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A note on dimension of local cohomology modules

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Abstract. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring and I be an ideal of R. In this paper first we find new results about the dimension of the local cohomology module $H_I^i(R)$. Then we will obtain new relations between the invariants such as arithmetic rank, cohomological dimension, krull dimension, and the height of an ideal of R.

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1 Introduction

Throughout this paper, R denotes a commutative Noetherian local ring with identity. For an R-module M, the i^{th} local cohomology module of M with respect to an ideal I is defined as

$$H_I^i(M) = \varinjlim_{n\geq 1} \operatorname{Ext}_R^i(R/I^n, M).$$

For each R module L ,we denote by $\operatorname{Assh}_R(L)$ the set $\{\mathfrak{p} \in \operatorname{Ass}_R(L) : \dim R/\mathfrak{p} = \dim L\}$. For any ideal $\mathfrak{b}$ of R, we denote $\{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$. For any ideal $\mathfrak{b}$ of R, the *radical* of $\mathfrak{b}$, denoted by $\sqrt{\mathfrak{b}}$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. Recall that, for an R- module M, the set $\operatorname{Min}\operatorname{Ass}_R(M)$ is defined as

$$\{\mathfrak{p} \in \operatorname{Ass}_R(M) : \nexists \mathfrak{q} \in \operatorname{Ass}_R(M), \mathfrak{q} \subsetneqq \mathfrak{p}\}.$$

The reader is referred to [1] for more details on local cohomology. Recall that, for an R-module M, the *cohomological dimension* of M *with respect* to I is defined as

$$\operatorname{Cd}(I, M) := \operatorname{Sup}\left\{i \in \mathbb{Z} : H_I^i(M) \neq 0\right\}.$$

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The cohomological dimension and arithmetic rank have been studied by several authors; see, for example, Dibaei and Yassemi [2], Grothendieck [5], Faltings [4], Hartshorne [6], Divaani-Aazar, Naghipour and Tousi [3], and Hellus–Stuckrad [7]. Also, for any proper ideal I of R, the *arithmetic rank* of I, denoted by $\operatorname{ara}(I)$, is the least number of elements of R required to generate an ideal which has the same radical as I.
In section two of this paper we present a new relation between, the dimension of ring, cohomological dimension, arithmetic rank and the height of an ideal of R. One of the main results of this paper is the following,

Theorem 1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and I be a proper ideal of R. Then there exist an integer i with $\operatorname{height} I \leq i \leq d$ and $\mathfrak{p} \in \operatorname{Supp} H^i_I(R)$, such that $\operatorname{height} \mathfrak{p} = i$ and $\dim R/\mathfrak{p} = d - i$. In particular, $\dim H^i_I(R) = d - i$.*

For any unexplained notation and terminology we refer the readers to [1] and [8].

2 Main results

The following lemmas will be quite useful in the proof of the main results.

Lemma 1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero R-module of dimension n. Let $x \in \mathfrak{m}$ be such that $H^1_{Rx}(M) \neq 0$. Then $\dim H^1_{Rx}(M) \leq \dim M - 1 = n - 1$.*

Proof. As $\operatorname{Supp} H^1_{Rx}(M) \subseteq \operatorname{Supp} M$, so it is enough to show that $\operatorname{Assh} M \cap \operatorname{Supp} H^1_{Rx}(M) = \emptyset$. Suppose on the contrary that, there exists $\mathfrak{p} \in \operatorname{Spec}(R)$, such that $\mathfrak{p} \in \operatorname{Assh} M \cap \operatorname{Supp} H^1_{Rx}(M)$. In this case, as $\operatorname{Assh} M \subseteq m \operatorname{Ass} M$, then $\dim_{R_\mathfrak{p}} M_\mathfrak{p} = 0$. Now, by [1, Theorem 4.3.2], $(H^1_{Rx}(M))_\mathfrak{p} = H^1_{R_\mathfrak{p} x}(M_\mathfrak{p}) = 0$ which is a contradiction. Therefore, from $\operatorname{Supp} H^1_{Rx}(M) \subseteq \operatorname{Supp} M$, we conclude that $\dim H^1_{Rx}(M) < \dim M$ and consequently $\dim H^1_{Rx}(M) \leq \dim M - 1 = n - 1$. □

Lemma 2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and I be a proper ideal of R. Then for each $0 \leq i \leq d$, $\dim H^i_I(R) \leq d - i$.*

Proof. Let $\mathfrak{p} \in \operatorname{Supp} H^i_I(R)$. Then $(H^i_I(R))_\mathfrak{p} \cong H^i_{IR_\mathfrak{p}}(R_\mathfrak{p}) \neq 0$. By [1, Theorem 6.1.2], $\dim R_\mathfrak{p} = \operatorname{height} \mathfrak{p} \geq i$ and so

$$\dim R/\mathfrak{p} \leq \dim R - \operatorname{height} \mathfrak{p} \leq d - i.$$

Therefore,

$$\dim H^i_I(R) = \operatorname{Sup}\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} H^i_I(R)\} \leq d - i.$$

□

We are now in a position to put all our previose Lemmas toghether to produce a proof of the main theorem of this paper.

Theorem 2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and I be a proper ideal of R. Then there exist an integer i with $\operatorname{height} I \leq i \leq d$ and $\mathfrak{p} \in \operatorname{Supp} H^i_I(R)$, such that $\operatorname{height} \mathfrak{p} = i$ and $\dim R/\mathfrak{p} = d - i$. In particular, $\dim H^i_I(R) = d - i$.*

Proof. We begin the proof by induction on $n = \dim R/I$. If $n = 0$, then I is a $\mathfrak{m}$-primary ideal and so we have $H_I^d(R) = H_{\mathfrak{m}}^d(R) \neq 0$ and $\operatorname{Supp} H_{\mathfrak{m}}^d(R) = \{\mathfrak{m}\}$. Therefore $\mathfrak{m} \in \operatorname{Supp} H_{\mathfrak{m}}^d(R)$ and $\operatorname{height} \mathfrak{m} = d$. In addition, $\dim R/\mathfrak{m} = d - d = 0$. Consequently, with $i = d$ and $\mathfrak{p} = \mathfrak{m}$ the result has been proved in this case. Now suppose, inductively, that $\dim R/I = n > 0$, and the result has been proved for $n-1$. By the prime avoidance theorem, $\mathfrak{m} \nsubseteq \cup_{\mathfrak{p} \in \operatorname{Min} \operatorname{Ass}_R R/I} \mathfrak{p}$. Hence there exists an element $x \in \mathfrak{m}$ such that $x \notin \cup_{\mathfrak{p} \in Min \operatorname{Ass}_R R/I} \mathfrak{p}$. Therefore $\dim R/I + Rx = n-1$ and by induction hypothesis there exists an integer $\operatorname{height}(I + Rx) \leq j \leq d$, such that $\dim R/q = d - j$ and $\operatorname{height} q = j$ for some $q \in \operatorname{Supp} H_{I+Rx}^j(R)$. By [10, Corollary 3.5], there exists an exact sequence as follows,

$$0 \to H_{Rx}^1(H_I^{j-1}(R)) \to H_{I+Rx}^j(R) \to H_{Rx}^0(H_I^j(R)) \to 0.$$

Now consider two cases:
1) If $q \in \operatorname{Supp} H_{Rx}^0(H_I^j(R))$, it follows from $H_{Rx}^0(H_I^j(R)) \cong \Gamma_{Rx}(H_I^j(R)) \subseteq H_I^j(R)$, that $q \in \operatorname{Supp} H_I^j(R)$. On the other hand, $\operatorname{height} q = j$, $\dim R/q = d - j$ and $\operatorname{height} I \leq \operatorname{height}(I + Rx) \leq j$, which shows that the proof is completed.
2) If $q \in \operatorname{Supp} H_{Rx}^1(H_I^{j-1}(R))$, then

$$0 \neq (H_{Rx}^1(H_I^{j-1}(R)))_q \cong H_{R_q x}^1(H_{IR_q}^{j-1}(R_q))$$

and so by Lemma 1, $\dim H_{IR_q}^{j-1}(R_q) \geq 1 + \dim H_{xR_q}^1(H_{IR_q}^{j-1}(R_q)) \geq 1$. Let $\mathfrak{p}R_q \in \operatorname{Assh} H_{IR_q}^{j-1}(R_q))$, then $\dim R_q/\mathfrak{p}R_q \geq 1$ and so $\operatorname{height} q/\mathfrak{p} \geq 1$. Hence $\dim R/\mathfrak{p} = \operatorname{height} \mathfrak{m}/\mathfrak{p} \geq \operatorname{height} \mathfrak{m}/q + \operatorname{height} q/\mathfrak{p} \geq \dim R/q + 1 = d - j + 1 = d - (j-1)$. Consequently, $\operatorname{height} \mathfrak{p} \leq \dim R - \dim R/\mathfrak{p} \leq d - (d - (j-1)) = j - 1$. Since $\mathfrak{p}R_q \in \operatorname{Assh} H_{IR_q}^{j-1}(R_q)$, it follows that $\mathfrak{p}R_q \in \operatorname{Supp} H_{IR_q}^{j-1}(R_q)$ and so $\mathfrak{p} \in \operatorname{Supp} H_I^{j-1}(R)$. Hence, $0 \neq (H_I^{j-1}(R))_{\mathfrak{p}} \cong H_{IR_{\mathfrak{p}}}^{j-1}(R_{\mathfrak{p}})$, and by Grothendieck's Vanishing Theorem, $\operatorname{height} \mathfrak{p} = \dim R_{\mathfrak{p}} \geq j - 1$. Also $\operatorname{height} \mathfrak{p} + \dim R/\mathfrak{p} \leq \dim R$, shows taht $\dim R/\mathfrak{p} \leq d - (j-1)$. Now we conclude that $\operatorname{height} \mathfrak{p} = j - 1$, $\dim R/\mathfrak{p} = d - (j-1)$ and $\mathfrak{p} \in \operatorname{Supp} H_I^{j-1}(R)$. The inductive step is complete. □

Lemma 3. *Let R be a Noetherian ring and I be a proper ideal of R. Then* $\operatorname{Cd}(I, R) \leq \operatorname{ara}(I) \leq \dim R$.

Proof. See [9, Theorem 2.11]. □

Lemma 4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I be a proper ideal of R such that* $\operatorname{Cd}(I, R) = \dim R$. *Then* $\operatorname{ara}(I) = \dim R$.

Proof. Follows from Lemma 3. □

Theorem 3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I be a proper ideal of R. Then there exisrs $\mathfrak{p} \in V(I)$ such that*

$$\operatorname{Cd}(IR_{\mathfrak{p}}, R_{\mathfrak{p}}) = \operatorname{ara}(IR_{\mathfrak{p}}) = \dim R_{\mathfrak{p}} = \operatorname{height} \mathfrak{p} = \dim R - \dim R/\mathfrak{p}.$$

Proof. By Theorem 2, there exists $i \geq \operatorname{height} I$ and $\mathfrak{p} \in \operatorname{Supp} H_I^i(R)$ such that $\dim R/\mathfrak{p} = \dim R - i$, $\operatorname{height} \mathfrak{p} = i$ and $i = \dim R_{\mathfrak{p}} = \operatorname{height} \mathfrak{p} = \dim R - \dim R/\mathfrak{p}$, Consequently $0 \neq$

$(H_I^i(R))_\mathfrak{p} \cong H^i_{IR_\mathfrak{p}}(R_\mathfrak{p})$. In this case $i = \dim R_\mathfrak{p} \leq \mathrm{Cd}(IR_\mathfrak{p}, R_\mathfrak{p}) \leq \dim R_\mathfrak{p} = i$ and we conclude that $\mathrm{Cd}(IR_\mathfrak{p}, R_\mathfrak{p}) = \dim R_\mathfrak{p} = i$. Hence, by Lemma 4, $\mathrm{ara}(IR_\mathfrak{p}) = \dim R_\mathfrak{p}$. Therefore, we have the relations as follows,

$$\mathrm{Cd}(IR_\mathfrak{p}, R_\mathfrak{p}) = \mathrm{ara}(IR_\mathfrak{p}) = \dim R_\mathfrak{p} = \mathrm{height}\, \mathfrak{p} = \dim R - \dim R/\mathfrak{p} = i.$$

This completes the proof. □

Corollary 1. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* $\mathfrak{p} \in \mathrm{Spec}(R)$*, such that*

$$\{q \in V(\mathfrak{p}) \mid \mathrm{height}\, q + \dim R/q = \dim R\} = \{\mathfrak{m}\}.$$

Then $\mathrm{Cd}(\mathfrak{p}, R) = \mathrm{ara}(\mathfrak{p}) = \dim(R)$.

Proof. Follows from Theorem 3. □

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