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# Another view of BZ-algebras

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**Abstract.** In this work, Sheffer stroke BZ-algebra (briefly, SBZ-algebra) is introduced and its properties are examined. Then a partial order is defined on SBZ-algebras. It is shown that a Cartesian product of two SBZ-algebras is an SBZ-algebra. After giving SBZ-ideals and SBZ-subalgebras, it is proved that any SBZ-ideal of an SBZ-algebra is an ideal of this SBZ-algebra and vice versa, and that it is also an SBZ-subalgebra. Also, a congruence relation on an SBZ-algebra is determined by an SBZ-ideal, and the quotient of an SBZ-algebra by a congruence relation on this algebra is constructed. Thus, it is proved that the quotient of the SBZ-algebra is an SBZ-algebra. Furthermore, we define SBZ-homomorphisms between SBZ-algebras and state that the kernel of an SBZ-homomorphism is an SBZ-ideal and so an SBZ-subalgebra. Hence, a new SBZ-homomorphism is described by means of the kernel of an SBZ-homomorphism. Finally, we show that some properties are preserved under SBZ-homomorphisms.

*Keywords*: BZ-algebra, Sheffer stroke, Congruence, SBZ-homomorphism.
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## 1 Introduction

In the two-valued propositional logic, the operators $\neg$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction) suffice to express any Boolean function or axiom. A system containing these operators is called a functionally complete system. E. L. Post gave its formal proof [19]. H. M. Sheffer introduced the Sheffer stroke operation. He demonstrated that this operation can define Boolean functions, and so a system including only this operation is functionally complete [22]. In $n$-valued

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logics, Post showed that two functions suffice to express any Boolean function or axiom [19], but D. L. Webb proved that a single function suffices for the aforementioned goal [23]. Therefore, many researchers look after Sheffer stroke operation since every Boolean function or axiom can be restated via this operation [11]. This operation induces reductions of axioms or formulas for many algebraic structures. Thereby, many scientists wish to apply such a reduction to several algebraic structures such as ortholattices [2], orthoimplication algebras [1], (fuzzy) filters of Sheffer stroke BL-algebras [18], Sheffer stroke Hilbert algebras [12], fuzzy filters [13] and neutrosophic $N$-structures [15], Sheffer stroke BE-algebras [6], Sheffer stroke UP-algebras [14], filters and neutrosophic $N$-structures of strong Sheffer stroke non-associative MV-algebras ( [17], [16]). These reductions are suitable for many studies in logic and related areas because a system containing only the Sheffer stroke operation is complete (completeness of a logical system).

To solve some problems on BCK-algebras, Y. Komori introduced a BCC-algebra (or BIK-algebras called by some mathematicians) which is an algebraic model of BCC-logic (or implicational logic) [8, 9]. The generalizations of this algebraic structure were studied by several researchers. An algebraic system that has the partial order defined as in BCC-algebras and BCK-algebras but has not the minimal element is called a BZ-algebra ( [5], [24, 25]), a weak-BCC-algebra [4]. However, the first name is more popular.

Besides, many scientists independently studied the algebraic structures, such as BCI-algebras [7], B-algebras [3], implication algebras, G-algebras, Hilbert algebras, vs. All of these algebras which have a single distinguished element and some common features are a generalization or a special case of BCC/BCH/BCI/BCK-algebras [20]. Therefore, BZ-algebras are related to many logical algebras and came to many researchers notice. They studied their closed, anti-group, strong, regular and associative ideals, T-ideals (or QA-ideals), H-ideals playing a crucial role in the theory of ideals, as well as their subalgebras, a congruence relation, atoms, T-type BZ-algebras ( [5], [10], [26]) filters [21], relations with groups [25], derivations.

We first introduce the fundamental definition of a Sheffer Stroke BZ-algebra, and we show that its axioms are independent. By giving basic notions about this algebraic structure, we define a partial order on it and present its properties. Then it is proved that a Sheffer Stroke BZ-algebra is a BZ-algebra with the condition $x.y := (x|(y|y))|(x|(y|y))$ and vice versa, and that the Cartesian product of two SBZ-algebras is an SBZ-algebra. After describing an SBZ-ideal and giving its features, it is showed that any SBZ-ideal of an SBZ-algebra is an ideal of this SBZ-algebra and vice versa. It is proved that the family of all SBZ-ideals of an SBZ-algebra forms a complete lattice, and that for a subset of an SBZ-algebra there exists a minimal SBZ-ideal containing this subset. By describing an SBZ-subalgebra, we demonstrate that any SBZ-ideal is an SBZ-subalgebra, but by counterexample, that the converse is not true. Then a congruence relation on an SBZ-algebra described by its SBZ-ideal and related notions is expressed. It is shown that a quotient of an SBZ-algebra by a congruence is a SBZ-algebra. Finally defining SBZ-homomorphisms, it is indicated that the mentioned concepts are preserved under SBZ-homomorphisms.

## 2 Preliminaries

In this section, we provide fundamental definitions and notions about the Sheffer stroke operation and BZ-algebra.

**Definition 1** ([2]). *Let $\mathcal{X} = \langle X, |\rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:*
$(S1)$ $x|y = y|x$,
$(S2)$ $(x|x)|(x|y) = x$,
$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

**Definition 2** ([5]). *A nonempty set $X$ with a binary operation . denoted by juxtaposition and a distinguished element $0$ is called a BZ-algebra if it satisfies in the following conditions, for all $x, y, z \in X$:*
$(BZ-1) : ((x.z).(y.z)).(x.y) = 0$,
$(BZ-2) : x.0 = x$, *and*
$(BZ-3) : x.y = y.x = 0$ *implies* $x = y$.

**Definition 3** ([5]). *Let $\mathcal{X} = \langle X; ., 0\rangle$ be a BZ-algebra. Then the binary relation $\leq$ defined by $x \leq y$ if and only if $x.y = 0$ is a partial order on $A$.*

**Definition 4** ([5]). *A nonempty subset $I$ of a BZ-algebra $\mathcal{X} = \langle X; ., 0\rangle$ is called a BZ-ideal of $X$ if it satisfies the following properties:*
*(1)* $0 \in I$, *and*
*(2) for any $x, y, z \in A$, $(x.y).z \in I$ and $y \in I$ imply $x.z \in I$.*

**Lemma 1** ([5]). *Let $\mathcal{X} = \langle X; ., 0\rangle$ be a BZ-algebra and $I$ be a BZ-ideal of $X$. Then $x.y \in I$ and $y \in I$ imply $x \in I$. Particularly, $x \leq y$ and $y \in I$ imply $x \in I$.*

## 3 Sheffer stroke BZ-algebras

In this section, we define the Sheffer Stroke BZ-algebra and give some notions about it.

**Definition 5.** *A Sheffer stroke BZ-algebra (briefly, SBZ-algebras) is a structure $\langle X; |, 0\rangle$ of type $(2, 0)$ such that the binary operation $|$ is a Sheffer stroke operation, $0$ is a distinguished element in $X$ and the following axioms are satisfied for all $x, y, z \in X$:*
$(SBZ-1)$ $(x|(y|y))|(((y|(z|z))|((x|(z|z))|(x|(z|z))))|((y|(z|z))|((x|(z|z))|(x|(z|z))))) = x|(x|x)$.
$(SBZ-2)$ $(x|(y|y))|(x|(y|y)) = (y|(x|x))|(y|(x|x)) = 0$ *imply* $x = y$.

**Lemma 2.** *The axioms $(SBZ-1)$ and $(SBZ-2)$ are independent.*

*Proof.* We construct a model for each axiom in which this axiom is false while the other is true.

$(i)$ To show that independency of $(SBZ-1)$, consider the set $X = \{0, x, y\}$ with Cayley table as below:

| $\mid$ | $0$ | $x$ | $y$ |
|---|---|---|---|
| $0$ | $x$ | $x$ | $y$ |
| $x$ | $x$ | $0$ | $0$ |
| $y$ | $y$ | $0$ | $y$ |

$(SBZ-2)$ holds while $(SBZ-1)$ does not, since
$(0|(0|0))|(((0|(y|y))|((0|(y|y))|(0|(y|y))))|((0|(y|y))|((0|(y|y))|(0|(y|y))))) = 0 \neq x = 0|(0|0)$.

$(ii)$ To demonstrate that independency of $(SBZ-2)$, consider the set $X = \{0, x, y\}$ with Cayley table as below:

| $\mid$ | $0$ | $x$ | $y$ |
|---|---|---|---|
| $0$ | $x$ | $x$ | $x$ |
| $x$ | $x$ | $0$ | $0$ |
| $y$ | $x$ | $0$ | $0$ |

$(SBZ-1)$ holds but $(SBZ-2)$ does not since $x \neq y$ when $(x|(y|y))|(x|(y|y)) = 0 = (y|(x|x))|(y|(x|x))$. □

**Example 1.** Given a structure $\langle X; |, 0\rangle$ with the set $X = \{0, x, y, z, t, u, v, 1\}$ and the Cayley table as below:

| $\mid$ | $0$ | $x$ | $y$ | $z$ | $t$ | $u$ | $v$ | $1$ |
|---|---|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $x$ | $1$ | $v$ | $1$ | $1$ | $v$ | $v$ | $1$ | $v$ |
| $y$ | $1$ | $1$ | $u$ | $1$ | $u$ | $1$ | $u$ | $u$ |
| $z$ | $1$ | $1$ | $1$ | $t$ | $1$ | $t$ | $t$ | $t$ |
| $t$ | $1$ | $v$ | $u$ | $1$ | $z$ | $v$ | $u$ | $z$ |
| $u$ | $1$ | $v$ | $1$ | $t$ | $v$ | $y$ | $t$ | $y$ |
| $v$ | $1$ | $1$ | $u$ | $t$ | $u$ | $t$ | $x$ | $x$ |
| $1$ | $1$ | $v$ | $u$ | $t$ | $z$ | $y$ | $x$ | $0$ |

Then this structure is a SBZ-algebra.

**Lemma 3.** *In a SBZ-algebra $\langle X; |, 0\rangle$, the following hold for all $x, y, z \in X$:*

1. $(x|(x|x))|(x|(x|x)) = 0$.
2. $x|x = x|(0|0)$.
3. $0|(x|x) = 0|0$.
4. $x|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))) = 0|0$.
5. $x|((y|(z|z))|(y|(z|z))) = y|((x|(z|z))|(x|(z|z)))$.

*Proof.* 1. Putting, simultaneously, $[y := x]$ and $[z|z := y]$ in $(SBZ-1)$, we have

$$(x|(x|x))|(x|(x|x)) \;=\; ((x|(x|x))|(((x|y)|((x|y)|(x|y)))|((x|$$

$$\begin{aligned} & y)|((x|y)|(x|y)))))|((x|(x|x))|(((x|y) \\ & |((x|y)|(x|y)))|((x|y)|((x|y)|(x|y))))) \\ = \; & ((x|y)|((x|y)|(x|y)))|((x|y)|((x|y)|(x|y))) \end{aligned}$$

from $(S1)-(S3)$. Then it follows from $(S1)$ that

$$\begin{aligned} (x|(x|x))|(x|(x|x)) &= ((x|y)|((x|y)|(x|y)))|((x|y)|((x|y)|(x|y))) \\ &= ((y|x)|((y|x)|(y|x)))|((y|x)|((y|x)|(y|x))) \\ &= (y|(y|y))|(y|(y|y)). \end{aligned}$$

Thus, the SBZ-algebra $\langle X;|,0\rangle$ satisfies the identity $(x|(x|x))$ $|(x|(x|x)) = (y|(y|y))|(y|(y|y))$ for all $x, y \in X$. It means that the SBZ-algebra $\langle X;|,0\rangle$ has a distinguished element which will be denoted by 0, and therefore it satisfies $(x|(x|x))|(x|(x|x)) = 0$ for all $x \in X$.

2. We conclude from (1), $(S1)$ and $(S2)$ that

$$\begin{aligned} x|(0|0) &= x|(((x|(x|x))|(x|(x|x)))|((x|(x|x))|(x|(x|x)))) \\ &= x|(x|(x|x)) \\ &= ((x|x)|(x|x))|((x|x)|x) \\ &= x|x \end{aligned}$$

for all $x \in X$.

3. It follows from (2), $(S1)$ and $(S2)$ that

$$0|(x|x) = 0|(x|(0|0)) = ((0|0)|(0|0))|((0|0)|x) = 0|0,$$

for all $x \in X$.

4. We get

$$\begin{aligned} x|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))) &= ((x|(y|y))|(x|(y|y)))|(x|(y|y)) \\ &= (((x|(y|y))|((x|(y|y))|(x|(y|y))))| \\ &\quad ((x|(y|y))|((x|(y|y))|(x|(y|y)))))| \\ &\quad (((x|(y|y))|((x|(y|y))|(x|(y|y))))| \\ &\quad ((x|(y|y))|((x|(y|y))|(x|(y|y))))) \\ &= 0|0 \end{aligned}$$

from $(S1)-(S3)$ and (1).

5. It follows from $(S1)$ and $(S3)$.

□

**Lemma 4.** *Let $\langle X;|,0\rangle$ be a SBZ-algebra. Then the binary relation $\leq$ defined by $x \leq y$ if and only if $(x|(y|y))|(x|(y|y)) = 0$ is a partial order on $X$, and 0 is the smallest element of $X$.*

*Proof.* Let $\langle X; |, 0\rangle$ be a SBZ-algebra.

- Reflective: it follows from Lemma 3 (1).
- Antisymmetric: Let $x \leq y$ and $y \leq x$, i.e., $(x|(y|y))|(x|(y|y)) = (y|(x|x))|(y|(x|x)) = 0$. Then we get $x = y$ from $(SBZ-2)$.
- Transitive: Let $x \leq y$ and $y \leq z$, i.e., $(x|(y|y))|(x|(y|y)) = 0$ and $(y|(z|z))|(y|(z|z)) = 0$. We obtain $x|(y|y) = 0|0$ and $y|(z|z) = 0|0$ from $(S2)$. Then we conclude that

$$\begin{aligned}
0 &= (x|(x|x))|(x|(x|x)) \\
&= ((x|(y|y))|(((y|(z|z))|((x|(z|z))|(x|(z|z))))|((y|(z|z)) \\
&\quad |((x|(z|z))|(x|(z|z))))))|((x|(y|y))|(((y|(z|z))|((x| \\
&\quad (z|z))|(x|(z|z))))|((y|(z|z))|((x|(z|z))|(x|(z|z)))))) \\
&= ((0|0)|(((0|0)|((x|(z|z))|(x|(z|z))))|((0|0)|((x|(z|z))|(x|(z|z))))))| \\
&\quad ((0|0)|(((0|0)|((x|(z|z))|(x|(z|z))))|((0|0)|((x|(z|z))|(x|(z|z)))))) \\
&= (x|(z|z))|(x|(z|z)),
\end{aligned}$$

i.e., $x \leq z$ from Lemma 3 (1)-(2), $(SBZ-1)$, $(S1)$ and $(S2)$. Therefore, this relation is a partial order on $X$.

Since we know $0 = (0|0)|(0|0) = (0|(x|x))|(0|(x|x))$ from $(S2)$ and Lemma 3 (3), it follows that $0 \leq x$ for all $x \in X$, that is, 0 is the smallest element of $X$. □

**Lemma 5.** *In any SBZ-algebra $\langle X; |, 0\rangle$, the following property hold for all $x, y, z \in X$*

$$x \leq y \text{ imply } (y|(z|z)) \leq (x|(z|z)) \text{ and } (z|(x|x)) \leq (z|(y|y)).$$

*Proof.* Let $\langle X; |, 0\rangle$ be a SBZ-algebra and let $x \leq y$, i.e., $(x|(y|y))|(x|(y|y)) = 0$. Then we obtain

$$\begin{aligned}
x|(x|x) &= (x|(y|y))|(((y|(z|z))|((x|(z|z))|(x|(z|z)))) \\
&\quad |((y|(z|z))|((x|(z|z))|(x|(z|z))))) \\
&= (0|0)|(((y|(z|z))|((x|(z|z))|(x|(z|z))))| \\
&\quad ((y|(z|z))|((x|(z|z))|(x|(z|z))))) \\
&= (y|(z|z))|((x|(z|z))|(x|(z|z)))
\end{aligned}$$

from $(SBZ-1)$, $(S2)$ and Lemma 3 (2). So, we have $((y|(z|z))|((x|(z|z))|(x|(z|z))))|((y|(z|z))|((x|(z|z))|(x|(z|z)))) = (x|(x|x))|(x|(x|x)) = 0$ from Lemma 3 (1). Thus, $(y|(z|z)) \leq (x|(z|z))$.

Substituting, simultaneously, $[x := y|y]$, $[y := x|x]$ and $[z|z := z]$ in $(SBZ-1)$, we get $y|(y|y) = (z|(x|x))|((z|(y|y))|(z|(y|y)))$ from $(S1)$, $(S2)$ and Lemma 3 (2). Therefore, it follows from Lemma 3 (1) that $((z|(x|x))|((z|(y|y))|(z|(y|y))))|((z|(x|x))|((z|(y|y))|(z|(y|y)))) = (y|(y|y))|(y|(y|y)) = 0$, i.e., $(z|(x|x)) \leq (z|(y|y))$. □

**Theorem 1.** *Let $\langle X; |, 0\rangle$ be a SBZ-algebra. If we define $x.y := (x|(y|y))|(x|(y|y))$, then $\langle X; ., 0\rangle$ is a BZ-algebra.*

*Proof.* Let $x, y, z$ be arbitrary elements in $X$.

$(BZ-1)$ : We have

$$\begin{aligned}((x.z).(y.z)).(x.y) &= ((x|(y|y))|(((y|(z|z))|((x|(z|z))|(x|(z|z))))|((y \\ &\quad |(z|z))|((x|(z|z))|(x|(z|z))))))|((x|(y|y))|(((y|( \\ &\quad z|z))|((x|(z|z))|(x|(z|z))))|((y|(z|z))|((x|(z|z)) \\ &\quad |(x|(z|z)))))) \\ &= (x|(x|x))|(x|(x|x)) \\ &= 0\end{aligned}$$

from $(S1)$, $(S2)$, $(SBZ-1)$ and Lemma 3 (1).

$(BZ-2)$ : We get $x.0 = (x|(0|0))|(x|(0|0)) = (x|x)|(x|x) = x$ from $(S2)$ and Lemma 3 (2).

$(BZ-3)$ : Let $x.y = y.x = 0$, i.e., $(x|(y|y))|(x|(y|y)) = (y|(x|x))|(y|(x|x)) = 0$. Then we obtain $x = y$ from $(SBZ-2)$. □

**Example 2.** Consider the SBZ-algebra $\langle X; |, 0\rangle$ in Example 1. Then a structure $\langle X; ., 0\rangle$ defined by this SBZ-algebra is a BZ-algebra with the following Cayley table:

| . | 0 | $x$ | $y$ | $z$ | $t$ | $u$ | $v$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x$ | $x$ | 0 | $x$ | $x$ | 0 | 0 | $x$ | 0 |
| $y$ | $y$ | $y$ | 0 | $y$ | $o$ | $y$ | 0 | 0 |
| $z$ | $z$ | $z$ | $z$ | 0 | $z$ | 0 | 0 | 0 |
| $t$ | $t$ | $y$ | $x$ | $t$ | 0 | $y$ | $x$ | 0 |
| $u$ | $u$ | $z$ | $u$ | $x$ | $z$ | 0 | $x$ | 0 |
| $v$ | $v$ | $v$ | $z$ | $y$ | $z$ | $y$ | 0 | 0 |
| 1 | 1 | $v$ | $u$ | $t$ | $z$ | $y$ | $x$ | 0 |

**Theorem 2.** *Let $\langle X; |_X, 0_X\rangle$ and $\langle Y; |_Y, 0_Y\rangle$ be SBZ-algebras. Then, $\langle X \times Y; |_{X\times Y}, 0_{X\times Y}\rangle$ is a SBZ-algebra where the set $X \times Y$ is the Cartesian product of $X$ and $Y$, the operation $|_{X\times Y}$ is defined by*

$$(x_1, y_1)|_{X\times X}(x_2, y_2) = (x_1|_X x_2, y_1|_Y y_2),$$

*and the distinguished element is $0_{X\times Y} = (0_X, 0_Y)$.*

*Proof.* Straightforward. □

## 4 On ideals of SBZ-algebras

In this section, we give some definitions and notions about ideals and subalgebras of a SBZ-algebra.Let $X$ be a SBZ-algebra, unless otherwise is stated.

**Definition 6.** *A nonempty subset $I \subseteq X$ is called a SBZ-ideal of $X$ if it satisfies*
*$(SBZi-1)$ $0 \in I$,*
*$(SBZi-2)$ $(((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z)) \in I$ and $y \in I$ imply $(x|(z|z))|(x|(z|z)) \in I$ for all $x, y, z \in X$.*

**Example 3.** Consider the SBZ-algebra in Example 1. Then it is clear that $X$ itself and $\{0\}$ are SBZ-ideals of $X$. Also, $\{0,x\}$, $\{0,y\}$, $\{0,z\}$, $\{0,x,y,t\}$, $\{0,x,z,u\}$ and $\{0,y,z,v\}$ are some SBZ-ideals of $X$.

**Lemma 6.** *If $I$ is a SBZ-ideal of $X$, then the following holds:*
$(SBZi-3)$ *For all $x,y \in X$, $(x|(y|y))|(x|(y|y)) \in I$ and $y \in I$ imply $x \in I$.*
$(SBZi-4)$ *For all $x,y \in X$, $x \leq y$ and $y \in I$ imply $x \in I$.*

*Proof.* Let $I$ be a SBZ-ideal of $X$.

$(SBZi-3)$ Putting simultaneously, $[z := 0]$ in $(SBZi-2)$, it follows from $(S2)$ and $(SBZ-2)$ that

$$\begin{aligned}(x|(y|y))|(x|(y|y)) &= (((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))|\\ &\quad (((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))\\ &= (((x|(y|y))|(x|(y|y)))|(0|0))|\\ &\quad (((x|(y|y))|(x|(y|y)))|(0|0)) \in I\end{aligned}$$

and $y \in I$ imply $x = (x|x)|(x|x) = (x|(0|0))|(x|(0|0)) \in I$.

$(SBZi-4)$ It follows from $(SBZi-3)$ and Lemma 4. □

**Lemma 7.** *Let $I$ be a subset of $X$ such that $0 \in I$.*

*$(i)$ If $(x|(y|y))|(x|(y|y)) \in I$ and $y \in I$ imply $x \in I$, then $I$ is a SBZ-ideal of $X$.*

*$(ii)$ If $x \leq y$ and $y \in I$ imply $x \in I$, then $I$ is a SBZ-ideal of $X$.*

*Proof.* $(i)$ Assume that $I$ is a subset of $X$ such that $0 \in I$, and that $(x|(y|y))|(x|(y|y)) \in I$ and $y \in I$ imply $x \in I$. Let $(((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z)) \in I$ and $y \in I$. We know $(((x|(z|z))|(x|(z|z)))|(y|y))|(((x|(z|z))|(x|(z|z)))|(y|y)) = (((x|(y|y))|(x|(y|y)))|(z|z))$ $|(((x|(y|y))|(x|(y|y)))|(z|z)) \in I$ from $(S1)$ and $(S3)$. Then we obtain $(x|(z|z))|(x|(z|z)) \in I$ by the assumption.

$(ii)$ Suppose that $I$ is a subset of $X$ such that $0 \in I$, and that $x \leq y$ and $y \in I$ imply $x \in I$, that is, $(x|(y|y))|(x|(y|y)) = 0 \in I$ and $y \in I$ imply $x \in I$. Thus, $I$ is a SBZ-ideal of $X$ from $(i)$. □

**Lemma 8.** *A subset $I$ is a SBZ-ideal of $X$ if and only if $0 \in I$ and $x \leq y$ and $y \in I$ imply $x \in I$.*

*Proof.* $(\Rightarrow)$ It follows from $(SBZi-1)$ and $(SBZi-4)$.

$(\Leftarrow)$ It follows from Lemma 7 (ii). □

**Theorem 3.** *The family $\Lambda_X$ of all SBZ-ideals of $X$ forms a complete lattice.*

*Proof.* Let $\{I_i\}_{i \in J}$ be a family of SBZ-ideals of $X$. Since we know $0 \in I_i$ for all $i \in J$, it follows $0 \in \bigcup_{i\in J} I_i$ and $0 \in \bigcap_{i\in J} I_i$.

$(i)$ Assume that $(((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z)) \in \bigcap_{i\in J} I_i$ and $y \in \bigcap_{i\in J} I_i$ hold for any $x,y,z \in X$. Therefore, we have $(((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z))$ $\in I_i$ and $y \in I_i$ hold for all $i \in J$. Then, we have $(x|(z|z))|(x|(z|z)) \in I_i$ for all $i \in J$, since every $I_i$ is a SBZ-ideal of $X$. Thus, $(x|(z|z))|(x|(z|z)) \in \bigcap_{i\in J} I_i$.

$(ii)$ Let $\Gamma$ be the family of all SBZ-ideals of $X$ contained in the union $\bigcup_{i\in J} I_i$. The $\bigcap \Gamma$ is an SBZ-ideal of $X$ from (i). If $\bigwedge_{i\in J} I_i = \bigcap_{i\in J} I_i$ and $\bigvee_{i\in J} I_i = \bigcap \Gamma$, then $(\Lambda_X, \bigwedge, \bigvee)$ is a complete lattice. □

**Corollary 1.** *Let $Y$ be a subset of an SBZ-algebra $X$. Then there is the minimal SBZ-ideal $\langle Y\rangle$ containing the subset $Y$.*

*Proof.* Let $C = \{I : I\ is\ an\ SBZ-ideal\ of\ X\ containing\ Y \subseteq X\}$. Then $\bigcap C$ is the minimal SBZ-ideal of $X$ containing $Y \subseteq X$. □

**Definition 7.** *A subset $Y$ of $X$ is called an SBZ-subalgebra of $X$ if the distinguished element $0$ of $X$ is in $Y$ and $\langle Y; |, 0\rangle$ forms an SBZ-algebra. Clearly, $X$ itself and $\{0\}$ are SBZ-subalgebras of $X$.*

**Lemma 9.** *Any SBZ-ideal of an SBZ-algebra $X$ is an SBZ-subalgebra of $X$.*

*Proof.* Let $I$ be an SBZ-ideal of $X$. We know $0 \in I$ from $(SBZi-1)$. Then the SBZ-ideal $I$ satisfies $(SBZ-1)-(SBZ-2)$ for all $x, y, z \in I$ because $I \subseteq X$ and $X$ is an SBZ-algebra. Thus, $\langle I; |, 0\rangle$ is an SBZ-algebra. □

However, the converse of Lemma 9 is not true.

**Example 4.** Given the SBZ-algebra $X$ in Example 1. Then a subset $S = \{0, x, v, 1\}$ of $X$ is an SBZ-subalgebra of $X$ but it is not an SBZ-ideal of $X$, since $(y|(x|x))|(y|(x|x)) = y \notin S$ when $(((y|(1|1))|(y|(1|1)))|(x|x))|(((y|(1|1))|(y|(1|1)))|(x|x)) = 0 \in S$ and $1 \in S$.

**Definition 8.** *Let $I$ be an SBZ-ideal of $X$. We define the binary relation $\sim_I$ on $X$ as follows: for all $x, y \in X$*

*$x \sim_I y\ if\ and\ only\ if\ (x|(y|y))|(x|(y|y)) \in I$ and $(y|(x|x))|(y|(x|x)) \in I$.*

**Example 5.** Consider the SBZ-ideal $I = \{0, x\}$ in Example 3. Then $\sim_I = \{(0,0),(x,x),(y,y),(z,z),(t,t),(u,u),(v,v),(1,1),(0,x),(x,0),(1,v),(v,1),(y,t),(t,y),(u,z),(z,u)\}$ is a binary relation on $X$. It can be seen easily that $\sim_I$ is an equivalence relation on $X$.

**Definition 9.** *If $x\rho y$ implies $(x|z)|(x|z)\rho(y|z)|(y|z)$, for all $x, y, z \in X$, then the equivalence relation $\rho$ is called a congruence relation on $X$.*

**Example 6.** Consider the SBZ-algebra in Example 1. Then the equivalence relation $\sim_I = \{(0,0),(x,x),(y,y),(z,z),(t,t),(u,u),(v,v),(1,1),(0,x),(x,0),(1,v),(v,1),(y,t),(t,y),(u,z),(z,u)\}$ is a congruence on $X$.

**Lemma 10.** *An equivalence relation $\rho$ is a congruence on $X$ if and only if $x\rho y$ and $u\rho v$ imply $(x|u)|(x|u)\rho(y|v)|(y|v)$, for all $x, y, u, v \in X$.*

*Proof.* Let $\rho$ be a congruence on $X$, and let $x, y, u, v$ be any elements in $X$ such that $x\rho y$ and $u\rho v$. Then it follows from $(S1)$ $(x|u)|(x|u)\rho(y|u)|(y|u)$ and $(y|u)|(y|u)\rho(y|v)|(y|v)$. Thus, we have $(x|u)|(x|u)\rho(y|v)|(y|v)$ from transitivity of $\rho$.

Conversely, assume that $x\rho y$ and $u\rho v$ imply $(x|u)|(x|u)\rho(y|v)|(y|v)$ for any $x, y, u, v \in X$. Let $x, y, z$ be any elements in $X$ such that $x\rho y$. Since $z\rho z$, $(x|z)|(x|z)\rho(y|z)|(y|z)$ from the assumption. Then $\rho$ is a congruence on $X$. □

**Lemma 11.** *Let $I$ be an SBZ-ideal of $X$ and the relation $\sim_I$ is defined as Definition 8. Then, $\sim_I$ is a congruence on $X$.*

*Proof.* We first show that the binary relation $\sim_I$ is an equivalence relation on $X$.

- Reflexive: Since we know $(x|(x|x))|(x|(x|x)) = 0$ from Lemma 3 (1) and $I$ is an SBZ-ideal of $X$, we have $(x|(x|x))|(x|(x|x)) = 0 \in I$, i.e., $x \sim_I x$ for all $x \in X$.
- Symmetric: Let $x, y$ be arbitrary elements in $X$ such that $x \sim_I y$, i.e., $(x|(y|y))|(x|(y|y)) \in I$ and $(y|(x|x))|(y|(x|x)) \in I$. Then we obtain $(y|(x|x))|(y|(x|x)) \in I$ and $(x|(y|y))|(x|(y|y)) \in I$, i.e., $y \sim_I x$.
- Transitive: Let $x, y, z$ be arbitrary elements in $X$ such that $x \sim_I y$ and $y \sim_I z$, i.e.,

$$(x|(y|y))|(x|(y|y)), (y|(x|x))|(y|(x|x)) \in I,$$

and

$$(y|(z|z))|(y|(z|z)), (z|(y|y))|(z|(y|y)) \in I.$$

Since we know $(((((x|(z|z))|(x|(z|z)))|(((y|(z|z))|(y|(z|z)))|((y|(z|z))|(y|(z|z)))))|(((x|(z|z))|(x|(z|z)))|(((y|(z|z))$ $|(y|(z|z)))|((y|(z|z))|(y|(z|z))))))|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y)))))|(((((x|(z|z))|(x|(z|z)$ $))|(((y|(z|z))|(y|(z|z)))|((y|(z|z))|(y|(z|z)))))|(((x|(z|z))|(x|(z|z)))|(((y|(z|z))|(y|(z|z)))|((y|(z|z))|$ $(y|(z|z))))))|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))) = ((x|(y|y))|(((y|(z|z))|((x|(z|z))|(x|(z|z))$ $))|((y|(z|z))|((x|(z|z))|(x|(z|z))))))|((x|(y|y))|(((y|(z|z))|((x|(z|z))|(x|(z|z))))|((y|(z|z))|((x|(z|z))|$ $(x|(z|z)))))) = (x|(x|x))|(x|(x|x)) = 0 \in I$ and $(y|(z|z))|(y|(z|z)) \in I$, we have $(((x|(z|z))|(x|(z|z)))$ $|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y)))))|(((x|(z|z))|(x|(z|z)))|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|$ $(x|(y|y))))) \in I$ by $(SBZi-2)$. By $(SBZi-2)$ and the fact that $(((x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y)))))|(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))))$ $|(z|z))|(((x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|()y|y)))))|(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y)$ $)))))|(z|z)) = (((x|(z|z))|(x|(z|z)))|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y)))))|(((x|(z|z))|(x|(z|z)$ $))|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))) \in I$ from $(S1)$ and $(S3)$, and $(x|(y|y))|(x|(y|y)) \in I$, we get $(x|(z|z))|(x|(z|z)) \in I$. Similarly, we obtain $(z|(x|x))|(z|(x|x)) \in I$ by putting, simultaneously, $[z := x]$ and $[x := z]$ in above statements. So, $x \sim_I z$. Thereby, $\sim_I$ is an equivalence relation on $X$.

Now, we demonstrate that the equivalence relation $\sim_I$ is a congruence on $X$. Let $x, y, u, v$ be any elements in $X$ such that $x \sim_I u$ and $y \sim_I v$, i.e., $(x|(u|u))|(x|(u|u)) \in I$, $(u|(x|x))|(u|(x|x)) \in I$, and $(y|(v|v))|(y|(v|v)) \in I$, $(v|(y|y))|(v|(y|y)) \in I$.

$(i)$ Since we know $((((x|y)|(((v|(y|y))|(v|(y|y)))|((v|(y|y))|(v|(y|y)))))|((x|y)|(((v|(y|y))|(v|(y|y)))|((v|(y|y))|(v|(y|y))$ $))))|((x|v)|(x|v)))|((((x|y)|(((v|(y|y))|(v|(y|y)))|((v|(y|y))|(v|(y|y)))))|((x|y)|(((v|(y|y))|(v|(y|y)))|$ $((v|(y|y))|(v|(y|y))))))|((x|v)|(x|v))) = ((x|((y|y)|(y|y)))|((((y|y)|((v|v)|(v|v)))|((x|((v|v)|(v|v)))|(x|$ $((v|v)|(v|v)))))|(((y|y)|((v|v)|(v|v)))|((x|((v|v)|(v|v)))|(x|((v|v)|(v|v)))))))|((x|((y|y)|(y|y)))|((((y|y)$ $|((v|v)|(v|v)))|((x|((v|v)|(v|v)))|(x|((v|v)|(v|v)))))|(((y|y)|((v|v)|(v|v)))|((x|((v|v)|(v|v)))|(x|((v|v)$ $|(v|v))))))) = (x|(x|x))|(x|(x|x)) = 0 \in I$ from $(S1)-(S3)$, $(SBZ-1)$ and Lemma 3 (1), we get $(((x|v)|(x|v))|(((x|y)|(x|y))|((x|y)|(x|y))))|(((x|v)|(x|v))|(((x|y)|(x|y))|((x|y)|(x|y)))) =$ $((x|y)|((x|v)|(x|v)))|((x|y)|((x|v)|(x|v))) \in I$ from $(S1)-(S2)$ and $(SBZi-2)$.

$(ii)$ Similarly, we have$(((x|y)|(x|y))|(((x|v)|(x|v))|((x|v)|(x|v))))|(((x|y)|(x|y))|(((x|v)|(x|v))|((x|v)|(x|v)))) = ((x|v)|((x|y)|(x|y)))|((x|v)|((x|y)|(x|y))) \in I$ by substituting, simultaneously, $[y := v]$ and $[v := y]$ in (i). Then $(x|y)|(x|y) \sim_I (x|v)|(x|v)$.

$(iii)$ By putting, simultaneously, $[x := v]$, $[y := x]$ and $[v := u]$ in $(i)$ and $(ii)$, we obtain $(((u|v)|(u|v))|(((x|v)|(x|v))|((x|v)|(x|v))))|(((u|v)|(u|v))|(((x|v)|(x|v))|((x|v)|(x|v)))) = ((x|v)|((u|v)|(u|v)))|((x|v)|((u|v)|(u|v))) \in I$ and $(((x|v)|(x|v))|(((u|v)|(u|v))|((u|v)|(u|v))))|(((x|v)|(x|v))|(((u|v)|(u|v))|((u|v)|(u|v)))) = ((u|v)|((x|v)|(x|v)))|((u|v)|((x|v)|(x|v))) \in I$ from $(S1) - (S2)$. Then $(x|v)|(x|v) \sim_I (u|v)|(u|v)$.

Thus, it follows $(x|y)|(x|y) \sim_I (u|v)|(u|v)$ from transitivity of $\sim_I$. □

**Theorem 4.** *If $I$ is an SBZ-ideal of $X$ and $\sim$ is the congruence on $X$ determined by $I$, then $X/I \equiv X/\sim = \{[x]_\sim : x \in X\}$ is also an SBZ-algebra with the operation $|_\sim$ defined by $[x]_\sim|_\sim[y]_\sim = [x|y]_\sim$ for all $x, y \in X$ and the distinguished element $I$.*

*Proof.* Suppose that $I$ is an SBZ-ideal of $X$ and $\sim$ is the congruence on $X$ determined by $I$. Let $X/I \equiv X/\sim = \{[x]_\sim : x \in X\}$ be a structure with the operation $|_\sim$ defined by $[x]_\sim|_\sim[y]_\sim = [x|y]_\sim$ for all $x, y \in X$.

First, we show $[0]_\sim = I$. For any $x \in [0]_\sim$, we get $x \sim 0$, i.e., $0 = (0|0)|(0|0) = (0|(x|x))|(0|(x|x)) \in I$ and $x = (x|x)|(x|x) = (x|(0|0))|(x|(0|0)) \in I$ from $(S2)$ and Lemma 3 (2)-(3). So, $[0]_\sim \subseteq I$. Because it follows from $(S2)$, and Lemma 3 (2)-(3) that $x = (x|x)|(x|x) = (x|(0|0))|(x|(0|0)) \in I$ and $0 = (0|0)|(0|0) = (0|(x|x))|(0|(x|x)) \in I$ for any $x \in I$, we obtain $x \sim 0$, i.e., $x \in [0]_\sim$. Then $I \subseteq [0]_\sim$.

Now, we demonstrate that the structure $X/I \equiv X/\sim = \{[x]_\sim : x \in X\}$ is an SBZ-algebra.

$(SBZ-1)$ : We have
$([x]_\sim|_\sim([y]_\sim|_\sim[y]_\sim))|_\sim((([y]_\sim|_\sim([z]_\sim|_\sim[z]_\sim))|_\sim(([x]_\sim|_\sim([z]_\sim|_\sim[z]_\sim))|_\sim([x]_\sim|_\sim([z]_\sim|_\sim[z]_\sim))))|_\sim(([y]_\sim|_\sim([z]_\sim|_\sim[z]_\sim))|_\sim(([x]_\sim|_\sim([z]_\sim|_\sim[z]_\sim))|_\sim([x]_\sim|_\sim([z]_\sim|_\sim[z]_\sim))))) = [(x|(y|y))|(((y|(z|z))|((x|(z|z))|(x|(z|z))))|((y|(z|z))|((x|(z|z))|(x|(z|z)))))]_\sim = [x|(x|x)]_\sim = [x]_\sim|_\sim([x]_\sim|_\sim[x]_\sim)$.

$(SBZ-2)$ : Let $([x]_\sim|_\sim([y]_\sim|_\sim[y]_\sim))|_\sim([x]_\sim|_\sim([y]_\sim|_\sim[y]_\sim)) = ([y]_\sim|_\sim([x]_\sim|_\sim [x]_\sim))|_\sim([y]_\sim|_\sim([x]_\sim|_\sim[x]_\sim)) = [0]_\sim$, i.e., $[(x|(y|y))|(x|(y|y))]_\sim = [(y|(x|x))|(y|(x|x))]_\sim = [0]_\sim$. Then, we obtain $((x|(y|y))|(x|(y|y))), ((y|(x|x))|(y|(x|x))) \in [0]_\sim = I$. So, it follows $x \sim y$, i.e., $[x]_\sim = [y]_\sim$. □

**Example 7.** Consider Example 6. Then

$$X/I \equiv X/\sim = \{[0]_\sim, [y]_\sim, [z]_\sim, [1]_\sim\}$$

is an SBZ-algebra with the following Cayley table and the distinguished element is $[0]_\sim = I$.

| $\vert_\sim$ | $[0]_\sim$ | $[y]_\sim$ | $[z]_\sim$ | $[1]_\sim$ |
|---|---|---|---|---|
| $[0]_\sim$ | $[1]_\sim$ | $[1]_\sim$ | $[1]_\sim$ | $[1]_\sim$ |
| $[y]_\sim$ | $[1]_\sim$ | $[z]_\sim$ | $[1]_\sim$ | $[z]_\sim$ |
| $[z]_\sim$ | $[1]_\sim$ | $[1]_\sim$ | $[y]_\sim$ | $[y]_\sim$ |
| $[1]_\sim$ | $[1]_\sim$ | $[z]_\sim$ | $[y]_\sim$ | $[0]_\sim$ |

## 5 SBZ-homomorphisms on SBZ-algebras

In this section, we introduce some definitions and notions about homomorphisms on SBZ-algebras.

**Definition 10.** *Let* $\langle X; |_X, 0_X\rangle$ *and* $\langle Y; |_Y, 0_Y\rangle$ *be SBZ-algebras. A mapping* $f : X \longrightarrow Y$ *is called an SBZ-homomorphism if*

$$f(x_1|_X x_2) = f(x_1)|_Y f(x_2)$$

*for all* $x, y \in X$.

**Lemma 12.** *Let* $\langle X; |_X, 0_X\rangle$ *and* $\langle Y; |_Y, 0_Y\rangle$ *be SBZ-algebras, and let the mapping* $f : X \longrightarrow Y$ *be an SBZ-homomorphism. Then* $f(X)$ *is an SBZ-ideal of* $Y$ *and* $Kerf$ *is an SBZ-ideal of* $X$. *Moreover,* $f(X)$ *is an SBZ-subalgebra of* $Y$ *and* $Kerf$ *is a SBZ-subalgebra of* $X$.

*Proof.* Let $\langle X; |_X, 0_X\rangle$ and $\langle Y; |_Y, 0_Y\rangle$ be SBZ-algebras, and let the mapping $f : X \longrightarrow Y$ be an SBZ-homomorphism.

• We show that $f(X)$ is an SBZ-ideal of $Y$.

$(SBZi-1)$ We have Since $f$ is a SBZ-homomorphism, we have

$$\begin{aligned} f(0_X) &= f((0_X|_X(0_X|_X 0_X))|_X(0_X|_X(0_X|_X 0_X))) \\ &= (f(0_X)|_Y(f(0_X)|_Y f(0_X)))|_Y(f(0_X)|_Y(f(0_X)|_Y f(0_X))) \\ &= 0_Y \end{aligned}$$

from Lemma 3 (1). Then $0_Y = f(0_X) \in f(X)$.

$(SBZi-2)$ Let $(((f(x_1)|_Y(f(x_2)|_Y f(x_2)))|_Y(f(x_1)|_Y(f(x_2)|_Y f(x_2))))|_Y(f(x_3)|_Y f(x_3)))|_Y(((f(x_1)|_Y (f(x_2)|_Y f(x_2)))|_Y(f(x_1)|_Y(f(x_2)|_Y f(x_2))))|_Y(f(x_3)|_Y f(x_3))) \in f(X)$ and $f(x_2) \in f(X)$. Then we get $f((((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))|_X(((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X (x_3|_X x_3))) \in f(X)$ and $f(x_2) \in f(X)$, that is, $(((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3)|_X x_3)|_X (((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))) \in X$ and $x_2 \in X$. Since $X$ is an SBZ-ideal of $X$, we obtain $(x_1|_X(x_3|_X x_3))|_X(x_1|_X(x_3|_X x_3)) \in X$. In other words, we get $(f(x_1)|_Y(f(x_3)|_Y f(x_3))) |_Y(f(x_1)|_Y(f(x_3|_Y f(x_3))) = f((x_1|_X(x_3|_X x_3))|_X(x_1|_X(x_3|_X x_3))) \in f(X)$.

• We demonstrate that $Kerf = \{x \in X : f(x) = 0_Y\}$ is an SBZ-ideal of $Y$.

$(SBZi-1)$ Since we know $f(0_X) = 0_Y$, we get $0_X \in Kerf$.

$(SBZi-2)$ Let $(((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))|_X(((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2))) |_X(x_3|_X x_3)) \in Kerf$ and $x_2 \in Kerf$, i.e., $(((f(x_1)|_Y(f(x_2)|_Y f(x_2)))|_Y(f(x_1)|_Y(f(x_2)|_Y f(x_2)))) |_Y(f(x_3)|_Y f(x_3)))|_Y(((f(x_1)|_Y(f(x_2)|_Y f(x_2)))|_Y(f(x_1)|_Y(f(x_2)|_Y f(x_2))))|_Y(f(x_3)|_Y f(x_3))) = f((((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))|_X(((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))) = 0_Y$ and $f(x_2) = 0_Y$. Then we have

$$\begin{aligned} 0_Y &= (((f(x_1)|_Y(0_Y|_Y 0_Y))|_Y(f(x_1)|_Y(0_Y|_Y 0_Y)))|_Y(f(x_3)|_Y f(x_3)))|_Y \\ &\quad (((f(x_1)|_Y(0_Y|_Y 0_Y))|_Y(f(x_1)|_Y(0_Y|_Y 0_Y)))|_Y(f(x_3)|_Y f(x_3))) \\ &= (f(x_1)|_Y(f(x_3)|_Y f(x_3)))|_Y(f(x_1)|_Y(f(x_3)|_Y f(x_3))) \\ &= f((x_1|_X(x_3|_X x_3))|_X(x_1|_X(x_3|_X x_3))) \end{aligned}$$

from Lemma 3 (2) and ($S2$). So, we obtain

$$(x_1|_X(x_3|_X x_3))|_X(x_1|_X(x_3|_X x_3)) \in Kerf.$$

Moreover, $f(X)$ is a SBZ-subalgebra of $Y$ and $Kerf$ is a SBZ-subalgebra of $X$ from Lemma 9. □

**Theorem 5.** *Let $\langle X; |_X, 0_X\rangle$ and $\langle Y; |_Y, 0_Y\rangle$ be SBZ-algebras, and let the mapping $f : X \longrightarrow Y$ be an SBZ-homomorphism. Then the followings are satisfied:*
($a$) *If $I$ is an SBZ-ideal of $X$, then $f(I)$ is an SBZ-ideal of $f(X)$.*
($b$) *If $K$ is an SBZ-ideal of $Y$, then $f^{-1}(K)$ is an SBZ-ideal of $X$.*

*Proof.* Let $\langle X; |_X, 0_X\rangle$ and $\langle Y; |_Y, 0_Y\rangle$ be SBZ-algebras, and let the mapping $f : X \longrightarrow Y$ be an SBZ-homomorphism.

($a$) Suppose that $I$ is an SBZ-ideal of $X$. Then we obtain $0_Y = f(0_X) \in f(I)$. Let $(((f(x_1)|_Y(f(x_2)|_Y f(x_2)))|_Y(f(x_1)|_Y(f(x_2)|_Y f(x_2)))|_Y(f(x_3)|_Y f(x_3)))|_Y(((f(x_1)|_Y(f(x_2)|_Y f(x_2)))$ $|_Y(f(x_1)|_Y(f(x_2)|_Y f(x_2))))|_Y(f(x_3)|_Y f(x_3))) \in f(I)$ and $f(x_2) \in f(I)$. It means that $f((((x_1|_X(x_2$ $|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))|_X(((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))) \in f(I)$ and $f(x_2) \in f(I)$. So, we get $(((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))|_X(((x_1|_X(x_2|_X x_2))$ $|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3)) \in I$ and $x_2 \in I$. Since $I$ is an SBZ-ideal of $X$, we have $(x_1|_X(x_3|_X x_3))$ $|_X(x_1|_X(x_3|_X x_3)) \in I$, i.e., $(f(x_1)|_Y(f(x_3)|_Y f(x_3)))|_Y(f(x_1)|_Y(f(x_3)|_Y f(x_3)))$ $= f((x_1|_X(x_3|_X x_3))|_X(x_1|_X(x_3|_X x_3))) \in f(I)$.

($b$) Assume that $K$ is an SBZ-ideal of $Y$. Because $f(0_X) = 0_Y \in K$, we have $0_X = f^{-1}(0_Y) \in f^{-1}(K)$. Let $(((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))|_X(((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))$ $|_X(x_3|_X x_3)) \in f^{-1}(K)$ and $x_2 \in f^{-1}(K)$. In means that $(((f(x_1)|_Y(f(x_2)|_Y f(x_2)))|_Y(f(x_1)|_Y(f(x_2)$ $|_Y f(x_2))))|_Y(f(x_3)|_Y f(x_3)))|_Y(((f(x_1)|_Y(f(x_2)|_Y f(x_2)))|_Y(f(x_1)|_Y(f(x_2)|_Y f(x_2))))$ $|_Y(f(x_3)|_Y f(x_3))) = f((((x_1|_X(x_2|_X x_2))|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))|_X(((x_1|_X(x_2|_X x_2))$ $|_X(x_1|_X(x_2|_X x_2)))|_X(x_3|_X x_3))) \in K$ and $f(x_2) \in K$. Since $K$ is an SBZ-ideal of $Y$, we obtain $f((x_1|_X(x_3|_X x_3))|_X(x_1|_X(x_3|_X x_3))) = (f(x_1)|_Y(f(x_3)|_Y f(x_3)))|_Y(f(x_1)|_Y(f(x_3)|_Y f(x_3))) \in K$, i.e., $(x_1|_X(x_3|_X x_3))|_X(x_1|_X(x_3|_X x_3)) \in f^{-1}(K)$. □

**Theorem 6.** *Let $f : X \longrightarrow Y$ be an SBZ-homomorphism between SBZ-algebras $\langle X; |_X, 0_X\rangle$ and $\langle Y; |_Y, 0_Y\rangle$. Then there exists an SBZ-homomorphism*

$$g : X/Kerf \longrightarrow f(X)$$

*such that $f = g \circ \pi$ where $\pi : X \longrightarrow X/Kerf$ is the canonical SBZ-epimorphism.*

*Proof.* Since $Kerf$ is an SBZ-ideal of $X$, we know that $\pi : X \longrightarrow X/Kerf$, $x \longmapsto [x]_\sim$ is a canonical SBZ-epimorphism by the definition of $\pi$. Then we get $g : X/Kerf \longrightarrow f(X)$ is an SBZ-homomorphism because we know $0_Y = f(0_X) = (g \circ \pi)(0_X) = g(\pi(0_X)) = g([0_X]_\sim) = g(Kerf)$ and $g([x_1]_\sim|_\sim[x_2]_\sim) = g([x_1|_X x_2]_\sim) = g(\pi(x_1|_X x_2)) = (g \circ \pi)(x_1|_X x_2) = f(x_1|_X x_2) = f(x_1)|_Y f(x_2) = (g \circ \pi)(x_1)|_Y(g \circ \pi)(x_2) = g(pi(x_1))|_Y g(\pi(x_2)) = g([x_1]_\sim)|_Y g([x_2]_\sim)$ for any elements $[x_1]_\sim, [x_2]_\sim \in X/Kerf$. □

**Theorem 7.** *Let $f : X \longrightarrow Y$ be a SBZ-homomorphism between SBZ-algebras $\langle X; |_X, 0_X\rangle$ and $\langle Y; |_Y, 0_Y\rangle$, and $I$ be an SBZ-ideal of $X$. $K$ is a SBZ-ideal of $X$ such that $I \subseteq K$ if and only if the set $K/I = \{[x]_I \in X/I : x \in K\}$ is an SBZ-ideal of an SBZ-algebra $X/I$.*

*Proof.* ($\Rightarrow$) We know that $\langle X/I, |_I, I\rangle$ is an SBZ-algebra and $I = [0]_I \in X/I$ is a distinguished element in $X/I$ from Theorem 4. Then we get $I = [0]_I \in K/I$ by the definition of $K/I$. Let $((([x_1]_I|_I([x_2]_I|_I[x_2]_I))|_I([x_1]_I|_I([x_2]_I|_I[x_2]_I)))|_I([x_3]_I|_I[x_3]_I))|_I((([x_1]_I|_I([x_2]_I|_I[x_2]_I))|_I([x_1]_I|_I([x_2]_I|_I[x_2]_I)))|_I([x_3]_I|_I[x_3]_I)) \in K/I$ and $[x_2]_I \in K/I$. That is, $[(((x_1|_X(x_2|_Xx_2))|_X(x_1|_X(x_2|_Xx_2)))|_X(x_3|_Xx_3))|_X(((x_1|_X(x_2|_Xx_2))|_X(x_1|_X(x_2|_Xx_2)))|_X(x_3|_Xx_3))]_I \in K/I$ and $[x_2]_I \in K/I$. Then we get $(((x_1|_X(x_2|_Xx_2))|_X(x_1|_X(x_2|_Xx_2)))|_X(x_3|_Xx_3))|_X(((x_1|_X(x_2|_Xx_2))|_X(x_1|_X(x_2|_Xx_2)))|_X(x_3|_Xx_3)) \in K$ and $x_2 \in K$. Since $K$ is an SBZ-ideal of $X$, we conclude $(x_1|_X(x_3|_Xx_3))|_X(x_1|_X(x_3|_Xx_3)) \in K$. Thus, it follows

$$([x_1]_I|_I([x_3]_I|_I[x_3]_I))|_I([x_1]_I|_I([x_3]_I|_I[x_3]_I)) = [(x_1|_X(x_3|_Xx_3))|_X(x_1|_X(x_3|_Xx_3))]_I \in K/I.$$

($\Leftarrow$) As in necessary condition, it is proved from the definition of $K/I$. □

**Theorem 8.** *Let $I$ and $K$ be SBZ-ideals of $X$ such that $I \subseteq K$. Then*

$$(X/I)/(K/I) \cong X/K.$$

*Proof.* We know $K/JI = \{[x]_I \in X/I : x \in K\}$ is an SBZ-ideal of a SBZ-algebra $X/I = \{[x]_I : x \in X\}$. The factor-set $(X/I)/(K/I) = \{[[x]_I]_{K/I} : [x]_I \in X/I\}$ can be properly defined as an SBZ-algebra on the SBZ-algebra $X/I$ by its SBZ-ideal $K/I$. We define $\varphi : (X/I)/(K/I) \longrightarrow X/K$, $[[x]_I]_{K/I} \longmapsto [x]_K$. Because

$$\begin{aligned}
\varphi([[x_1]_I]_{K/I}|_{K/I}[[x_2]_I]_{K/I}) &= \varphi([[x_1]_I|_I[x_2]_I]_{K/I}) \\
&= \varphi([[x_1|_Xx_2]_I]_{K/I}) \\
&= [x_1|_Xx_2]_K \\
&= [x_1]_K|_K[x_2]_K \\
&= \varphi([[x_1]_I]_{K/I})|_K\varphi([[x_2]_I]_{K/I})
\end{aligned}$$

for arbitrary elements $[[x_1]_I]_{K/I}, [[x_2]_I]_{K/I} \in (X/I)/(K/I)$ and $K = [0]_K = \varphi([[0]_I]_{K/I})$, we have that $\varphi$ is an SBZ-homomorphism.

- $\varphi$ is an SBZ-monomorphism: Let $[[x]_I]_{K/I} \in Kerf$, i.e., $K = [0]_K = \varphi([[x]_I]_{K/I}) = [x]_K$. So, we have $x \sim_K 0$. Then we obtain $x = (x|_Xx)|_X(x|_Xx) = (x|_X(0|_X0))|_X(x|_X(0|_X0)) \in K$ and $0 = (0|_X0)|_X(0|_X0) = (0|_X(x|_Xx))|_X(0|_X(x|_Xx)) \in K$ from Lemma 3 (2)-(3) and ($S2$), i.e., $[x]_I = [(x|_X(0|_X0))|_X(x|_X(0|_X0))]_I = ([x]_I|_I([0]_I|_I[0]_I))|_I([x]_I|_I([0]_I|_I[0]_I)) \in K/I$ and $[0]_I = [(0|_X(x|_Xx))|_X(0|_X(x|_Xx))]_I = ([0]_I|_I([x]_I|_I[x]_I))|_I([0]_I|_I([x]_I|_I[x]_I)) \in K/I$. So, $[x]_I \sim_{K/I} [0]_I$, i.e., $[[x]_I]_{K/I} = [[0]_I]_{K/I}$. Thus, it follows $Kerf = \{[[0]_I]_{K/I}\}$.
- $\varphi$ is an SBZ-epimorphism: By the definition of $\varphi$, $\varphi$ is an SBZ-epimorphism.

Hence, $\varphi$ is an SBZ-isomorphism, i.e., $(X/I)/(K/I) \cong X/K$. □

# 6 Conclusion

In the present paper, we have given a Sheffer Stroke BZ-algebra (shortly, SBZ-algebra), and study the Cartesian product, when an SBZ-algebra is a BZ-algebra and vice versa, SBZ-ideals, whether SBZ-ideals are ideals or ideals are SBZ-ideals, SBZ-subalgebras, a relationship between SBZ-ideals and SBZ-subalgebras, a congruence relation, SBZ-homomorphisms, whether SBZ-algebras, SBZ-ideals and related notions are preserved under SBZ-homomorphisms, and many properties in SBZ-algebras. After introducing an SBZ-algebra and presenting basic notions about this algebraic structure, we define a partial order on it and give its properties. It is proved that a SBZ-algebra is a BZ-algebra with the condition $x.y := (x|(y|y))|(x|(y|y))$ and vice versa, and that the Cartesian product of two SBZ-algebras is an SBZ-algebra. Also, by describing an SBZ-ideal, we show that any SBZ-ideal of an SBZ-algebra is an ideal of this algebra and vice versa. It is proved that the family of all SBZ-ideals of an SBZ-algebra forms a complete lattice, and that for a subset of an SBZ-algebra there exists the minimal SBZ-ideal containing this subset. By describing an SBZ-subalgebra, we demonstrate that any SBZ-ideal is an SBZ-subalgebra but the converse is not true. Besides, it is given a congruence relation on an SBZ-algebra described by its SBZ-ideal, and shown that a quotient of an SBZ-algebra by a congruence is an SBZ-algebra. Finally, SBZ-homomorphisms are defined and it is indicated that the mentioned concepts are preserved under SBZ-homomorphisms.

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