Journal of Algebra and Related Topics
Vol. 14, No. 1, 2026, pp. 77-86. Research Article

Ring in which every element is sum of two 6−potent elements

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Abstract. In this paper, we prove the following results. Every element of a ring R is a sum of two commuting $6-$potent elements if and only if R is isomorphic to $R_1 \times R_2 \times R_3$, where R_1 is isomorphic to a subdirect product of Z_2's, R_2 is isomorphic to a subdirect product of Z_3's and R_3 is isomorphic to a subdirect product of Z_{11}'s. Also, if every element of a ring R is the sum of two 6-potent and one nilpotent all commute with each other, then R is isomorphic to $R_1 \times R_2 \times R_3$, where $J(R_1)$ is nil and $R_1/J(R_1)$ is a subdirect product of rings isomorphic to either of the rings $Z_2, F_4, M_2(F_2)$ and $M_2(F_4)$, $a^{81} - a$ is nilpotent for every $a \in R_2$, $J(R_3)$ is nil and $R_3/J(R_3)$ is a subdirect product of Z_{11}'s.

Keywords: 4-potents, 6-potents, Chinese remainder theorem.
AMS Subject Classification 2010: 16S60, 16S10, 16N20, 16N40, 16U10,16U60.

1 Introduction

In the paper [3] in 1988, the authors proved that in a ring R every element is a sum of two commuting idempotents if and only if R has the identity $x^3 = x$. Then, after a long break in 2016 [4],the authors proved that every element of a ring R is a sum of two commuting tripotents if and only if $R \cong R_1 \times R_2 \times R_3$, where $R_1/J(R_1)$ is Boolean with $u^2 = 1$ for every $u \in U(R_1)$, R_2 is a subdirect product of Z_3's, and R_3 is a subdirect product of Z_5's. The authors in [4] also questioned about rings in which every element is a sum of two commuting $p-$potents. We try to investigate it and find the structure of a ring in which every element is the sum of two commuting $4-$potent elements. Then, we find the equivalence condition for a ring in which every element is a sum of two commuting $6-$potent elements. Also, we discuss the ring in which every element is a sum of two $6-$potent elements and one nilpotent element, all of which commute with each other. All rings considered here are associative with unity. The Jacobson radical, finite field

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Received: 23 November 2023/ Revised: 25 May 2024/ Accepted: 28 May 2024
DOI: 10.22124/JART.2024.26096.1605

with n elements, group of units, the set of nilpotent elements are denoted by $J(R), F_n, U(R)$ and $Nil(R)$ respectively. $M_n(R)$ stands for $n \times n$ matrix ring. In a reduced ring there is no nonzero nilpotent elements.

2 Results ans discussion

Lemma 1. *Suppose $R \cong \prod R_i$, where $\prod R_i$ is a subdirect product of rings R_i's. Then every element of R is a sum of two commuting $p-$potent elements if and only if every element of R_i is a sum of two commuting $p-$potent elements. We can also extend the result i.e, if every element of R is a sum of two $p-$potent and one nilpotent, all commute with each other if and only if every element of R_i is a sum of two $p-$potent and one nilpotent, all commute with each other.*

Proof. It can be easily proved by isomorphism property of ring. □

Lemma 2 ([2], Lemma 3.2)**.** *Let p be a prime. The following are equivalent for a ring R:*

1. *$p \in Nil(R)$ and $a^p - a$ is nilpotent for all $a \in R$.*
2. *$J(R)$ is nil and $R/J(R)$ is a subdirect product of Z_p's.*

Lemma 3. *$\binom{p^k}{a}$ where $1 \leq a \leq p^k - 1$ is always divisible by p.*

Proof. We have $\binom{p^k}{a} = \frac{(p^k)!}{a!(p^k-a)!}$. Now,
Power of p in $(p^k)!$ is $[\frac{p^k}{p}] + [\frac{p^k}{p^2}] + [\frac{p^k}{p^3}] + \cdots = p^{k-1} + p^{k-2} + \cdots + p^2 + p + 1 = \frac{p^k-1}{p-1}$. Power of p in $a!$ is $[\frac{a}{p}] + [\frac{a}{p^2}] + [\frac{a}{p^3}] + \cdots$. Power of p in $(p^k - a)!$ is $[\frac{p^k-a}{p}] + [\frac{p^k-a}{p^2}] + [\frac{p^k-a}{p^3}] + \cdots$. For any $a, b \in R$ we have $[a + b] \geq [a] + [b]$. So, $[p^k/p^l] \geq [a/p^l] + [(p^k - a)/p^l]$ for $0 \leq a \leq p^k$. Now, $1 = [p^k/p^k] > [a/p^k] + [(p^k - a)/p^k] = 0$ for $1 \leq a \leq p^k - 1$. So power of p in $(p^k)!$ is atleast one greater than the combine power of p in $a!$ and $(p^k - a)!$. So $\binom{p^k}{a}$ is always divisible by p for $1 \leq a \leq p^k - 1$. For example $\binom{8}{1}$, $\binom{8}{2}$, $\binom{8}{3}$ and $\binom{8}{4}$ are all divisible by 2. □

Lemma 4 ([2], Theorem 3.5)**.** *The following are equivalent for a ring R.*

1. *$a^4 - a$ is nilpotent for all $a \in R$.*
2. *$J(R)$ is nil and $R/J(R)$ is a subdirect product of rings isomorphic to Z_2 or F_4 or $M_2(Z_2)$.*

Lemma 5 ([2], Proposition 3.16)**.** *The following are equivalent for a ring R.*

1. *$a^{16} - a$ is nilpotent for all $a \in R$.*
2. *$J(R)$ is nil and $R/J(R)$ is a subdirect product of rings isomorphic to either of the rings $Z_2, F_4, F_{16}, M_2(F_2)$ and $M_2(F_4)$.*

Theorem 1. *If R is a ring in which every element is sum of two commuting $4-$ potent elements. Then for every $k \in R$ we have*

$$(k - 2)(k - 1)k(k + 1)(k^2 - k + 1)(k^2 + k + 1)(k^2 + 2k + 4) = 0.$$

Also, we have $R \cong R_1 \times R_2 \times R_3$ where,

1. *R_1 is zero or $J(R_1)$ is nil with $R_1/J(R_1)$ is a subdirect product of rings isomorphic to Z_2 or F_4 or $M_2(Z_2)$. For every $k \in R_1$, $k^8 = k^2$. For every $n \in Nil(R)$ and $u \in U(R_1)$ we have $n^2 = 0$, $2n = 0$ and $u^6 = 1$.*

2. *R_2 is zero or R_2 is isomorphic to a subdirect product of Z_3's.*

3. *R_3 is zero or R_3 is isomorphic to a subdirect product of Z_7's.*

Proof. Let $k \in R$ then there exists $e, f \in R$ with $e^4 = e, f^4 = f, ef = fe$ such that $k = e + f$. Now, $k^4 = e^4 + f^4 + 4(e^3f + ef^3) + 6e^2f^2$. So,

$$k^4 - k = 4(e^3f + ef^3) + 6e^2f^2, \tag{1}$$

implies that

$$k^4 - k = (k^3 - 1)(e + f) \Rightarrow (k^4 - k)e^3f^3 = (k^3 - 1)(ef^3 + e^3f).$$

Again,

$$(k^4 - k)e^3f^3 = 4(e^6f^4 + e^4f^6) + 6e^5f^5 = 4(e^3f + ef^3) + 6e^2f^2 = k^4 - k.$$

Therefore, $k^4 - k = (k^3 - 1)(ef^3 + e^3f)$.
Using 1,

$$(k^3 - 5)(e^3f + ef^3) - 6e^2f^2 = 0. \tag{2}$$

Now, multiplying 2 by $k^2e^2f^2$ we have

$$\begin{aligned}
&(k^3 - 5)k^2(e^5f^3 + e^3f^5) - 6k^2e^4f^4\\
&= (k^3 - 5)(e^2 + f^2 + 2ef)(e^2f^3 + e^3f^2) - 6(e^2 + f^2 + 2ef)ef\\
&= (k^3 - 5)(e^4f^3 + e^5f^2 + e^2f^5 + e^3f^4 + 2e^3f^4 + 2e^4f^3) - 6(e^3f + ef^3 + 2e^2f^2)\\
&= (k^3 - 5)(ef^3 + e^2f^2 + e^2f^2 + e^3f + 2e^3f + 2ef^3) - 6(e^3f + ef^3 + 2e^2f^2)\\
&= (k^3 - 5)[3(e^3f + ef^3) + 2e^2f^2] - 6(e^3f + ef^3 + 2e^2f^2)\\
&= 0.
\end{aligned}$$

Thus,

$$[3(k^3 - 5) - 6](e^3f + ef^3) + 2[(k^3 - 5) - 6]e^2f^2 = 0. \tag{3}$$

As a consequence of equations 2 and 3,

$$\begin{aligned}
&[(k^3 - 5)^2 - 6(k^3 - 5) + 9(k^3 - 5) - 18](e^3f + ef^3)\\
&= [k^6 - 10k^3 + 25 - 6k^3 + 30 + 9k^3 - 45 - 18](k^3 - 1)(e^3f + ef^3)\\
&= (k^6 - 7k^3 - 8)(k^4 - k) = 0 \Rightarrow (k^3 - 8)(k^3 + 1)k(k^3 - 1)\\
&= (k - 2)(k - 1)k(k + 1)(k^2 - k + 1)(k^2 + k + 1)(k^2 + 2k + 4)\\
&= 0.
\end{aligned}$$

Putting $k = 3$, $2^3 \times 3 \times 7 \times 13 \times 19 = 0$. Again putting $k = -2$, $(-4) \times (-3) \times (-2) \times (-1) \times 7 \times 3 \times 8 = 0$. So, $2^6 \times 3 \times 7 = 0$. Now taking g.c.d $(2^3 \times 7 \times 13 \times 19, 2^6 \times 3 \times 7)$ we get

$2^3 \times 3 \times 7 = 0$. Now, 8 divides $(k-2)(k-1)k(k+1)$ for integer value of k (i.e $k = k.1_R$, where, 1_R is the multiplicative identity of R, here $1_R = 1$). Similarly, 3 divides $(k-1)k(k+1)$ for integer value of k. Again 7 divides $k^7 - k = k(k^3+1)(k^3-1)$ for integer value of k. Therefore, $2^3 \times 3 \times 7$ divides $(k^3-8)(k^3+1)k(k^3-1)$ for any integer value of k.
Now, $2^3 \times 3 \times 7 = 0$. So, by using the Chinese Remainder Theorem, $R \cong R_1 \times R_2 \times R_3$, where $R_1 \cong \frac{R}{2^3 R}$, $R_2 \cong \frac{R}{3R}$, and $R_3 \cong \frac{R}{7R}$.

Assume that $R_1 \neq 0$. In R_1, $2^3 = 8 = 0$. Suppose $4 = e + f$ where, $e^4 = e, f^4 = f, ef = fe$. Then,

$$\begin{aligned}
0 = 4^2 = e^2 + f^2 + 2ef &\Rightarrow e^2 + f^2 = -2ef \Rightarrow (e^2+f^2)^2 = (-2ef)^2 \\
&\Rightarrow e^4 + f^4 + 2e^2f^2 = 4e^2f^2 \Rightarrow e + f = 2e^2f^2 \\
&\Rightarrow 4 = 2e^2f^2 \Rightarrow 4e^4f^4 = 0 \Rightarrow 4ef = 0.
\end{aligned}$$

But $e^2 + f^2 = -2ef \Rightarrow 4e(e^2+f^2) = 4e(-2ef) \Rightarrow 4e^3 + 4ef^2 = -8ef \Rightarrow 4e^3 = 0 \Rightarrow 4e = 0$. Similarly we can show $4f = 0$.
Also, $e + f = 4 \Rightarrow e^2 + ef = 4e = 0 \Rightarrow e^2 = -ef$. Similarly, $f^2 = -ef$. At this point, $e^2 = -ef = f^2 \Rightarrow e^4 = f^4 \Rightarrow e = f$. So, $e^2 = -ef = -e^2 \Rightarrow 2e^2 = 0$. Therefore, $4 = 2e^2f^2 = 0$. As for any $k \in R_1$ we have

$$k^4 - k = (e+f)^4 - (e+f) = e^4 + 4e^3f + 6e^2f^2 + 4ef^3 + f^4 - e - f = 2e^2f^2,$$

where, $ef = fe, e^4 = e, f^4 = f$ so $k^4 - k = 2e^2f^2 \Rightarrow (k^4-k)^2 = 0$ as $4 = 0$. Now, if $n \in N(R_1)$ then $(n^4-n)^2 = n^2(n^3-1)^2 = 0$. So, $n^2 = 0$ as $n^3 - 1 \in U(R_1)$. Also, $2(n^4-n) = 2n(n^3-1) = 0$. So, $2n = 0$.
At this point for $n = a + b$ with $a^4 = a, b^4 = b, ab = ba$, $n^4 - n = 2a^2b^2 \Rightarrow n = -2a^2b^2 \Rightarrow n = 2a^2b^2$. So, 2 divides every nilpotent element. Now, $2(k^4-k) = 0$. So, $2k^5 = 2k^2$. Thus, $(k^4-k)^2 = k^8 - 2k^5 + k^2 = k^8 - 2k^2 + k^2 = 0$, and so, $k^8 = k^2$. Also for $u \in U(R_1)$, $u^8 = u^2 \Rightarrow u^2(u^6-1) = 0 \Rightarrow u^6 = 1$.

Suppose $R_2 \neq 0$. In R_2, $3 = 0$. Let $k^2 = 0$ for $k \in R_2$. There exist $e, f \in R_2$ with $e^4 = e, f^4 = f, ef = fe$. At this point, $0 = k^2 = e^2 + f^2 + 2ef$. So, $ef = e^2 + f^2 + 3ef = e^2 + f^2$. Therefore, $e^2f^2 = e^4 + f^4 + 2e^2f^2 = e + f + 2e^2f^2$. Thus, $k = -e^2f^2$, $k^2 = e^4f^4 = ef$, and $ef = 0$.

Now $k = e + f, k^4 = e^4 + f^4 = e + f = k$, so $k = 0$ as $ef = 0$. Thus, R_2 is a reduced ring and also a subdirect product of domains $\{R_\alpha\}$. Since R_α has only the trivial 4−potent $0, 1$ (as $e^4 = e$ in R_2 imply $e = 0$ or $e = 1$ or $e^2+e+1 = 0$. So, $e^2+e-2 = (e+2)(e-1) = 0 \Rightarrow e = -2, 1 \Rightarrow e = 1$ as $3 = 0$) we infer that $R_\alpha = \{0, 1, 2\}$. As $3 = 0$ in R_α, R_α is isomorphic to Z_3. Hence, R_2 is isomorphic to a subdirect product of Z_3's.

Suppose $R_3 \neq 0$. In R_3, $7 = 0$. Let $k^2 = 0$ for $k \in R_2$. There exist $e, f \in R_2$ with $e^4 = e, f^4 = f, ef = fe$. Now, $0 = k^7 = e^7 + f^7 + 7F(e,f) = e + f = k \Rightarrow k = 0$, where $F(e,f)$ is a function of e, f. So, R_3 is a reduced ring, and R_3 is a subdirect product of the domains $\{R_\alpha\}$. Now, if $k \in R_\alpha$ with $k^4 - k = 0$ then $k(k-1)(k^2+k+1) = 0 \Rightarrow k = 0, 1, 2, 4$ as $7 = 0$ in R_α ($k^2 + k + 1 = 0 \Rightarrow k^2 + k - 6 = 0 \Rightarrow (k+3)(k-2) = 0 \Rightarrow k = -3, 2 = 4, 2$ as $7 = 0$). Therefore, $R_\alpha = \{0, 1, 2, 3, 4, 5, 6\}$ as $7 = 0$. Clearly, $R_\alpha \cong Z_7$. Hence, R_3 is isomorphic to a subdirect product of Z_7's.

Clearly, R_1 has the identity $x^8 = x^2$, R_2 has the identity $x^3 = x$ and R_3 has the identity $x^7 = x$. So, R has the identity $x^8 = x^2$. Hence, for every $n \in Nil(R)$ and $u \in U(R)$ we have $n^2 = 0$ and $u^6 = 1$.

□

Example 1. $M_2(Z_2)$, F_4, Z_7, $Z_2 \times Z_3 \times Z_3 \times Z_7 \times Z_7$ etc. are some ring with above property. For instance, $Z_7 = \{0+0, 0+1, 1+1 = 0+2, 1+2, 2+2 = 0+4, 1+4, 2+6\}$, where $0^4 = 0, 1^4 = 1, 2^4 = 2, 4^4 = 4$.

Corollary 1. *If* $R \cong R_1 \times R_2 \times R_3$*, where* R_1 *is a subdirect product of* Z_2*'s,* R_2 *is a subdirect product of* Z_3*'s and* R_3 *is a subdirect product of* Z_7*'s. Then every element of the ring is the sum of two commuting* $4-$*potent elements.*

Proof. Here, R_1 is a subdirect product of $\{R_\alpha : \alpha \in \bigwedge\}$ where $R_\alpha = Z_2$ for all $\alpha \in \bigwedge$. Therefore, R_1 is a subring of $\prod_{\alpha \in \bigwedge} R_\alpha$. Let $x = (x_\alpha) \in R_1$. Then $\bigwedge$ is a disjoint union of $\bigwedge_0$ and $\bigwedge_1$ such that $x_\alpha = i$ if and only if $\alpha \in \bigwedge_i$ for $i = 0, 1$. Without loss of generality, we can denote $x = (0_{\bigwedge_0}, 1_{\bigwedge_1})$. Clearly, $x^4 = x, 0^4 = 0, 0x = x0$ and $x = x + 0$ which is the sum of two commuting $4-$potent elements.
Again, R_2 is a subdirect product of $\{R_\alpha : \alpha \in \bigwedge\}$ where $R_\alpha = Z_3$ for all $\alpha \in \bigwedge$. Therefore, R_2 is a subring of $\prod_{\alpha \in \bigwedge} R_\alpha$. Let $x = (x_\alpha) \in R_2$. Then $\bigwedge$ is a disjoint union of $\bigwedge_0$, $\bigwedge_1$ and $\bigwedge_2$ such that $x_\alpha = i$ if and only if $\alpha \in \bigwedge_i$ for $i = 0, 1, 2$. Without loss of generality, we can denote $x = (0_{\bigwedge_0}, 1_{\bigwedge_1}, 2_{\bigwedge_2})$. Then suppose $y := x^2 = (0_{\bigwedge_0}, 1_{\bigwedge_1}, 1_{\bigwedge_2})$. Let $z := x - x^2 = (0_{\bigwedge_0}, 0_{\bigwedge_1}, 1_{\bigwedge_2})$. Clearly $y^4 = y$, $z^4 = z$, $yz = zy$ and $x = y + z$ which is the sum of two commuting $4-$potent elements.
Again suppose R_3 is a subdirect product of $\{R_\alpha : \alpha \in \bigwedge\}$ where $R_\alpha = Z_7$ for all $\alpha \in \bigwedge$. Therefore, R_3 is a subring of $\prod_{\alpha \in \bigwedge} R_\alpha$. Let $x = (x_\alpha) \in R_3$. Then $\bigwedge$ is a disjoint union of $\bigwedge_0$, $\bigwedge_1$,$\bigwedge_2$, $\bigwedge_3$, $\bigwedge_4$, $\bigwedge_5$ and $\bigwedge_6$ such that $x_\alpha = i$ if and only if $\alpha \in \bigwedge_i$ for $i = 0, 1, 2, 3, 4, 5$ and 6. Without loss of generality, we can denote $x = (0_{\bigwedge_0}, 1_{\bigwedge_1}, 2_{\bigwedge_2}, 3_{\bigwedge_3}, 4_{\bigwedge_4}, 5_{\bigwedge_5}, 6_{\bigwedge_6})$. Now, as we know $0^4 = 0, 1^4 = 1, 2^4 = 2$ and $4^4 = 4$ in Z_7, so considering the two elements $y = (0_{\bigwedge_0}, 1_{\bigwedge_1}, 1_{\bigwedge_2}, 2_{\bigwedge_3}, 2_{\bigwedge_4}, 4_{\bigwedge_5}, 4_{\bigwedge_6})$ and $z = (0_{\bigwedge_0}, 1_{\bigwedge_1}, 1_{\bigwedge_2}, 1_{\bigwedge_3}, 2_{\bigwedge_4}, 1_{\bigwedge_5}, 2_{\bigwedge_6})$. Then we have $y^4 = y$, $z^4 = z$, $yz = zy$ and $x = y + z$ which is a sum of two commuting $4-$potent elements. As $R \cong R_1 \times R_2 \times R_3$ so using the Lemma 1, we have every element of R is a sum of two commuting $4-$potent elements. □

Remark 1. *If* R_1, R_2 *and* R_3 *are as given in the Corollary 1 then* R_1 *has the identity* $x^2 = x$*,* R_2 *has the identity* $x^3 = x$ *and* R_3 *has the identity* $x^7 = x$*. Combining all these threes we get* R *has the identity* $x^7 = x$*.*

Corollary 2. *Let* R *be a domain in which every element is a sum of two commuting four potent elements; then* R *is isomorphic to* Z_2 *or* Z_3 *or* Z_7*'s.*

Proof. From Theorem 1 we get in R we have $2^3 \times 3 \times 7 = 0$. As R is a domain so $2 = 0$ or $3 = 0$ or $7 = 0$.Then clearly R is isomorphic to Z_2 or Z_3 or Z_7.

In the paper [1] the author proved that in a ring in which every element is sum of two 4−potent and one nilpotent, all commute with each other then 42 is nilpotent in that ring.

□

Theorem 2. *The following conditions are equivalent.*

1. *R is a ring in which every element is sum of two commuting* $6-$*potent elements.*
2. *For every* $k \in R$ *we have* $(k^5 - 32)(k^{10} + 11k^5 - 1)(k^6 - k) = 0$ *and* $R \cong R_1 \times R_2 \times R_3$ *where,*
   (a) R_1 *is zero or* R_1 *is isomorphic to a subdirect product of* Z_2*'s.*
   (b) R_2 *is zero or* R_2 *is isomorphic to a subdirect product of* Z_3*'s.*
   (c) R_3 *is zero or a is isomorphic to a subdirect product of* Z_{11}*'s.*

Proof. (1) ⇒ (2). Let $k \in R$ then there exists $e, f \in R$ with $e^6 = e, f^6 = f, ef = fe$ such that $k = e + f$. Suppose $T_1 = ef^5 + e^5 f$, $T_2 = e^4 f^2 + e^2 f^4$, $T_3 = 2e^3 f^3$. We have $e^5 f^5 T_i = T_i$ for $i = 1, 2, 3, 4$. Now, $k^6 - k = 6T_1 + 15T_2 + 10T_3$. Again $k^6 - k = (k^5 - 1)(e + f) \Rightarrow (k^6 - k)e^5 f^5 = (k^5 - 1)T_1$. Also, $(k^6 - k)e^5 f^5 = 6T_1 + 15T_2 + 10T_3$ which implies that $k^6 - k = (k^5 - 1)T_1$. Therefore,

$$(k^5 - 7)T_1 - 15T_2 - 10T_3 = 0. \tag{4}$$

Again,

$$\begin{aligned} k^2 e^4 f^4 T_1 &= k^2 e^4 f^4 (ef^5 + e^5 f) = k^2(e^5 f^9 + e^9 f^5) \\ &= (e^2 + f^2 + 2ef)(e^5 f^4 + e^4 f^5) \\ &= e^7 f^4 + e^6 f^5 + e^5 f^6 + e^4 f^7 + 2(e^6 f^5 + e^5 f^6) \\ &= (e^2 f^4 + e^4 f^2) + 3(ef^5 + e^5 f) = T_2 + 3T_1. \end{aligned}$$

Similarly, we can show $k^2 e^4 f^4 T_2 = T_1 + 2T_2 + T_3$ and $k^2 e^4 f^4 T_3 = 2T_2 + 2T_3$.

Now, multiplying the equation 4 by $k^2 e^4 f^4$,

$$(k^5 - 7)(3T_1 + T_2) - 15(T_1 + 2T_2 + T_3) - 10(2T_2 + 2T_3) = 0,$$

which implies that

$$(3k^5 - 36)T_1 + (k^5 - 57)T_2 - 35T_3 = 0. \tag{5}$$

We have,

$$\begin{aligned} k^4 e^3 f^3 T_1 &= (e^4 + 4e^3 f + 6e^2 f^2 + 4ef^3 + f^4)(e^4 f^3 + e^3 f^4) \\ &= (e^3 f^3 + 4e^2 f^4 + 6ef^5 + 4e^5 f + e^4 f^2) + (e^2 f^4 + 4ef^5 \\ &\quad + 6e^5 f + 4e^4 f^2 + e^3 f^3) \\ &= 10(e^5 f + ef^5) + 5(e^2 f^4 + e^4 f^2) + 2e^3 f^3 = 10T_1 + 5T_2 + T_3. \end{aligned}$$

Similarly, we can show $k^4e^3f^3T_2 = 5T_1 + 7T_2 + 4T_3$ and $k^4e^3f^3T_3 = T_1 + 4T_2 + 3T_3$. Now, multiplying the equation 4 by $k^4e^3f^3$,

$$(k^5 - 7)(10T_1 + 5T_2 + T_3) - 15(5T_1 + 7T_2 + 4T_3) - 20(T_1 + 4T_2 + 3T_3) = 0,$$

which implies that

$$(10k^5 - 165)T_1 + (5k^5 - 220)T_2 + (k^5 - 127)T_3 = 0. \tag{6}$$

Using equations 4, 5 and 6 we have,

$$\begin{vmatrix} k^5 - 7 & -15 & -10 \\ 3k^5 - 36 & k^5 - 57 & -35 \\ 10k^5 - 165 & 5k^5 - 220 & k^5 - 127 \end{vmatrix} T_1 = 0.$$

So,

$$(k^{15} - 21k^{10} - 353k^5 + 32)T_1 = 0.$$

Since $k^6 - k = (k^5 - 1)T_1$, then

$$(k^{15} - 21k^{10} - 353k^5 + 32)(k^6 - k) = (k^5 - 32)(k^{10} + 11k^5 - 1)(k^6 - k) = 0.$$

Putting $k = -1, 11$ we have $2 \times 3 \times 11^2 = 0$ and $(11^5 - 32)(11^{10} + 11^6 - 1)11(11^5 - 1) = 0$. Taking g.c.d of $(2 \times 3 \times 11^2$, $(11^5 - 32)(11^{10} + 11^6 - 1)11(11^5 - 1))$, we get $2 \times 3 \times 11 = 0$. Using the Chinese Remainder Theorem, $R \cong R_1 \times R_2 \times R_3$ where, $R_1 \cong \frac{R}{2R}$, $R_2 \cong \frac{R}{3R}$, and $R_3 \cong \frac{R}{11R}$.

Consider $R_1 \neq 0$. In R_1 we have $2 = 0$. Let $k^2 = 0$ for some $k \in R_1$ where, $k = e + f$, $e^6 = e, f^6 = f, ef = fe$. Now, $k^2 = 0 \Rightarrow e^2 + f^2 + 2ef = 0 \Rightarrow e^2 = -f^2 \Rightarrow e^6 = -f^6 \Rightarrow e + f = 0 \Rightarrow k = 0$. So, R_1 is a reduced ring and also is a subdirect product of domains $\{R_\alpha\}$'s. In R_α $0, 1$ are the only trivial $6-$potent element. Hence, $R_\alpha = \{0, 1, 2\} = \{0, 1\}$ which is isomorphic to Z_2 as $2 = 0$ in R_α. So, R_1 is isomorphic a subdirect product of Z_2's.
Consider $R_2 \neq 0$. In R_2 we have $3 = 0$. Suppose $k^2 = 0$ for some $k \in R_2$ where $k = e + f$, $e^6 = e, f^6 = f, ef = fe$. Now, $k^2 = 0$ for some $k \in R_2$. Now, $k^3 = 0 \Rightarrow e^3 + f^3 + 3e^2f + 3ef^2 = 0 \Rightarrow e^3 + f^3 = 0 \Rightarrow e^3 = -f^3 \Rightarrow e^6 = f^6 \Rightarrow e = f$. Therefore, $2e^3 = 0 \Rightarrow 2e = 0 \Rightarrow k = 0$. So R_2 is a reduced ring and a subdirect product of domains $\{R_\alpha\}$'s. In R_α $0, 1$ are the only trivial $6-$potent. So, $R_\alpha = \{0, 1, 2\}$ which is isomorphic to Z_3 as $3 = 0$ in R_α. Hence, R_3 is isomorphic a subdirect product of Z_3's.

Consider $R_3 \neq 0$. In R_3 we have $11 = 0$. Suppose $k^2 = 0$ for some $k \in R_3$ where $k = e + f$, $e^6 = e, f^6 = f, ef = fe$. Now, $k^{11} = e^{11} + f^{11} + 11F(e, f) \Rightarrow 0 = e + f \Rightarrow k = 0$ where $F(e, f)$ is a function of e, f and $e^{11} = e^6e^5 = e^6 = e$. So, R_3 is a reduced ring and a subdirect product of domains $\{R_\alpha\}$'s. In R_α we have $11 = 0$. Suppose $e(e^5 - 1) = 0$ in R_α for some $e \in R_\alpha$ then $e = 0, 1, 3, 4, 5, 9$ i.e these are the only $6-$potent elements. Then $R_\alpha = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ which is isomorphic to Z_{11}'s. So, R_3 is isomorphic to a subdirect product of Z_{11}'s.

(2) $\Rightarrow$ (1). Let R_1, R_2, R_3 be defined as in (2). Suppose R_1 is a subdirect product of $\{R_\alpha : \alpha \in \bigwedge\}$ where, $R_\alpha = Z_2$ for all $\alpha \in \bigwedge$ (Actually $R_\alpha \cong Z_2$, but if we take $R_\alpha = Z_2$ there is no effect in the proof. So in later proof of R_2,R_3 we take as well). Therefore, R_1 is a subring of $\prod_{\alpha\in\bigwedge} R_\alpha$. Let $x = (x_\alpha) \in R_1$. Then $\bigwedge$ is a disjoint union of $\bigwedge_0$ and $\bigwedge_1$ such that $x_\alpha = i$ if and only if $\alpha \in \bigwedge_i$ for $i = 0, 1$. Without loss of generality, we can denote $x = (0_{\bigwedge_0}, 1_{\bigwedge_1})$. Then clearly, $x^6 = x$, $0^6 = 0$. So, $x = 0 + x$ which is the sum of two commuting $6-$potent elements.
Assume that R_2 is a subdirect product of $\{R_\alpha : \alpha \in \bigwedge\}$ where, $R_\alpha = Z_3$ for all $\alpha \in \bigwedge$. Therefore, R_2 is a subring of $\prod_{\alpha\in\bigwedge} R_\alpha$. Let $x = (x_\alpha) \in R_2$. Then $\bigwedge$ is a disjoint union of $\bigwedge_0$, $\bigwedge_1$ and $\bigwedge_2$ such that $x_\alpha = i$ if and only if $\alpha \in \bigwedge_i$ for $i = 0, 1, 2$. Without loss of generality, we can denote $x = (0_{\bigwedge_0}, 1_{\bigwedge_1}, 2_{\bigwedge_2})$. Then suppose $y := x^2 = (0_{\bigwedge_0}, 1_{\bigwedge_1}, 1_{\bigwedge_2})$. Let $z := x - x^2 = (0_{\bigwedge_0}, 0_{\bigwedge_1}, 1_{\bigwedge_2})$. Now, clearly $y^6 = y$, $z^6 = z$, $yz = zy$ and $x = y+z$ which is the sum of two commuting $6-$potent elements.
Again, assume that R_3 is a subdirect product of $\{R_\alpha : \alpha \in \bigwedge\}$ where $R_\alpha = Z_{11}$ for all $\alpha \in \bigwedge$. Therefore, R_3 is a subring of $\prod_{\alpha\in\bigwedge} R_\alpha$. Let $x = (x_\alpha) \in R_3$. Then $\bigwedge$ is a disjoint union of $\bigwedge_0$, $\bigwedge_1$,$\bigwedge_2$, $\bigwedge_3$, $\bigwedge_4$, $\bigwedge_5$, $\bigwedge_6$,$\bigwedge_7$,$\bigwedge_8$,$\bigwedge_9$ and $\bigwedge_{10}$ such that $x_\alpha = i$ if and only if $\alpha \in \bigwedge_i$ for $i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and 10. Without loss of generality, we can denote $x = (0_{\bigwedge_0}, 1_{\bigwedge_1}, 2_{\bigwedge_2}, 3_{\bigwedge_3}, 4_{\bigwedge_4}, 5_{\bigwedge_5}, 6_{\bigwedge_6}, 7_{\bigwedge_7}, 8_{\bigwedge_8}, 9_{\bigwedge_9}, 10_{\bigwedge_{10}})$. Now as we know $0^6 = 0, 1^6 = 1, 3^6 = 3, 4^6 = 4$ and $9^6 = 9$ in Z_{11} , so considering the two elements $y = (0_{\bigwedge_0}, 1_{\bigwedge_1}, 1_{\bigwedge_2}, 3_{\bigwedge_3}, 0_{\bigwedge_4}, 1_{\bigwedge_5}, 3_{\bigwedge_6}, 3_{\bigwedge_7}, 4_{\bigwedge_8}, 9_{\bigwedge_9}, 1_{\bigwedge_{10}})$ and $z = (0_{\bigwedge_0}, 0_{\bigwedge_1}, 1_{\bigwedge_2}, 0_{\bigwedge_3}, 4_{\bigwedge_4}, 4_{\bigwedge_5}, 3_{\bigwedge_6}, 4_{\bigwedge_7}, 4_{\bigwedge_8}, 0_{\bigwedge_9}, 9_{\bigwedge_{10}})$. Then we have $y^6 = y$, $z^6 = z$, $yz = zy$ and $x = y + z$ which is a sum of two commuting $6-$potent elements. As $R \cong R_1 \times R_2 \times R_3$, so using the Lemma 1, we have every element of R is a sum of two commuting $6-$potent elements.
If R_1, R_2 and R_3 are as given in the Theorem 2, then R_1 has the identity $x^2 = x$, R_2 has the identity $x^3 = x$ and R_3 has the identity $x^{11} = x$. Combining all these threes we get that R has the identity $x^{11} = x$.

□

Example 2. $Z_2 \times Z_{11}$, Z_3, $Z_2 \times Z_3 \times Z_{11}$ are rings with above property. For illustration in Z_{11}, $6-$potents are $0, 1, 3, 4, 5, 9$. Now, $Z_{11} = \{0+0, 0+1, 1+1, 0+3, 0+4, 1+5, 3+3, 3+4, 4+4, 0+9, 1+9\}$

Corollary 3. *If in a ring R every element is a sum of two $6-$potent and a nilpotent, all commute each other, then $R \cong R_1 \times R_2 \times R_3$ where,*

1. *R_1 is zero or $J(R_1)$ is nil and $R_1/J(R_1)$ is a subdirect product of rings isomorphic to either of the rings Z_2, F_4, $M_2(F_2)$ and $M_2(F_4)$.*
2. *R_2 is zero or $a^{81} - a$ is nilpotent for every $a \in R_2$.*
3. *R_3 is zero or $J(R_3)$ is nil and $R_3/J(R_3)$ is a subdirect product of Z_{11}'s*

Proof. Let $k \in R$. Then k can be expressed as $k = e + f + n$, where $e^6 = e, f^6 = f, n \in Nil(R)$ and $ef = fe, en = ne, fn = nf$. Now, $k - n = e + f$ is a sum of two commuting $6-$potent elements. Therefore, using Theorem 2, we have $\{(k-n)^5 - 32\}\{(k-n)^{10} + 11(k-n)^5 - 1\}\{(k-n)^6 - (k-n)\} = 0 \Rightarrow (k^5 - 32)(k^{10} + 11k^5 - 1)(k^6 - k) \in Nil(R)$ as k commute with e, f, n.

Now, putting $k = -1, 11$ we get $2 \times 3 \times 11^2$ and $(11^5 - 32)(11^{10} + 11^6 - 1)11(11^5 - 1)$ are nilpotent, taking g.c.d of both, we get $2 \times 3 \times 11$ is nilpotent. So, for some $m \in N$ we have $2^m \times 3^m \times 11^m = 0$. Hence, using the Chinese Remainder Theorem, $R \cong R_1 \times R_2 \times R_3$, where $R_1 \cong \frac{R}{2^m R}$, $R_2 \cong \frac{R}{3^m R}$ and $R_1 \cong \frac{R}{11^m R}$.

Suppose $R_1 \neq 0$. In R_1 we have $2^m = 0$ i.e $2 \in Nil(R_1)$. Clearly, $2 + n$ is nilpotent for $n \in Nil(R_1)$. Let $a \in R_1$ so $a = e + f + n$ where $e^6 = e, f^6 = f, n \in Nil(R), ef = fe, en = ne, fn = nf$. Therefore, using Lemma 3, we have $a^2 = e^2 + f^2 + n_1$, $n_1 \in Nil(R_1)$ which implies that $a^8 = e^8 + f^8 + n_2 = e^3 + f^3 + n_2 \Rightarrow a^{16} = e^6 + f^6 + n_3 = e + f + n_3 = a + n_4 \Rightarrow a^{16} - a = n_4$, where n_2, n_3, n_4 are nilpotents and functions of e, f, n. So, $a^{16} - a$ is nilpotent for every $a \in R_1$. Hence, using Lemma 5, we have $J(R_1)$ is nil and $R_1/J(R_1)$ is a subdirect product of rings isomorphic to either of the rings Z_2, F_4, F_{16}, $M_2(F_2)$ or $M_2(F_4)$.

Assume that $R_2 \neq 0$. In R_2 we have $3^m = 0$ i.e $3 \in Nil(R_2)$. Clearly, $3 + n$ is nilpotent for $n \in Nil(R_2)$. Let $a \in R_2$ so, $a = e + f + n$ where $e^6 = e, f^6 = f, n \in Nil(R), ef = fe, en = ne, fn = nf$. Therefore, using Lemma 3, we have $a^3 = e^3 + f^3 + n_1 \Rightarrow a^9 = e^9 + f^9 + n_2 = e^4 + f^4 + n_2 \Rightarrow a^{27} = e^{12} + f^{12} + n_3 = (e^6)^2 + (f^6)^2 + n_4 = e^2 + f^2 + n_4 \Rightarrow a^{81} = e^6 + f^6 + n_5 = e + f + n_5 = a + n_5 \Rightarrow a^{81} - a = n_5$ which implies that $a^{81} - a$ is nilpotent for every $n \in Nil(R_2)$. Assume that $R_3 \neq 0$. In R_3 we have $11^m = 0$ i.e $11 \in Nil(R_3)$. Clearly, $11 + n$ is nilpotent for $n \in Nil(R_3)$ Let $a \in R_3$ so, $a = e + f + n$ where $e^6 = e, f^6 = f, n \in Nil(R), ef = fe, en = ne, fn = nf$. Therefore, using Lemma 3, we have $a^{11} = e^{11} + f^{11} + n_1 = e^6 + f^6 + n_1 = a + n_1$, where $n_1 \in Nil(R_3)$ which implies that $a^{11} - a$ is nilpotent for every $a \in R_3$. So, using Lemma 2, we have $J(R_3)$ is nil and $R_3/J(R_3)$ is a subdirect product of Z_{11}'s.

□

We finish our study with these open problem.

Open Problem 1 Find the equivalence condition for a ring in which every element is a sum of two commuting $4-$potent elements.

Acknowledgments

The first author would like to thank UGC for granting fund. The authors would like to thank the referees for careful reading.

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