \overleftarrow{W_0}$$

and

$$\overrightarrow{W_0} = u_2T, \ldots, u_{2k-1}T,$$

which is equal to empty word when $k = 1$.

Proof. We apply the Theorem 1 with $A_1 = P_k$ and $A_i = P_k^{2^{i-2}}$, $i \geq 2$. Note that in this case, the sequence $(z_i)_{i\geq 1}$ is constant and equal to 1. So we will not obtain an usual continued fraction expansion. However, the usual continued fraction expansion can be deduced from it by using the following lemma

Lemma 2. *Let* $\alpha \in \mathbb{F}_q[T]$, $\beta \in \mathbb{F}_q^*$ *and* $\gamma \in \mathbb{F}_q(T)$. *Then*

$$[\alpha, \beta, \gamma] = [\alpha + \beta^{-1}, -\beta^2\gamma - \beta].$$

In fact, we have

$$\begin{aligned}
\frac{Q_k}{P_k} + \frac{1}{P_k^2} &= [0, u_1T, \overrightarrow{W_0}, u_{2k}T, 1, -u_{2k}T, -\overleftarrow{W_0}, -u_1T] \\
&= [0, u_1T, \overrightarrow{W_0}, u_{2k}T+1, u_{2k}T-1, \overleftarrow{W_0}, u_1T]\,, \\
\frac{Q_k}{P_k} + \frac{1}{P_k^2} + \frac{1}{P_k^4} &= [0, u_1T, \overrightarrow{W_1}, u_1T, 1, -u_1T, -\overleftarrow{W_1}, -u_1T] \\
&= [0, u_1T, \overrightarrow{W_1}, u_1T+1, u_1T-1, \overleftarrow{W_1}, u_1T]\,.
\end{aligned}$$

Then, for all $j \geq 2$

$$\begin{aligned}
&\frac{Q_k}{P_k} + \frac{1}{P_k^2} + \ldots + \frac{1}{P_k^{2^{j+1}}} \\
&= [0, u_1T, \overrightarrow{W_j}, u_1T, 1, -u_1T, -\overleftarrow{W_j}, -u_1T] \\
&= [0, u_1T, \overrightarrow{W_j}, u_1T+1, u_1T-1, \overleftarrow{W_j}, u_1T]\,.
\end{aligned}$$

So we obtain the desired result.

□

We consider the Fibonacci sequence of polynomials $(F_n(T))_n$ in $\mathbb{F}_q[T]$, when q is a power of an odd prime number p, defined by the recurrence relation

$$F_0 = 1, F_1 = T, \quad and \quad F_n = TF_{n-1} + F_{n-2}$$

for all $n \geq 2$. We know that $\frac{F_n(T)}{F_{n-1}(T)} = [\overbrace{T, T, \cdots, T}^{nT's}]$ because:

$$\frac{F_n(T)}{F_{n-1}(T)} = T + \frac{1}{\frac{F_{n-1}(T)}{F_{n-2}(T)}}.$$

We have: $F_2(T) = T^2 + 1$, $F_3(T) = T^3 + 2T$, $F_4(T) = T^4 + 3T^2 + 1, \ldots$.

Corollary 4. *Let $n \geq 2$. The continued fraction of the series*

$$\alpha = \frac{F_{n-1}}{F_n} + \sum_{i=2}^{\infty} \frac{1}{F_n^{2^i-1}} \tag{6}$$

is given by

$$\begin{aligned}\alpha &= \prod_{i=2}^{\infty} \psi_{(-1)^n F_n}\big(\psi_{-F_n}\big([0, \underbrace{T, T, \ldots, T}_{n}]\big)\big) \\ &= [0, T, T, \ldots, T, (-1)^n F_n(T), -T, \ldots, -T, -F_n(T), T, \ldots, T, \\ &\quad (-1)^{n+1} F_n(T), -T, \ldots, -T, -F_n(T), \ldots].\end{aligned}$$

Proof. Similar to the proof of the Corollary 2 with $A_i = F_n^{2^{i-1}}$, $i \geq 1$. □

Corollary 5. *Let $n \geq 2$. The continued fraction of the series*

$$\alpha = \frac{F_{n-1}}{F_n} + \sum_{i=2}^{\infty} \frac{1}{F_n^{2^{i-1}}} \tag{7}$$

is given by

$$\alpha = [0, T, W_\infty],$$

where W_∞ denote the infinite sequence belong by W_j for all $j \geq 1$ where

$$\overrightarrow{W_j} = \overrightarrow{W_{j-1}}, T+1, T-1, \overleftarrow{W_{j-1}} \quad (j \geq 2)$$

with

$$\overrightarrow{W_1} = \overrightarrow{W_0}, T+1, T-1, \overleftarrow{W_0}$$

and

$$\overrightarrow{W_0} = \underbrace{T, \ldots, T}_{n-2},$$

which is equal to empty word if $n = 2$.

Proof. Similar to the proof of the Corollary 3 with $A_1 = F_n$ and $A_i = F_n^{2^{i-2}}$, $i \geq 2$. □

Remark 1. *Note that:*

1. *In the equality (4) and (6) we have respectively* $\alpha = \xi_2(P_k) + \dfrac{Q_k - 1}{P_k}$ *and* $\alpha = \xi_2(F_{n-1}) + \dfrac{F_{n-1} - 1}{F_n}$.

2. *In the equality (5) and (7) we have respectively* $\alpha = \xi_1(P_k) + \dfrac{Q_k - 1}{P_k}$ *and* $\alpha = \xi_1(F_{n-1}) + \dfrac{F_{n-1} - 1}{F_n}$.

3 Conclusion

This article has presented significant results regarding the continued fraction expansions of transcendental series within the framework of polynomials over finite fields. The corollaries provide an enriched perspective on the relationships between these series and the Fibonacci polynomials, thereby contributing to a better understanding of the structures and properties of continued fractions in an algebraic context.

The research perspectives that emerge from this study are both varied and promising. It would be pertinent to consider generalizing these results to higher dimensions or to other types of polynomials, particularly those with coefficients in various algebraic structures. Additionally, a thorough exploration of the computational aspects of continued fractions related to transcendental series could lead to the development of more efficient calculation algorithms. Furthermore, the implications of these findings in number theory, especially in relation to Diophantine equations and transcendence theory, also deserve in-depth examination.

By addressing these different issues, we hope to deepen our understanding of the interactions between continued fractions and polynomial structures in various mathematical contexts, thus paving the way for significant new advancements in this field.

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