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An extension of commutativity degree of finite groups

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Abstract. Let G be a finite group. The commutativity degree of G, written $d(G)$, is defined as the ratio

$$\frac{|\{(x,y)|x,y \in G, xy = yx\}|}{|G|^2}.$$

In this paper, we first extend this concept of finite groups to the commutativity degree of fuzzy subgroups. Then, by using the numerical solutions of the equation $xy - zu \equiv t(mod\ n)$, we give explicit formulas for the commutativity degree of fuzzy subgroups of 2-generated groups of nilpotency class 2. Finally we show that this method also works for a large class of finite groups, including metabelian groups.

Keywords: Nilpotent groups, Commutativity degree, Fuzzy groups.
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1 Introduction

In the last years there has been a growing interest in the use of probability in finite group theory. One of the most important aspects that have been studied is the probability that two elements of a finite group G commute. This is denoted by $d(G)$ and is called the commutativity degree of G. In obtaining the properties of $d(G)$, Gustafson [2] proved that for a non-abelian finite group G, $d(G) \leq \frac{5}{8}$ and P. Lescot [7] studied the groups where $d(G) \geq \frac{1}{2}$ and classified these groups. In [3], M. Hashemi gave some explicit formulas of $d(G)$ for some particular finite groups G.

A fuzzy subset of the group G is a function from G into $[0,1]$. The set of all fuzzy subsets of G is called the fuzzy power set of G and is denoted by $FP(G)$. The fuzzy subset $\lambda \in FP(G)$

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is called fuzzy subgroup of G if for every $x, y \in G$;
(1) $\lambda(xy) \geq min\{\lambda(x),\ \lambda(y)\}$.
(2) $\lambda(x^{-1}) \geq \lambda(x)$.
Then for all $x \in G$ we get $\lambda(x^{-1}) = \lambda(x)$ and $\lambda(e) \geq \lambda(x)$. The set of all fuzzy subgroups of G is denoted by $F(G)$. For each $t \in [0,1]$, we define the level subset: $P^t(\lambda) = \{x \in G;\ \lambda(x) \geq t\}$. These subsets allow us to characterize the fuzzy subgroups of G, as follows: λ is a fuzzy subgroup of G if and only if its level subsets are subgroups in G. The fuzzy subgroups of G can be classified up to some natural equivalence relations on the set consisting of all fuzzy subsets of G. One of them (used in [8, 9], too) is defined by

$$\mu \sim \lambda \,\text{if and only if}\, \mu(x) \geq \mu(y) \Leftrightarrow \lambda(x) \geq \lambda(y) \,\text{for all}\, x, y \in G$$

and two fuzzy subgroups of G are called distinct if $\mu \nsim \lambda$.
Let $\lambda : G \to [0,1]$ be a fuzzy subset of G, where $\lambda(G) = \{\theta_1,\ \theta_2,\ \ldots,\ \theta_n\}$ and $\theta_1 >\ \theta_2 >\ \cdots >\ \theta_n$. Then by above we get

Theorem 1. *$\lambda \in FP(G)$ is a fuzzy subgroup of G if and only if there is a chain of subgroups of G as; $P_1(\lambda) < P_2(\lambda) < \cdots < P_n(\lambda) = G$ such that λ can be written as*

$$\lambda(x) = \begin{cases} \theta_1 & \text{if } x \in \mathrm{P}_1(\lambda); \\ \theta_2 & \text{if } x \in \mathrm{P}_2(\lambda) - \mathrm{P}_1(\lambda); \\ \vdots & \vdots \\ \theta_n & if\ x \in P_n(\lambda) - P_{n-1}(\lambda). \end{cases}$$

Hence there exists a one-to-one correspondence between the collection of the equivalence classes of fuzzy subgroups of G and the collection of chains of subgroups of G which end in G.

This paper is organized as follows: In Section 2 we state some results that are required in later sections and in Section 3 we introduce a generalization of the commutativity degree of groups to the fuzzy subgroups. Section 4 is devoted to compute the commutativity degree of fuzzy subgroups of some classes of finite groups.

2 Preliminaries and Results

For the integers $m, n, k \geq 2$ where $k|(m,n)$, we consider the finitely presented groups G_{mn} and $H(m,n,k)$, as follows;
$G_{mn} = \langle a, b | a^m = b^n = 1,\ [a,b]^a = [a,b],\ [a,b]^b = [a,b] \rangle$,
$H(m,n,k) = \left\langle a, b, c | a^m = b^n = c^k = 1, [a,b] = c,\ [a,c] = [b,c] = 1 \right\rangle \cdot$
This section is devoted to explain some results concerned to these groups. First, we state a lemma without proof that establishes some properties of groups of nilpotency class 2.

Lemma 1. *If G is a group and $G' \subseteq Z(G)$, then the following hold for every integer k and $u, v, w \in G$, where $[u,v] := u^{-1}v^{-1}uv$:*

(i) $[uv, w] = [u,w][v,w]$ *and* $[u, vw] = [u,v][u,w]$.

(ii) $[u^k, v] = [u, v^k] = [u,v]^k$.

(iii) $(uv)^k = u^k v^k [v,u]^{k(k-1)/2}$.

(iv) If $G = \langle a, b \rangle$ *then* $G' = \langle [a,b] \rangle \cdot$

The following results can be seen in [1], [4] and [5]:

Proposition 1. *Let $G = G_{mn}$ and $d = (m, n)$. Then*
(i) $G' = \langle [a, b] \rangle$ and $|G| = mnd$.
(ii) Every element of G is in the form $a^i b^j g$ where $0 \le i \le m-1, 0 \le j \le n-1$ and $g \in G'$.
(iii) $Z(G) \cong \langle x, y, z | x^{m/d} = y^{n/d} = z^d = [x, y] = [x, z] = [y, z] = 1 \rangle$.

For the particular case, consider $m = n$ then for $m \ge 2$ we get

$$G_m = G_{mm} = \left\langle a, b | a^m = b^m = 1, [a, b]^a = [a, b], [a, b]^b = [a, b] \right\rangle .$$

By the Proposition 1, we have

Corollary 1. (i) *Every element of G_m can be written uniquely in the form $a^r b^s [b, a]^t$ where $0 \le r, s, t \le m-1$.*

(ii) $|G_m| = m^3, Z(G_m) = G'_m = \langle [a, b] \rangle$ *and* $|Z(G_m)| = m$.

We recall the Heisenberg group

$$H(\mathbb{Z}) = \{ \begin{pmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} | r, s, t \in \mathbb{Z} \}.$$

By [6](Section 2 of Chapter 5), we get

Proposition 2. *(i) $H(\mathbb{Z}) \cong \langle a, b, c | [a, b] = c, \ [a, c] = [b, c] = 1 \rangle$.*

(ii) Every element of $H(\mathbb{Z})$ may be written uniquely in the form $a^i b^j c^k$, where $i, j, k \in \mathbb{Z}$.

(iii) $Z(H(\mathbb{Z})) = H'(\mathbb{Z}) = \langle c \rangle$.

In particular, for $n \ge 2$, we get

$$H(n, n, n) \cong H(\frac{\mathbb{Z}}{n\mathbb{Z}}) = \{ \begin{pmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} | r, s, t \in \frac{\mathbb{Z}}{n\mathbb{Z}} \} \le SL(3, \frac{\mathbb{Z}}{n\mathbb{Z}}).$$

Now, we consider the group
$T = H(m, n, k) \times H(m, n, k) \cong \langle X_1 \cup X_2 | R_1 \cup R_2 \cup S \rangle$, where
$X_i = \{a_i, b_i, c_i\}$ generates the $i-$th factor of T,
$R_i = \{a_i^m = b_i^n = c_i^k = 1, [a_i, b_i] = c_i, \ [a_i, c_i] = [b_i, c_i] = 1\}$ and $S = \{[x, y] = e | x \in X_1, y \in X_2\}$.
Then we obtain the following.

Proposition 3. *For $G = H(m, n, k)$ and $T = G \times G$, we have*
(i) every element of G may be written uniquely in the form $a^r b^s c^t$, where $0 \le r < m, 0 \le s < n$ and $0 \le t < k$.

(ii) $Z(G) = G' = \langle c \rangle$ *and* $|G| = mnk$.
(iii) Every element of T *is uniquely expressible in the form;*

$$a_1^{r_{11}} b_1^{s_{11}} c_1^{t_{11}} a_2^{r_{12}} b_2^{s_{12}} c_2^{t_{12}},$$

where $0 \leq r_{11}, r_{12} < m, 0 \leq s_{11}, s_{12} < n$ *and* $0 \leq t_{11}, t_{12} < k$.
(iv) $Z(T) = T' = \langle c_1, c_2 \rangle$ *and* $|T| = (mnk)^2$.

By the Corollary 1 and Proposition 3, we see that G_{mn} and $H(m, n, k)$ are finite.

The following lemma is crucial for the aims of this paper.

Lemma 2. *For two constant integers* t, n *and variables* x, y, u *and* z*, the number of solutions of the equation* $xy - uz \equiv t \ (mod\ n)$ *is*

$$\sum_{d|n} [\sum_{d_2|(d,t)} (\frac{n^2}{d}\phi(\frac{n}{d})\phi(\frac{d}{d_2}) \times d_2)].$$

Proof. Let $d = (n, x)$. Then the equation $xy - uz \equiv t \ (mod\ n)$ is reduced to $y \equiv (\frac{x}{d})^*(\frac{uz+t}{d}) \ (mod\ \frac{n}{d})$ and this equation has a solution if and only if $uz + t \equiv 0 \ (mod\ d)$, where k^* is the arithmetic inverse of k respect to $\frac{n}{d}$. By these facts, we solve the sub equation $uz + t \equiv 0 \ (mod\ d)$. For this, consider $d_1 = (d, t)$ and $d_2 = (d, u)$. Then the equation $uz + t \equiv 0 \ (mod\ d)$ has a solution if and only if $d_2|t$ (i.e $d_2|d_1$). In this case, $z \equiv (\frac{u}{d_2})^*(\frac{-t}{d_2})(mod \frac{d}{d_2})$ is a solution. Then for $d_2|d_1$, the solution set of the equation is $A = \{(u, z)|(u, d) = d_2, z \in \{a, a + \frac{d}{d_2}, \ldots, a + (d_2 - 1) \times (\frac{d}{d_2})\}\}$ where $a = (\frac{u}{d_2})^*(\frac{-t}{d_2})$. Hence the number of solutions of the equation $uz + t \equiv 0 \ (mod\ d)$ is

$$\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2,$$

where $d_1 = (d, t)$.
As an immediate consequence of these we get for $d|n$, (x, y, u, z) is a solution of $y \equiv (\frac{x}{d})^*(\frac{uz+t}{d}) \ (mod\ \frac{n}{d})$ if and only if $d = (x, n), y = (\frac{x}{d})^*(\frac{uz+t}{d})$ and $(u, z) \in A$. So that, for $d|n$, the number of solutions of $y \equiv (\frac{x}{d})^*(\frac{uz+t}{d}) \ (mod\ \frac{n}{d})$ is

$$\phi(\frac{n}{d})(\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2).$$

This leads us to; the number of solutions of $xy - uz \equiv t \ (mod\ n)$ is equal to

$$\sum_{d|n} [\phi(\frac{n}{d}) \times d \times (\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2 \times \frac{n^2}{d^2})] = \sum_{d|n} [\frac{n^2}{d}\phi(\frac{n}{d})(\sum_{d_2|(d,t)} \phi(\frac{d}{d_2}) \times d_2)].$$

As required. □

Let $\beta = \sum_{d|n} [\sum_{d_2|(d,t)} (\frac{n^2}{d}\phi(\frac{n}{d})\phi(\frac{d}{d_2}) \times d_2)]$. In Table 1, we presented the number of solutions computed by the formula β for different values of n and t. It is noted that the MATLAB Software confirms this Table.

By elementary concepts of number theory, we have the following corollary:

Corollary 2. *Let t, m, n and k be integers, where $k|(m,n)$. Also let x, y, u and v be variables when $0 \leq x, u < m$ and $0 \leq y, v < n$. Then, the number of solutions of the equation $xy - uv \equiv t \ (mod\ k)$ is*

$$(\frac{m}{k})^2(\frac{n}{k})^2 \sum_{d|k}[\sum_{d_2|(d,t)}(\frac{k^2}{d}\phi(\frac{k}{d})\phi(\frac{d}{d_2}) \times d_2)].$$

3 The commutativity degree of fuzzy subgroup

In this section we introduce and study the concept of commutativity degree on fuzzy subgroups of finite group G. First by Theorem 1, we note that every $\lambda \in F(G)$ is corresponding to a chain of subgroups of G as; $P_1(\lambda) < P_2(\lambda) < \cdots < P_n(\lambda) = G$.

Definition 1. *Let G be a finite group with identity e. For a fuzzy subgroup λ of the group G; we define the commutativity degree of λ, denoted by $d(\lambda)$, as follows:*

$$d(\lambda) = \frac{|\{(x,y) \in G \times G;\ \lambda([x,y]) = \lambda(e)\}|}{|G \times G|}.$$

Clearly for every $\lambda \in F(G)$, we have $0 < d(G) \leq d(\lambda) \leq 1$ where $d(G)$ is the commutativity degree of G. Also if G is abelian, then $d(G) = d(\lambda) = 1$.

Lemma 3. *For fuzzy subgroup λ of the group G, suppose $\rho_g = \{(x,y) \in G \times G;\ [x,y] = g\}$. Then*

$$d(\lambda) = \frac{\sum_{g \in G' \bigcap P_1(\lambda)} |\rho_g|}{|G \times G|}.$$

Proof. Let λ be fuzzy subgroup of G. Since $e \in P_1(\lambda)$, we have $\lambda([x,y]) = \lambda(e)$ if and only if $[x,y] \in P_1(\lambda)$. Then

$$\begin{aligned}\{(x,y) \in G \times G;\ \lambda([x,y]) = \lambda(e)\} &= \{(x,y) \in G \times G;\ [x,y] \in P_1(\lambda)\} \\ &= \{(x,y) \in G \times G;\ [x,y] \in G' \bigcap P_1(\lambda)\}.\end{aligned}$$

So that

$$d(\lambda) = \frac{|\{(x,y) \in G \times G;\ [x,y] \in G' \bigcap P_1(\lambda)\}|}{|G \times G|} = \frac{\sum_{g \in G' \bigcap P_1(\lambda)} |\rho_g|}{|G \times G|}.$$

Thus the result holds. □

Corollary 3. *For two fuzzy subgroups λ and μ of the group G, if $P_1(\lambda) = P_1(\mu)$ then $d(\lambda) = d(\mu)$.*

Proof. This is a direct consequence of Lemma 3. □

For $H \leq G'$, we consider $F_H = \{\lambda \in F(G);\ P_1(\lambda) \bigcap G' = H\}$ and

$$\lambda_H(x) = \begin{cases} \theta_1 & if\ x \in H; \\ \theta_2 & if\ x \in G - H; \end{cases}$$

where θ_1 and θ_2 are some constants in $[0,1]$ with $\theta_1 > \theta_2$. Then $\lambda_H \in F_H$ and $F(G) = \bigcup_{H \leq G'} F_H$. Also for every $\lambda \in F_H$, we have $d(\lambda) = d(\lambda_H)$.

So this leads to:

Proposition 4. *For every $\lambda \in F(G)$ there exists $H \leq G'$ such that $d(\lambda) = d(\lambda_H)$.*

Proof. The result follows by considering $H = P_1(\lambda) \bigcap G'$. □

Now using the formulas $d(G) = \frac{|\rho_e|}{|G|^2} = \frac{k(G)}{|G|}$ and $|G^2| = \sum_{g \in G'} |\rho_g|$, we find the commutativity degree of fuzzy subgroups of S_3 and S_4. We know that these groups are not nilpotent.

Example 1. To calculate the commutativity degree of fuzzy subgroups of S_3 and S_4, we note that
$S_3 = \{\epsilon,\ \alpha_1,\ \alpha_2,\ \alpha_4,\ \tau_7,\ \tau_8\}$;
$A_4 = \{\epsilon,\ \sigma_2,\ \sigma_5,\ \sigma_8,\ \tau_1,\ \tau_2,\ \tau_3,\ \tau_4,\ \tau_5,\ \tau_6,\ \tau_7,\ \tau_8\}$;
$S_4 = \{\epsilon,\ \sigma_1, \ldots,\ \sigma_9,\ \tau_1, \ldots,\ \tau_8,\ \alpha_1, \ldots,\ \alpha_6\}$;
where
$\sigma_1 = (1,2,3,4),\ \sigma_2 = (1,3)(2,4),\ \sigma_3 = (1,4,3,2),\ \sigma_4 = (1,2,4,3)$,
$\sigma_5 = (1,4)(2,3),\ \sigma_6 = (1,3,4,2),\ \sigma_7 = (1,3,2,4),\ \sigma_8 = (1,2)(3,4)$,
$\sigma_9 = (1,4,2,3),\ \tau_1 = (2,3,4),\ \tau_2 = (2,4,3),\ \tau_3 = (1,3,4), \tau_4 = (1,4,3),\ \tau_5 = (1,2,4),\ \tau_6 = (1,4,2),\ \tau_7 = (1,2,3),\ \tau_8 = (1,3,2), \alpha_1 = (1,2), \alpha_2 = (1,3),\ \alpha_3 = (1,4),\ \alpha_4 = (2,3),\ \alpha_5 = (2,4),\ \alpha_6 = (3,4)$.
Also, the following are all subgroups of A_4
$B_0 = \{\epsilon\},\ B_1 = \{\epsilon,\ \sigma_2\},\ B_2 = \{\epsilon,\ \sigma_5\},\ B_3 = \{\epsilon,\ \sigma_8\},\ C_4 = \{\epsilon,\ \tau_1,\ \tau_2\}, C_5 = \{\epsilon,\ \tau_3,\ \tau_4\},\ C_6 = \{\epsilon,\ \tau_5,\ \tau_6\},\ C_7 = \{\epsilon, \tau_7, \tau_8\},\ C_8 = \{\epsilon, \sigma_2, \sigma_5, \sigma_8\}, C_9 = A_4$.

Since for conjugate elements g and h we have $|\rho_g(G)| = |\rho_h(G)|$, it follows that there exist constants r_σ and r_τ such that $|\rho_{\sigma_i}| = r_\sigma$ for all $i \in \{2,\ 5,\ 8\}$ and $|\rho_{\tau_j}| = r_\tau$ for all j. Also we note that $|\rho_e| = \frac{k(G)}{|G|} \times |G|^2$ and $|G^2| = \sum_{g \in G'} |\rho_g|$. Using these facts, we have

$|\rho_\epsilon(S_3)| = d(S_3) \times |S_3|^2 = \frac{k(S_3)}{|S_3|}|S_3|^2 = 3|S_3| = 18$;
$|r_\tau(S_3)| = \frac{|\rho_{\tau_7}(S_3)|+|\rho_{\tau_8}(S_3)|}{2} = \frac{|S_3|^2 - |\rho_\epsilon(S_3)|}{2} = 9$.
Then by Lemma 3, $d(\lambda_{B_0}) = \frac{18}{36}, d(\lambda_{A_3}) = 1$.

In order to compute the commutativity degree of fuzzy subgroups of S_4, we consider $H = C_8 = \{\epsilon,\ \sigma_2,\ \sigma_5,\ \sigma_8\}$. Then H is normal in S_4 and $S_4/H \cong S_3$. Using this fact (together with $S_4' = A_4$), we get

$$\begin{aligned}
|\rho_\epsilon(S_4)| =& |S_4|^2 d(S_4) = \frac{k(S_4)}{|S_4|}|S_4|^2 = 5|S_4| = 120.\\
r_\sigma(S_4) =& \frac{(|\rho_\epsilon(S_4)| + |\rho_{\sigma_2}(S_4)| + |\rho_{\sigma_5}(S_4)| + |\rho_{\sigma_8}(S_4)|) - |\rho_\epsilon(S_4)|}{3}\\
=& \frac{d(S_4/H)|S_4|^2 - |\rho_\epsilon|}{3} = \frac{d(S_3)|S_4|^2 - |\rho_\epsilon|}{3} = \frac{\frac{k_{S_3}}{|S_3|}|S_4|^2 - |\rho_\epsilon|}{3}\\
=& \frac{\frac{1}{2}24^2 - 120}{3} = 56.
\end{aligned}$$

$$\begin{aligned}
r_\tau(S_4) =& \frac{1}{8}\sum_{i=1}^{8}|\rho_{\tau_i}| = \frac{|S_4|^2 - (|\rho_\epsilon(S_4)| + |\rho_{\sigma_2}(S_4)| + |\rho_{\sigma_5}(S_4)| + |\rho_{\sigma_8}(S_4)|)}{8}\\
=& \frac{24^2 - (120 + 3 \times |r_\sigma(S_4)|)}{8} = \frac{24^2 - (120 + 3 \times 56)}{8} = 36.
\end{aligned}$$

Now by using the Lemma 3, we obtain

$d(\lambda_{B_0}) = \frac{\sum_{g \in B_0} |\rho_g(S_4)|}{|G \times G|} = \frac{120}{576}$, $d(\lambda_{B_1}) = \frac{|\rho_\epsilon(S_4)| + |\rho_{\sigma_2}(S_4)|}{576} = \frac{120+56}{576}, \dots,$

$d(\lambda_{C_8}) = \frac{|\rho_\epsilon(S_4)| + |\rho_{\sigma_2}(S_4)| + |\rho_{\sigma_5}(S_4)| + |\rho_{\sigma_8}(S_4)|}{576} = \frac{120+168}{576} = \frac{288}{576}$,

$d(\lambda_{A_4}) = \frac{|\rho_\epsilon(S_4)| + \cdots + |\rho_{\sigma_8}(S_4)|}{576} = \frac{120+\cdots+56}{576} = 1.$

Then for $\lambda \in F(S_4)$, $d(\lambda)$ is obtained by Proposition 4.

4 Computations on finite groups

In this section we study the commutativity degree of fuzzy subgroups for a finite group G. We first prove Theorem 2; which is a crucial result for calculating $\rho_g(G)$ when G is finite 2-generated group of nilpotency class 2. Then for integers $m, n, k \geq 2$ where $k|(m, n)$, we consider the finite groups G_m, D_m and $H(m, n, k)$, as follows;

$G_m = \langle a, b | a^m = b^m = 1, [a, b]^a = [a, b], [a, b]^b = [a, b] \rangle$;

$D_m = \langle a, b | a^m = b^2 = (ab)^2 = 1 \rangle$;

$H(m, n, k) = \langle a, b, c | a^m = b^n = c^k = 1, [a, b] = c,\ [a, c] = [b, c] = 1 \rangle$.

By applying Lemma 2 and Theorem 2, we compute the commutativity degrees of fuzzy subgroup for these groups and A_4. Then one can see that the method obtained in Theorem 2 can be extended to a larger family, such as the family of finite metabelian groups.

Theorem 2. *For a finite 2-generated group $G = \langle a, b\rangle$ of nilpotency class 2 and $g = [a, b]^t \in G'$, $|\rho_g(G)|$ is a multiple of the number of solutions of the equation $ri - sj \equiv t \ (mod\ d)$ where $d = |[a, b]|$.*

Proof. Let $n = |aG'|$, $m = |bG'|$ and $d = |[a, b]|$. Furthermore, by using the fundamental theorem of finite abelian groups, we assume that $\frac{G}{G'} = \{a^r b^s G' | 0 \le r < n,\ 0 \le s < m\}$. So that for every $g \in G$, we have $g = a^{r_1} b^{s_1} [b, a]^{t_1}$ where $0 \le r_1 < n, 0 \le s_1 < m,\ 0 \le t_1 < d$. Let $h = a^{r_1} b^{s_1} [b, a]^{t_1} = a^{r_2} b^{s_2} [b, a]^{t_2}$, where $0 \le r_1, r_2 < n,\ 0 \le s_1, s_2 < m,\ 0 \le t_1, t_2 < d$. Then $a^{r_1 - r_2} b^{s_1 - s_2} G' = G'$, by uniqueness of presentation of elements $\frac{G}{G'}$, we obtain $r_1 = r_2,\ s_1 = s_2$. So that $t_1 = t_2$. Consequently, for $x = a^{r_1} b^{s_1} [b, a]^{t_1},\ y = a^{r_2} b^{s_2} [b, a]^{t_2}$ and $g = [a, b]^t \in G'$, we have

$$\begin{aligned}
|\rho_g| =& |\{(x, y) \in G \times G;\ [x, y] = g\}| \\
=& |\{(x, y) \in G \times G;\ [a, b]^{r_1 s_2 - r_2 s_1} = [a, b]^t\}| \\
=& |\{(r_1, s_1, t_1, r_2, s_2, t_2);\ r_1 s_2 - r_2 s_1 \equiv t(mod\ d)\}|.
\end{aligned}$$

Then the result holds. □

We are now in a position to obtain the commutativity degree of fuzzy subgroup of finite 2-generated group $G = \langle a, b\rangle$ of nilpotency class 2. By the above, we see that $G = \langle a, b\rangle$ is a quotient of some G_{rs}. Then there are m, n such every element of G can be written in the form $a^{r_1} b^{s_1} [b, a]^{t_1}$ where $0 \le r_1 \le m - 1, 0 \le s_1 \le n - 1$ and $0 \le t_1 \le |[a, b]| - 1$.
Now let $\lambda \in F(G)$, then $G' \bigcap P_1(\lambda) \le \langle [a, b]\rangle$. So that there is t such that $t|k = |[a, b]|$ and

$$G' \bigcap P_1(\lambda) = \langle [a, b]^t\rangle = \{[a, b]^{ti};\ i = 1, 2, \dots, \frac{k}{t}\}.$$

These lead us to:

Theorem 3. *For a finite 2-generated group $G = \langle a, b\rangle$ of nilpotency class 2 and $\lambda \in F(G)$,*

$$d(\lambda) = \frac{1}{t^2} \sum_{d|t} [\sum_{d_2|d} \phi(\frac{t}{d}) \phi(\frac{d}{d_2}) \frac{d_2}{d}],$$

where $G' \bigcap P_1(\lambda) = \langle [a, b]^t\rangle$.

Proof. For $g = [a, b]^{ti} \in G'$, by the Proof of Theorem 2, we get $|L_i| = |\rho_g| = |\{(r_1, s_1, t_1, r_2, s_2, t_2);\ r_1 s_2 - r_2 s_1 \equiv ti(mod\ k)\}|$. Then by Corollary 2 and since t_1 and t_2 admit k values, we have

$$\begin{aligned}
\alpha = & \sum_{g \in G' \bigcap P_1(\lambda)} |\rho_g| = \sum_{i=1}^{\frac{k}{t}} |L_i| = \sum_{i=1}^{\frac{k}{t}} |\{(r_1, s_1, t_1, r_2, s_2, t_2); \\
& 0 \le r_1, r_2 \le m - 1,\ 0 \le s_1, s_2 \le n - 1,\ 0 \le t_1, t_2 \le k - 1, \\
& r_1 s_2 - r_2 s_1 \equiv ti(mod\ k)\}| = |\{(r_1, s_1, t_1, r_2, s_2, t_2); \\
& 0 \le r_1, r_2 \le m - 1,\ 0 \le s_1, s_2 \le n - 1,\ 0 \le t_1, t_2 \le k - 1, \\
& r_1 s_2 - r_2 s_1 \equiv 0(mod\ t)\}| = \frac{(mnk)^2}{t^4} \sum_{d|t} [\sum_{d_2|d} \frac{t^2}{d} \phi(\frac{t}{d}) \phi(\frac{d}{d_2}) \times d_2]
\end{aligned}$$

$$= \frac{(mnk)^2}{t^2}\sum_{d|t}[\sum_{d_2|d}\phi(\frac{t}{d})\phi(\frac{d}{d_2})\frac{d_2}{d}].$$

Since $|G| = mnk$, the result follows from Lemma 3. □

For example, let $G = G_m$, $\lambda \in F(G)$ and $G' \bigcap P_1(\lambda) = \langle [a,b]^t \rangle$. Then, by considering Theorem 3, we have

$$d(\lambda) = \frac{1}{t^2}\sum_{d|t}[\sum_{d_2|d}\phi(\frac{t}{d})\phi(\frac{d}{d_2})\frac{d_2}{d}].$$

We note that $H(m,n,k)$ is 2-generated group(by a and b), then Theorem 3 can be applied for finding the commutativity degree of its fuzzy subgroups. In what follow, by using Theorem 2, Proposition 3 and Lemmas 2, 3, we calculate the commutativity degree of fuzzy subgroups of $H(m,n,k) \times H(m,n,k)$. So that this method can be used for finite groups of nilpotency class 2. Let $x, y \in T = H(m,n,k) \times H(m,n,k)$. Then by the Lemma 2.5-(iii), $x = a_1^{r_{11}} b_1^{s_{11}} c_1^{t_{11}} a_2^{r_{12}} b_2^{s_{12}} c_2^{t_{12}}$, $y = a_1^{r_{21}} b_1^{s_{21}} c_1^{t_{21}} a_2^{r_{22}} b_2^{s_{22}} c_2^{t_{22}}$ and $[x,y] = c_1^{s_{11}r_{21} - r_{11}s_{21}} c_2^{s_{12}r_{22} - r_{12}s_{22}} \in T' = \langle c_1, c_2 \rangle$.

By using these facts, we prove the following theorem:

Theorem 4. *For $\lambda \in F(T)$, suppose $T' \bigcap P_1(\lambda) = \langle c_1^r,\ c_2^s \rangle$. Then $d(\lambda) = \frac{\alpha}{(mnk)^4}$ where*

$$\alpha = \frac{(mn)^4}{(k)^2}(\sum_{i=1}^{|c_1^r|}[\sum_{d|k}\sum_{d_2|(d,ri)}(\frac{k}{d}\phi(\frac{k}{d})\phi(\frac{d}{d_2}) \times d_2)])$$
$$\times (\sum_{j=1}^{|c_2^s|}[\sum_{d|k}\sum_{d_2|(d,sj)}(\frac{k}{d}\phi(\frac{k}{d})\phi(\frac{d}{d_2}) \times d_2)]).$$

Proof. For $g = c_1^{t_1} c_2^{t_2} \in T'$, we have

$$\begin{aligned}|\rho_g| =& |\{(x,y) \in T \times T;\ [x,y] = g\}| \\ =& |\{(x,y) \in T \times T;\ c_1^{s_{11}r_{21} - r_{11}s_{21}} c_2^{s_{12}r_{22} - r_{12}s_{22}} = c_1^{t_1} c_2^{t_2}\}| \\ =& |\{(r_{11}, s_{11}, t_{11}, r_{12}, s_{12}, t_{12}, r_{21}, s_{21}, t_{21}, r_{22}, s_{22}, t_{22}); \\ & s_{11}r_{21} - r_{11}s_{21} \equiv t_1 (mod\ k),\ s_{12}r_{22} - r_{12}s_{22} \equiv t_2 (mod\ k)\}|.\end{aligned}$$

By Corollary 2 and since t_{11}, t_{12}, t_{21} and t_{22} admit k values, we have

$$|\rho_g| = k^6 (\frac{m}{k})^4 (\frac{n}{k})^4 (\sum_{d|k}[\sum_{d_2|(d,t_1)}(\frac{k}{d}\phi(\frac{k}{d})\phi(\frac{d}{d_2}) \times d_2)])$$
$$\times (\sum_{d|k}[\sum_{d_2|(d,t_2)}(\frac{k}{d}\phi(\frac{k}{d})\phi(\frac{d}{d_2}) \times d_2)]).$$

Now for $\lambda \in F(T)$ and $H = T' \bigcap P_1(\lambda) = \langle c_1^r,\ c_2^s \rangle$. We get

$$\alpha = \sum_{g \in H} |\rho_g| = \sum_{i=1}^{|c_1^r|}\sum_{j=1}^{|c_2^s|} |L_{ij}| = \frac{(mn)^4}{(k)^2}(\sum_{i=1}^{|c_1^r|}[\sum_{d|k}\sum_{d_2|(d,ri)}(\frac{k}{d}\phi(\frac{k}{d})\phi(\frac{d}{d_2}) \times d_2)])$$

$$\times(\sum_{j=1}^{|c_2^s|}[\sum_{d|k}\sum_{d_2|(d,sj)}(\frac{k}{d}\phi(\frac{k}{d})\phi(\frac{d}{d_2})\times d_2)]).$$

The theorem is proved. □

In the rest of this section, we compute the commutativity degree of fuzzy subgroups of the metabelian groups D_m $(m \geq 3)$ and A_4 which are not nilpotent. Our method may be generalized for the finite metabelian groups.

For the dihedral group $D_m = \langle a, b | a^m = b^2 = (ab)^2 = 1\rangle$ where $m \geq 3$, we have;
1- $D_m = \{a^i b^j | 0 \leq i \leq m-1,\ 0 \leq j \leq 1\}$.
2- $|D_m| = 2m$ and

$$D'_m = \begin{cases} \langle a^2\rangle & \text{if m = 2k;} \\ \langle a\rangle & \text{if m = 2k + 1 .} \end{cases}$$

Now, let $x = a^{i_1}b^{j_1},\ y = a^{i_2}b^{j_2} \in D_m$. Then $[x, y] = a^{\alpha}$, where

$$\alpha = (-1)^{j_1+j_2}(-i_2 + i_1(1-(-1)^{j_2})) + (-1)^{j_2} i_2.$$

Hence

$$\alpha = \begin{cases} 0 & \text{if } \mathrm{j}_1 = \mathrm{j}_2 = 0; \\ 2(i_1 - i_2) & \text{if } \mathrm{j}_1 = \mathrm{j}_2 = 1; \\ -2i_1 & \text{if } _1 = 0,\ \mathrm{j}_2 = 1; \\ 2i_2 & \text{if } \mathrm{j}_1 = 1,\ \mathrm{j}_2 = 0. \end{cases}$$

By using above information, we prove that;

Theorem 5. *For the group $G = D_m$ and $g \in D_m$, we have*

1-if m is odd then

$$|\rho_g| = \begin{cases} m^2 + 3m & \text{if g = e;} \\ 3m & \text{if g} \neq \text{e;} \end{cases}$$

2-if m is even then

$$|\rho_g| = \begin{cases} m^2 + 6m & \text{if g = e;} \\ 6m & \text{if g} \neq \text{e.} \end{cases}$$

So that using this theorem, one can obtain explicit formulas for the commutativity degree of fuzzy subgroups of D_m.

Proof. Let m be odd. It is sufficient that we find $|\rho_g|$ for every $g \in D'_m = \{a^i |\ i = 0, 1, \ldots, m-1\}$. For $g = a^t$, we consider $\rho_g = \{(x, y) \in D_m \times D_m;\ [x, y] = a^t\}$. Then

$$\rho_g = \{(x, y) \in D_m \times D_m; a^{\alpha} = a^t\} = \{(x, y) \in D_m \times D_m;\ \alpha \equiv t(mod\ m)\}.$$

Hence, for the element $g = a^t \in D'_m$ we have $|\rho_g|$ is equal to the solutions number of $\alpha \equiv t(mod\ m)$. So that $|\rho_e| = m^2 + 3m$ and for every $1 \leq t \leq m-1,\ \ |\rho_{a^t}| = 3m$.
When m is even, the result follows from a similar argument as above. □

Example 2. To obtain the commutativity degree of fuzzy subgroups of A_4, by notations of Example 1, we consider $a = \tau_7, b = \sigma_8, c = \sigma_2$. Clearly $A_4 = \{a^{i_1}b^{i_2}c^{i_3} | -1 \leq i_1 \leq 1, 0 \leq i_2, i_3, \leq 1\}$. Then $A_4 = \langle a, b, c | a^3 = b^2 = c^2 = [b, c] = 1, [a^i, b] = b^{|i|}c^r, [a^i, c] = b^{|i|}c^s\rangle$, where $r = |i|(\frac{1}{2} - \frac{i}{2}), s = |i|(\frac{1}{2} + \frac{i}{2})$. Now let $x = a^{i_1}b^{i_2}c^{i_3},\ y = a^{j_1}b^{j_2}c^{j_3} \in A_4$. Then by $uv = vu[u, v]$ and the relations of A_4, we get

$$\begin{aligned}[x, y] =& [a^{-i_1}, c]^{j_3}[a^{-i_1}, b]^{j_2}[b, a^{j_1}]^{i_2}[c, a^{j_1}]^{i_3}\\ =& b^{\frac{|i_1|(2j_3+j_2+i_1j_2)+|j_1|(2i_3+i_2-j_1i_2)}{2}} \times c^{\frac{|i_1|(2j_2+j_3-i_1j_3)+|j_1|(2i_2+i_3+j_1i_3)}{2}}.\end{aligned}$$

For $g \in A_4' = \langle b, c\rangle$, we have $g = b^{r_1}c^{r_2}$ where $r_1, r_2 \in \{0, 1\}$. Then $|\rho_g(A_4)| = |\{(x, y)|[x, y] = g\}| = (number\ of\ (i_1, i_2, i_3, j_1, j_2, j_3)\ with$ $-1 \leq i_1, j_1 \leq 1,\ 0 \leq i_2, i_3, j_2, j_3 \leq 1)$ is a solution of system (1). So that $(x, y) \in \rho_g(A_4)$ if and only if $(i_1, i_2, i_3, j_1, j_2, j_3)$ is a solution of the following system:

$$\begin{cases} \frac{|i_1|(2j_3+j_2+i_1j_2)+|j_1|(2i_3+i_2-j_1i_2)}{2} & \equiv r_1\ (mod\ 2);\\ \frac{|i_1|(2j_2+j_3-i_1j_3)+|j_1|(2i_2+i_3+j_1i_3)}{2} & \equiv r_2\ (mod\ 2);\\ \quad -1 \leq i_1, j_1 \leq 1,\ 0 \leq i_2, i_3, j_2, j_3 \leq 1. \end{cases} \tag{1}$$

For the calculating of $|\rho_e|$, we must count the solutions of (1) where $r_1, r_2 = 0$. To do this, we consider the following cases:
1-Let $i_1 = j_1 = 0$. Since $0 \leq i_2, i_3, j_2, j_3 \leq 1$ the number of solutions of system (1) is equal to 16.
2-Let $i_1 = 0,\ j_1 = -1, 1$, then the system (1) reduces to the following (respectively for $j_1 = -1$ and $j_1 = 1$):

$$\begin{cases} i_3 + i_2 & \equiv 0\ (mod\ 2);\\ i_2 & \equiv 0\ (mod\ 2);\\ \quad 0 \leq i_2, i_3, j_2, j_3 \leq 1. \end{cases}$$

$$\begin{cases} i_3 & \equiv 0\ (mod\ 2);\\ i_3 + i_2 & \equiv 0\ (mod\ 2);\\ \quad 0 \leq i_2, i_3, j_2, j_3 \leq 1. \end{cases}$$

So that the number of solutions of these systems is 8.
Using similar arguments in step (2), one can show that the number of solutions of systems (1) is equal to 8 in every remaining case (that is $i_1 = -1,\ |j_1| = 1;\ i_1 = 1,\ |j_1| = 1$ and $j_1 = 0,\ |i_1| = 1$). Combining of all these, we get $|\rho_e| = 48$.
Using analogous arguments, we see that $|\rho_b| = |\rho_c| = |\rho_{bc}| = 32$. Then we have

$$|\rho_\epsilon(A_4)| = 48,\ |\rho_{\sigma_2}(A_4)| = |\rho_{\sigma_5}(A_4)| = |\rho_{\sigma_8}(A_4)| = 32.$$

Combining of all these and Lemma 3, we have
$d(\lambda_{B_0}) = \frac{48}{144}, d(\lambda_{B_1}) = d(\lambda_{B_2}) = d(\lambda_{B_3}) = \frac{48+32}{144} = \frac{80}{144},\ d(\lambda_{A_4'}) = 1.$

At the end of this section, according to obtained results in this paper, we believe that if every element of finite group G is uniquely expressible as $x_1^{i_1} \cdots x_r^{i_r}$ with $0 \leq i_j < |x_j|$, where $G = \langle x_1, \ldots, x_r | R\rangle$ then our method can be applied for calculating the commutativity degree of fuzzy subgroups of G.

Table 1: The number of solutions of the equation $xy - uz \equiv t \ (mod\ n)$

$n \setminus t$	1	3	4	5	6	7	8	12
6	144	198	240	144	330	144	240	330
8	384	384	672	384	576	384	736	672
9	648	864	648	648	864	648	648	864
10	720	720	1200	870	1200	720	1200	1200
12	1152	1584	2112	1152	2376	1152	2112	2904
14	2016	2016	3360	2016	3360	2310	3360	3360
18	3888	5184	6480	3888	8640	3888	6480	8640
27	17496	23328	17496	17496	23328	17496	17496	23328
36	31104	41472	57024	31104	62208	31104	57024	76032
48	73728	101376	129024	73728	152064	73728	138240	177408

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