Journal of Algebra and Related Topics
Vol. , No. -, 2026, pp. -. Research Article

Line comaximal graphs

Soheila Khojasteh[†*]

[†]*Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran*
Emails: soheila.khojasteh@iau.ac.ir, khojaste.soheila@gmail.com

Abstract. Let R be a commutative ring with non-zero identity. The comaximal graph is a graph with vertices all elements of R and two distinct vertices x and y are adjacent if and only if $Rx + Ry = R$. Let $\Gamma_2(R)$ be the subgraph of the comaximal graph with vertex-set $W^*(R)$, where $W^*(R)$ is the set of all non-zero and non-unit elements of R. In this paper, we investigate when the graph $\Gamma_2(R)$ is a line graph. We completely present all commutative rings which their comaximal graphs are line graphs. Also, we study when the comaximal graph is the complement of a line graph.

Keywords: Comaximal graph; line graph; complement of a graph.
AMS Subject Classification 2020: 05C25, 05C76, 13H.

1 Introduction

There are many papers on assigning a graph to a ring R, for instance, see [2, 3, 5, 9] and [10]. In [12], Sharma and Bhatwadekar, defined the comaximal graph of a commutative ring R, $\Gamma(R)$ with vertices all elements of R and two distinct vertices x and y are adjacent if and only if $Rx + Ry = R$. In [11, 13, 14] a subgraph of the comaximal graph with non-unit elements of R as vertices, was studied. In this subgraph, the zero element of R is an isolated vertex. In [1], the authors omitted the zero vertex. Let $W^*(R)$ be the set of all non-zero and non-unit elements of R. The vertex set of $\Gamma_2(R)$ is $W^*(R)$, and two distinct vertices x and y in $W^*(R)$ are adjacent if and only if $Rx + Ry = R$, where Rz is the ideal generated by the element z in R. The graph $\Gamma_2(R)$ is the induced subgraph of $\Gamma(R)$ by $W^*(R)$. In [6], the planar index and outerplanar index of the comaximal graph was studied. Motivated by the previous works on the comaximal graph, in this paper we study line comaximal graphs. Throughout this paper, all graphs are simple with no loops and multiple edges and R is a commutative ring with non-zero identity. We denote the set of all zero-divisor elements and the set of all unit elements of R by $\mathrm{Z}(R)$ and

[*]Corresponding author

Received: 16 November 2024/ Revised: 21 December 2024/ Accepted: 06 January 2025
DOI: 10.22124/JART.2025.28997.1732

 http://jart.guilan.ac.ir

$U(R)$, respectively. If R has a unique maximal ideal $\mathfrak{m}$, then R is said to be a local ring and it is denoted by $(R, \mathfrak{m})$.

For basic definitions on graphs, one may refer to [8]. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. If x is adjacent to y, then we write $x—y$ or $\{x, y\} \in E(G)$. A graph G is *complete* if each pair of distinct vertices is joined by an edge. For a positive integer n, we use K_n to denote the complete graph with n vertices. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. Note that a graph whose vertex set is empty is an *empty graph*. The *complement* of G, denoted by $\overline{G}$ is a graph on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. If $|V(G)| \geq 2$, then a *path* from x to y is a series of adjacent vertices $x — x_1 — x_2 — \cdots — x_n — y$. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertex are distinct. We use P_n and C_n to denote the path and the cycle with n vertices, respectively. Suppose that H is a non-empty subset of $V(G)$. The subgraph of G whose vertex set is H and whose edge set is the set of those edges of G with both ends in H is called the subgraph of G *induced* by H. For every positive integer r, an r-partite graph is one whose vertex set can be partitioned into r subsets, or parts, in such a way that no edge has both ends in the same part. An r-partite graph is *complete r-partite* if any two vertices in different parts are adjacent. We denote the complete r-partite graph, with part sizes $n_1, \ldots, n_r$ by $K_{n_1,\ldots,n_r}$. For every $n \geq 2$, the *star graph* with n vertices is the complete bipartite graph with part sizes 1 and $n - 1$. Also, a *double-star graph* is a union of two star graphs with centers x and y such that x is adjacent to y. The *line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G.

Here, is a brief summary of the present paper. In this paper, we investigate when the comaximal graph is a line graph. Also, we study when the comaximal graph is the complement of a line graph. In Sec. 2, we characterize all finite rings whose comaximal graphs are line graphs. In Sec. 3, we characterize all finite rings whose comaximal graphs are complements of line graphs.

2 When the comaximal graph is a line graph

In this section, we study when the graph $\Gamma_2(R)$ is a line graph. We determine all finite commutative rings whose cozero-divisor graphs are line graphs. We will use one of the characterizations of line graphs which was proved in [7].

Theorem 1. *Let G be a graph. Then G is the line graph of some graph if and only if none of the nine graphs in Fig.* 1 *is an induced subgraph of G.*

Throughout the paper R is a finite commutative ring. By the structure theorem of Artinian rings [4, Theorem 8.7], there exists positive integer n such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ and $(R_i, \mathfrak{m}_i)$ is a local ring for all $1 \leq i \leq n$. We use this theorem in the rest of the paper. Also, let e_i be the $1 \times n$ vector whose ith component is 1 and the other components are 0.

We first present the following result.

Lemma 1. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for all $1 \leq i \leq n$. If $\Gamma_2(R)$ is a line graph, then $n \leq 3$.*

Proof. By contradiction, suppose that $n \geq 4$. It is easy to see that the induced subgraph by the set $\{e_1 + e_2 + e_3, \sum_2^n e_i, \sum_3^n e_i, \sum_4^n e_i\}$ is isomorphic to $K_{1,3}$. Therefore, by Theorem 1, $\Gamma_2(R)$ is not a line graph, a contradiction. □

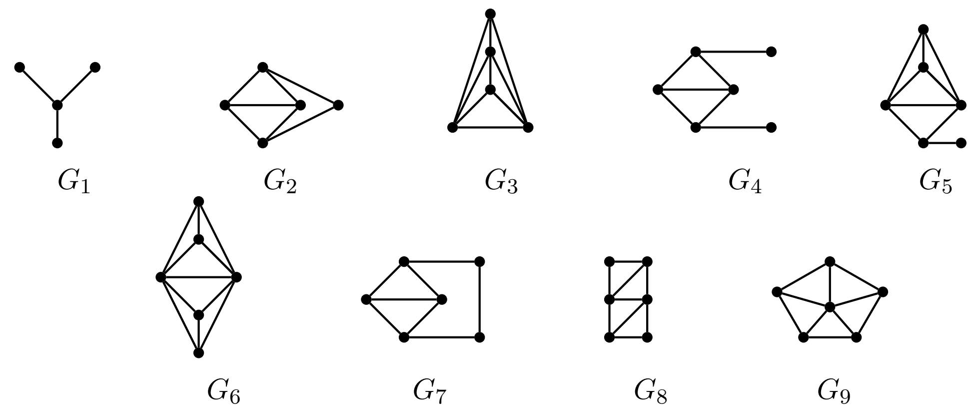


Fig. 1. Forbidden induced subgraphs of line graphs.

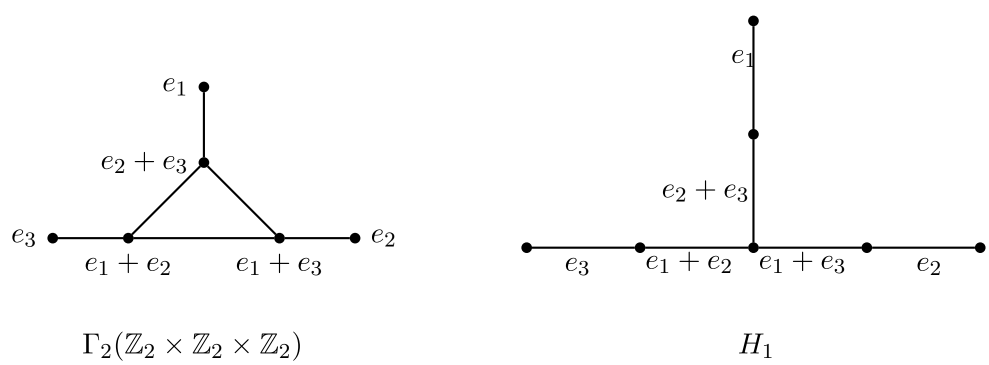


Fig. 2. $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is the line graph of H_1.

Lemma 2. *Let $R \cong R_1 \times R_2 \times R_3$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for $i = 1, 2, 3$. Then $\Gamma_2(R)$ is a line graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Let $\Gamma_2(R)$ be a line graph. If $x \in R_1 \setminus \{0, 1\}$, then the induced subgraph by the set $\{e_1 + e_2, e_3, e_1 + e_3, xe_1 + e_3\}$ is isomorphic to $K_{1,3}$, a contradiction. Therefore $R_1 \cong \mathbb{Z}_2$ and similarly, $R_2 \cong R_3 \cong \mathbb{Z}_2$. This implies that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We draw the graph $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ in Fig. 2. One can easily see that the graph $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is the line graph of H_1 which is drawn in Fig 2. The proof of converse is clear. □

Lemma 3. *Let $R \cong R_1 \times R_2$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for $i = 1, 2$. Then $\Gamma_2(R)$ is a line graph if and only if R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. Assume that $\Gamma_2(R)$ is a line graph. We claim that $|U(R_1)| \leq 2$. By contradiction, suppose that $u_1, u_2, u_3 \in U(R)$. Then the set $\{e_2, u_1e_1, u_2e_1, u_3e_1\}$ implies that $\Gamma_2(R)$ has a $K_{1,3}$ as an induced subgraph, which is impossible. Therefore, the claim is proved. By the same argument, we have $|U(R_2)| \leq 2$. We note that $\mathfrak{m}_1 = Z(R_1)$ and $\mathfrak{m}_2 = Z(R_2)$. Using [3, Remark 1], we find that R_i is isomorphic to one of the rings $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and $\mathbb{Z}_2[x]/(x^2)$, for $i = 1, 2$. Therefore, R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2)$ and $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2)$. Clearly, $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong K_2$ and so $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2) = L(P_3)$, $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_3) \cong K_{1,2}$ and so $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_3) = L(P_4)$, $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)) \cong C_4 \cup K_1 = L(C_4 \cup K_2)$. Also, $\Gamma_2(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong C_4 = L(C_4)$. Moreover, if R is isomorphic to one of the rings $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$, then the induced subgraph by the set $\{e_1 + 2e_2, e_2, 2e_1 + 3e_2, 2e_1 + e_2\}$ is isomorphic to $K_{1,3}$, a contradiction. For three rings $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2)$ and $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2)$, let $\overline{x} = x + (x^2) \in \mathbb{Z}_2[x]/(x^2)$. Then the set $\{e_1 + \overline{x}e_2, e_2, 2e_1 + \overline{(1 + x)}e_2, 2e_1 + e_2\}$ implies that $\Gamma_2(R)$ has the induced subgraph $K_{1,3}$, which is impossible. The proof of converse is clear. □

Remark 1. For the last case of our discussion, we must assume that $n = 1$. So, R is a local ring. Let $\mathfrak{m}$ be the only maximal ideal of R. We note that if $(R, \mathfrak{m})$ is a local ring with $\mathfrak{m} \neq 0$, then $V(\Gamma_2(R))$ is totally disconnected which implies that $\Gamma_2(R)$ is is the line graph of the graph $(|\mathfrak{m}| - 1)K_2$. Also, if R is a field, then it is an empty graph and so it is the line graph of the graph K_1.

In the following theorem, we characterize all commutative rings such that their comaximal graphs are line graphs.

Theorem 2. *Let R be a commutative ring. Then $\Gamma_2(R)$ is a line graph if and only if R is a local ring or it is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

3 When the comaximal graph is the complement of a line graph

In this section, we investigate when the graph $\Gamma_2(R)$ is the complement of a line graph. We use the following version of Theorem 1.

Theorem 3. *A graph G is the complement of a line graph if and only if none of the nine graphs $\overline{G_i}$ of Fig. 3 is an induced subgraph of G.*

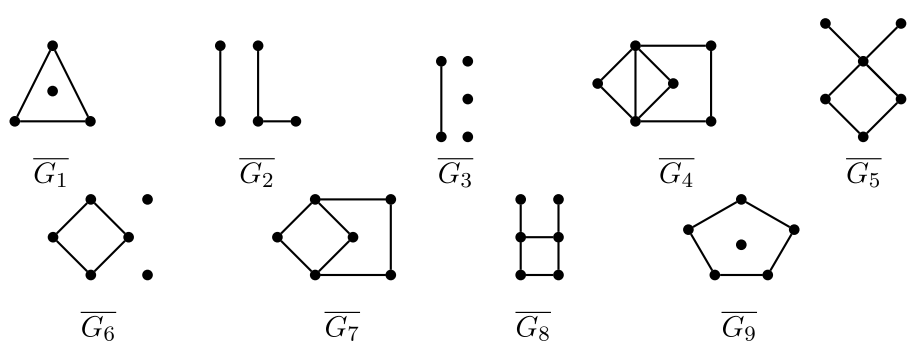


Fig. 3. Forbidden induced subgraphs of complement of line graphs.

Lemma 4. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for all $1 \leq i \leq n$. If $\Gamma_2(R)$ is the complement of a line graph, then $n \leq 3$.*

Proof. By contradiction, suppose that $n \geq 4$. Then the set $\{\sum_4^n e_i, \sum_{i\neq 1} e_i, \sum_{i\neq 2} e_i, \sum_{i\neq 3} e_i\}$ is isomorphic to $\overline{G_1}$. This is a contradiction. Hence $n \leq 3$. □

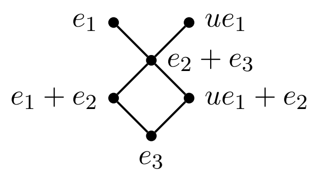
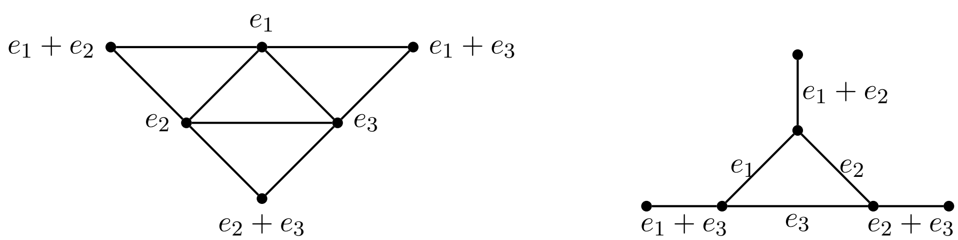


Fig. 4



$\overline{\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)}$ H_2

Fig. 5. $\overline{\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)}$ is the complement of line graph H_2

Lemma 5. *Let $R \cong R_1 \times R_2 \times R_3$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for $i = 1, 2, 3$. Then $\Gamma_2(R)$ is the complement of a line graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. One side is clear. For the other side, let $\Gamma_2(R)$ be the complement of a line graph. We prove that $U(R_1) = \{1\}$. By contradiction, suppose that $u \in U(R_1) \setminus \{1\}$. Then the induced subgraph by the set $\{e_1, ue_1, e_2 + e_3, e_1 + e_2, ue_1 + e_2, e_3\}$ is isomorphic to $\overline{G_5}$ (see Fig. 4), a contradiction. Therefore $U(R_1) = \{1\}$. Similarly $U(R_2) = U(R_3) = \{1\}$. This implies that $R_i \cong \mathbb{Z}_2$, for $i = 1, 2, 3$ and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The graph $\overline{\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)}$ was drawn in Fig. 5. It is not hard to see that it is the complement of the line graph of the graph H_2 (see Fig. 5). This completes the proof. □

Lemma 6. *Let $R \cong R_1 \times R_2$ and let $(R_i, \mathfrak{m}_i)$ be a local ring for $i = 1, 2$. Then $\Gamma_2(R)$ is the complement of a line graph if and only if R_1 is a field and $|\mathfrak{m}_2| \leq 2$.*

Proof. Assume that $\Gamma_2(R)$ is the complement of a line graph. Let $A = \mathfrak{m}_1 \times U(R_2)$, $B = U(R_1) \times \mathfrak{m}_2$ and let $C = \mathfrak{m}_1 \times \mathfrak{m}_2 \setminus \{0\}$. Clearly, $W^*(R) = A \cup B \cup C$, C is the set of all isolated vertices of $\Gamma_2(R)$ and $\Gamma_2(R) \setminus C$ is complete bipartite graph with parts A and B. Let $|U(R_i)| = s_i$ and $|\mathfrak{m}_i| = t_i$, for $i = 1, 2$. It is not hard to see that $|A| = s_2 t_1, |B| = s_1 t_2$ and $|C| = t_1 t_2 - 1$.

First, suppose that $t_1 t_2 \geq 3$. If $min\{s_1 t_2, s_2 t_1\} \geq 2$, then $\Gamma_2(R)$ has a $\overline{G_6}$ as an induced subgraph, which is impossible. Therefore, we may assume that $s_1 t_2 = 1$. This yields that R_1

is a field and $R_2 \cong \mathbb{Z}_2$. Also, we deduce that $C = \emptyset$ and $\overline{\Gamma_2(R)} \cong \overline{K_{s_1t_2,s_2t_1}} = K_{s_1t_2} \cup K_{s_2t_1} = L(K_{1,s_1t_2} \cup K_{1,s_2t_1})$. Now, assume that $t_1t_2 = 1$. Then R_1 and R_2 are fields. $\overline{\Gamma_2(R)} \cong \overline{K_{s_1,s_2}} = K_{s_1} \cup K_{s_2} = L(K_{1,s_1} \cup K_{1,s_2})$. Finally, assume that $t_1t_2 = 2$. With no loss of generality, assume that $t_1 = 1, t_2 = 2$. Clearly, $\Gamma_2(R) = \overline{K_1} \cup K_{2s_1,s_2}$ and so it is the complement of line graph of double star graph with part sizes $2s_1$ and s_2. The proof of converse is clear. □

Remark 2. The only remaining case is that R is a local ring. As we mentioned in the previous section, if $(R, \mathfrak{m})$ is a local ring, with $\mathfrak{m} \neq 0$, then $\Gamma_2(R)$ is totally disconnected. It follows that $\Gamma_2(R)$ is the complement of the line graph of the graph $K_{1,|\mathfrak{m}|-1}$. Also, if R is a field, then it is the complement of the line graph of the graph K_1.

Now, we have the following conclusion which completely characterizes all finite commutative rings R whose comaximal graphs are the complement of line graphs.

Theorem 4. *Let R be a commutative ring. Then $\Gamma_2(R)$ is the complement of a line graph if and only if R is a local ring or R is isomorphic to one of the following rings $F \times R_1, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where F is a field, $(R_1, \mathfrak{m}_1)$ is a local ring and $|\mathfrak{m}_1| \leq 2$.*

Acknowledgments

I would like to thank the referee(s) for valuable and fruitful comments.

References

[1] M. Afkhami and K. Khashyarmanesh, *On the cozero-divisor graphs and comaximal graphs of commutative rings*, Journal of Algebra and Its Applications, (3) **12** (2013), 1250173.

[2] S. Akbari and S. Khojasteh, *Some criteria for the finiteness of cozero-divisor graphs*, Journal of Algebra and Its Applications, (8) **12** (2013), 1350056.

[3] S. Akbari and A. Mohammadian, *Zero-divisor graphs of non-commutative rings*, J. Algebra, **296** (2006), 462-479.

[4] M. F. Atiyeh and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, Inc, 1969.

[5] Z. Barati, *Line zero divisor graphs*, Journal of Algebra and Its Applications, (2021) 2150154.

[6] Z. Barati and M. Afkhami, *The planar index and outerplanar index of the comaximal and cozero-divisor graphs*, Southeast Asian Bulletin of Mathematics, **44** (2020), 465–478.

[7] L. W. Beineke, *Characterizations of derived graphs*, J. Comb. Theory, **9** (1970), 129–135.

[8] J. A. Bondy and U.S.R. Murty, *Graph Theory*, Graduate Texts in Mathematics, 244 Springer, New York, 2008.

[9] S. Khojasteh, *The complement of the intersection graph of ideals of a poset*, Journal of Algebra and Its Applications, (11) **22** (2023), 2350236.

[10] S. Khojasteh, *Line cozero-divisor graphs*, Le Matematiche, (2) **77** (2022), 293–306.

[11] H. R. Maimani, M. Salimi, A. Sattari and S. Yassemi, *Comaximal graph of commutative rings*, Journal of Algebra, **319** (2008), 1801–1808.

[12] P. K. Sharma and S. M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra, **176** (1995), 124–127.

[13] M. M. Slavko and Z. Z. Petrovic, *On the structure of comaximal graphs of commutative rings with identity*, Bull. Aust. Math. Soc., **83** (2011), 11–21.

[14] H. J. Wang, *Graphs associated to co-maximal ideals of commutative rings*, J. Algebra, **320** (2008), 2917–2933.