Journal of Algebra and Related Topics
Vol. 13, No. 02, 2025, pp. 43-52. Research Article

Berwald and Douglas spaces of a Finsler space with deformed Berwald-infinite series metric

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Abstract. In the present paper, we have studied the basic properties of the Berwald, and Douglas spaces of a Finsler space with the deformed Berwald-Infinite series metric and examined the condition under which the Finsler space with the deformed Berwald-Infinite series metric will be a Berwald and Douglas space.

Keywords: Berwald - Infinite series, Finsler space, Berwald space, Douglas space, $(\alpha,\ \beta)$-metric.
AMS Subject Classification 2010: 53B40, 53C60.

1 Introduction

Berwald [4] introduced a very interesting Finsler metric in 1929 that was defined on the unit ball $B^n(1)$ with all straight line segments and geodesics having constant flag curvature $K = 0$ in the form of

$$L = \frac{\{\sqrt{1-|x|^2|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle\}^2}{\{1-|x|^2\}^2\sqrt{1-|x|^2|y|^2+\langle x,y\rangle^2}}. \tag{1}$$

In modern terms, above Berwald's metric is defined as $\frac{(\alpha+\beta)^2}{\alpha}$ [9] and belongs to a special type of Finsler metric called Berwald-type metric, and the authors of the papers introduced very interesting geometrical properties with respect to this metric in the field of Finsler geometry. Lee and Park [5] introduced an r-th series of (α,β)-metrics in 2004 as

$$L(\alpha,\beta) = \beta\sum_{k=0}^{r}(\frac{\alpha}{\beta})^k, \tag{2}$$

*Corresponding author
Received: 18 January 2024/ Revised: 06 May 2024/ Accepted: 25 May 2024
DOI: 10.22124/JART.2024.26510.1622

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by considering $\alpha < \beta$. Further, if $r = 1$ then above equation reduces to Randers metric. For $r = 2$, equation (2) becomes a combination of the Randers metric and the Kropina metric. If $r = \infty$ then the above metric is expressed as

$$L(\alpha, \beta) = \frac{\beta^2}{\beta - \alpha}. \tag{3}$$

The metric described above is known between the infinite series (α, β)-metric. This metric is very remarkable because it is the difference of Randers and Matsumoto metrics. In 2020, Tripathi [10], combined Berwald and Infinite series metrics and named them Berwald-Infinite series metrics. They studied the basic properties of Finsler space with this important metric and various hypersurfaces with this metric. In this paper, we derive the conditions under which the Finsler space F^n endowed with the Berwald-infinite series metric becomes a Berwald space and a Douglas space.

2 Preliminaries

Berwald-Infinite series metric is a combination of Berwald metric and Infinite series metric which we defined as

Definition 1 ([10]). *Let F^n be an n-dimensional Finsler space consisting of an n-dimensional differentiable manifold M^n equipped with a fundamental function L defined as*

$$L(\alpha, \beta) = \frac{(\alpha + \beta)^2}{\alpha} + \frac{\beta^2}{(\beta - \alpha)}, \tag{4}$$

then the metric L is known as Berwald-Infinite series metric and the Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$ equiped with this metric is known as Berwald-infinite series Finsler space.

On the other hand, the geodesics of a Finsler space $F^n = (M^n, L)$ are given by the system of differential equations including the function

$$4G^i(x, y) = g^{ij}(y^r \dot{\partial}_j \partial_r L^2 - \partial_j L^2).$$

For an (α, β)-metric $L(\alpha, \beta)$ the space $R^n = (M^n, \alpha)$ is called the associated Riemannian space with $F^n = \{M^n, L(\alpha, \beta)\}$ [1, 6]. The covariant differentiation with respect to the Levi-Civita connection $\gamma^i_{jk}(x)$ of R^n is denoted by $(;)$. We put $(a^{ij}) = (a_{ij})^{-1}$, and use the symbols as follows:

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}),\ s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}),\ r^i_j = a^{ir} r_{rj},\ s^i_j = a^{ir} s_{rj},\ r_j = b_r r^r_j,$$

$$s_j = b_r s^r_j,\ b^i = a^{ir} b_r,\ b^2 = a^{rs} b_r b_s.$$

According to [7], if $\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$, where $\gamma^2 = b^2\alpha^2 - \beta^2$, then the function $G^i(x, y)$ of F^n with an (α, β)-metric is written in the form

$$2G^i = \gamma^i_{00} + 2B^i, \tag{5}$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha} s^i_0 + C^* \left\{ \frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i \right) \right\},$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$ *and* $L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha \partial \alpha}$, the subscript '0' means the contraction by y^i and we put

$$C^* = \frac{\alpha\beta(r_{00} L_\alpha - 2 s_0 \alpha L_\beta)}{2(\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha})}. \tag{6}$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity. For example, γ^i_{00} is $hp(2)$.
From the former of (5) the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of F^n with an (α, β)-metric is given by

$$G^i_j = \dot{\partial}_j G^i = \gamma^i_{0j} + B^i_j,$$
$$G^i_{jk} = \dot{\partial}_k G^i_j = \gamma^i_{jk} + B^i_{jk},$$

where we put $B^i_j = \dot{\partial}_j B^i$ and $B^i_{jk} = \dot{\partial}_k B^i_j$. $B^i(x, y)$ is called the *difference vector* [7]. On account of [7], B^i_{jk} is determined by

$$L_\alpha B^t_{ji} y^j y_t + \alpha L_\beta (B^t_{ji} b_t - b_{j;i}) y^j = 0, \tag{7}$$

where $y_k = a_{ik} y^i$.

A Finsler space F^n with an (α, β)-metric is a Douglas space, if and only if $B^{ij} = B^i y^j - B^j y^i$ is $hp(3)$, and various authors [3,8] studied the properties of this space in details. From latter of (5) B^{ij} is written as follows:

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha} (s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i). \tag{8}$$

Lemma 1 ([2]). *If $\alpha^2 \equiv 0 \,(mod\, \beta)$, that is, $a_{ij}(x) y^i y^j$ contains $b_i(x) y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case, we have $\delta = d_i(x) y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

3 The condition for F^n to be a Berwald space

In the present section, we find the condition under which a Finsler space F^n with an deformed Berwald-Infinite series metric will be a Berwald space. In a n-dimensional Finsler space F^n with deformed Berwald-Infinite series metric, we have the following results:

$$L_\alpha = \frac{(\alpha^2 - \beta^2)(\beta - \alpha)^2 + \alpha^2 \beta^2}{\alpha^2 (\beta - \alpha)^2}, \tag{9}$$

$$L_\beta = \frac{2(\alpha + \beta)(\beta - \alpha)^2 + \alpha \beta^2 - 2\alpha^2 \beta}{\alpha (\beta - \alpha)^2},$$

$$L_{\alpha\alpha} = \frac{2\left\{\alpha^3 + (\beta-\alpha)^3\right\}\beta^2}{\alpha^3(\beta-\alpha)^3},$$
$$L_{\beta\beta} = \frac{2\left\{\alpha^3 + (\beta-\alpha)^3\right\}}{\alpha(\beta-\alpha)^3}.$$

Substituting (9) into (7), we have

$$\begin{aligned}&\{(\alpha^2\beta^2 + \alpha^4 - \beta^4)B^t_{ji}y^j y_t + (2\alpha^2\beta^3 - 4\alpha^4\beta)(B^t_{ji}b_t - b_{j;i})y^j\}\\ &+\alpha\{(2\beta^3 - 2\alpha^2\beta)B^t_{ji}y^j y_t + (2\alpha^4 - \alpha^2\beta^2)(B^t_{ji}b_t - b_{j;i})y^j\} = 0.\end{aligned} \tag{10}$$

Assume that F^n is a Berwald space, that is $G^i_{jk} = G^i_{jk}(x)$. Then we have $B^t_{ji} = B^t_{ji}(x)$. Since α is irrational in (y^i), from (10) we have

$$(\alpha^2\beta^2 + \alpha^4 - \beta^4)B^t_{ji}y^j y_t + (2\alpha^2\beta^3 - 4\alpha^4\beta)(B^t_{ji}b_t - b_{j;i})y^j = 0$$

and

$$(2\beta^3 - 2\alpha^2\beta)B^t_{ji}y^j y_t + (2\alpha^4 - \alpha^2\beta^2)(B^t_{ji}b_t - b_{j;i})y^j = 0,$$

which can be written in matrix form of homogeneous linear equations,

$$AX = 0$$

$$\begin{bmatrix}(\alpha^2\beta^2 + \alpha^4 - \beta^4) & (2\alpha^2\beta^3 - 4\alpha^4\beta)\\ (2\beta^3 - 2\alpha^2\beta) & (2\alpha^4 - \alpha^2\beta^2)\end{bmatrix}\begin{bmatrix}B^t_{ji}y^j y_t\\ (B^t_{ji}b_t - b_{j;i})y^j\end{bmatrix} = 0.$$

Let,

$$A = \begin{bmatrix}(\alpha^2\beta^2 + \alpha^4 - \beta^4) & (2\alpha^2\beta^3 - 4\alpha^4\beta)\\ (2\beta^3 - 2\alpha^2\beta) & (2\alpha^4 - \alpha^2\beta^2)\end{bmatrix},$$

where

$$|A| = -7\alpha^6\beta^2 + 9\alpha^4\beta^4 + 2\alpha^8 - 3\alpha^2\beta^6 \neq 0,$$

which implies

$$B^t_{ji}y^j y_t = 0 \quad \text{and} \quad (B^t_{ji}b_t - b_{j;i})y^j = 0,$$

which show

$$B^t_{ji}a_{th} + B^t_{hi}a_{tj} = 0 \quad \text{and} \quad (B^t_{ji}b_t - b_{j;i})y^j = 0.$$

The former yields $B^t_{ji} = 0$ by the well-known Christoffel process, which gives $b_{j;i} = 0$. Hence $r_{ij} = 0$ and $s_{ij} = 0$. Conversely, if $b_{j;i} = 0$ then $B^t_{ji} = 0$ are uniquely determined from (10). Therefore we have,

Theorem 1. *The Finsler space F^n with the deformed Berwald-Infinite series metric is a Berwald space if and only if $b_{j;i} = 0$, and then the Berwald connection is essentially Riemannian $(\gamma^i_{jk}, \gamma^i_{0j}, 0)$.*

Theorem 2. *The Finsler space F^n with the deformed Berwald-Infinite series metric is a Berwald space if and only if $r_{ij} = 0$ and $s_{ij} = 0$.*

4 The condition for F^n to be a Douglas space

In the present section, we consider the condition that a Finsler space F^n with the deformed Berwald-Infinite series metric be a Douglas space. Substituationg (9) into (8), we obtain

$$\begin{aligned}
&\{-37\alpha^4\beta^5 - 16b^2\alpha^4\beta^5 + 27\alpha^5\beta^4 + 2b^2\alpha^5\beta^4 + 20b^2\alpha^6\beta^3 \\
&+ 4\alpha^3\beta^6 + 10b^2\alpha^3\beta^6 + 22\alpha^2\beta^7 - 2b^2\alpha^2\beta^7 - 10\alpha^7\beta^2 \\
&- 18b^2\alpha^7\beta^2 + 5\alpha^8\beta + 6b^2\alpha^8\beta - 15\alpha\beta^8 - \alpha^9 + 3\beta^9\}B^{ij} \\
&- \alpha^2\{-41\alpha^3\beta^5 - 14b^2\alpha^3\beta^5 + 55\alpha^4\beta^4 + 10b^2\alpha^4\beta^4 \\
&- 7\alpha^5\beta^3 + 22b^2\alpha^5\beta^3 - 11\alpha^2\beta^6 + 4b^2\alpha^2\beta^6 + 21\alpha\beta^7 \\
&- 17\alpha^6\beta^2 - 36b^2\alpha^6\beta^2 - 2\alpha^8 - 6\beta^8 + 10\alpha^7\beta + 12b^2\alpha^7\beta\}(s^i_0y^j - s^j_0y^i) \\
&- \alpha^2\beta\{r_{00}(-8\alpha^2\beta^4 + \alpha^3\beta^3 + 10\alpha^4\beta^2 - \beta^6 + 5\alpha\beta^5 - 9\alpha^5\beta + 3\alpha^6) \\
&- 2s_0\alpha^2(11\alpha^3\beta^2 + 6\alpha^5 - 18\alpha^4\beta + 2\beta^5 + 5\alpha^2\beta^3 - 7\alpha\beta^4)\}(b^iy^j - b^jy^i) = 0.
\end{aligned} \tag{11}$$

It is noted that $\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$.

Suppose that F^n is a Douglas space, that is, B^{ij} are $hp(3)$. Separating (11) in the rational and irrational terms of y^i because α is irrational in (y^i), we have

$$\begin{aligned}
&\{-37\alpha^4\beta^5 - 16b^2\alpha^4\beta^5 + 20b^2\alpha^6\beta^3 + 22\alpha^2\beta^7 - 2b^2\alpha^2\beta^7 + 5\alpha^8\beta + 6b^2\alpha^8\beta + 3\beta^9\}B^{ij} \\
&- \alpha^2\{55\alpha^4\beta^4 + 10b^2\alpha^4\beta^4 - 11\alpha^2\beta^6 + 4b^2\alpha^2\beta^6 - 17\alpha^6\beta^2 - 36b^2\alpha^6\beta^2 - 2\alpha^8 - 6\beta^8\}(s^i_0y^j - s^j_0y^i) \\
&- \alpha^2\beta\{r_{00}(-8\alpha^2\beta^4 + 10\alpha^4\beta^2 - \beta^6 + 3\alpha^6) - 2s_0\alpha^2(-18\alpha^4\beta + 5\alpha^2\beta^3 + 2\beta^5)\}(b^iy^j - b^jy^i) \\
&+ \alpha[\{27\alpha^4\beta^4 + 2b^2\alpha^4\beta^4 + 4\alpha^2\beta^6 + 10b^2\alpha^2\beta^6 - 10\alpha^6\beta^2 - 18b^2\alpha^6\beta^2 - 15\beta^8 - \alpha^8\}B^{ij} \\
&- \alpha^2\{-41\alpha^2\beta^5 - 14b^2\alpha^2\beta^5 - 7\alpha^4\beta^3 + 22b^2\alpha^4\beta^3 + 21\beta^7 + 10\alpha^6\beta + 12b^2\alpha^6\beta\}(s^i_0y^j - s^j_0y^i) \\
&- \alpha^2\beta\{r_{00}(\alpha^2\beta^3 + 5\beta^5 - 9\alpha^4\beta) - 2s_0\alpha^2(11\alpha^2\beta^2 + 6\alpha^4 - 7\beta^4)\}(b^iy^j - b^jy^i)] = 0.
\end{aligned} \tag{12}$$

Hence the equation (12) is divided into two equations as follows:

$$\begin{aligned}
&\{-37\alpha^4\beta^5 - 16b^2\alpha^4\beta^5 + 20b^2\alpha^6\beta^3 + 22\alpha^2\beta^7 - 2b^2\alpha^2\beta^7 + 5\alpha^8\beta + 6b^2\alpha^8\beta + 3\beta^9\}B^{ij} \\
&- \alpha^2\{55\alpha^4\beta^4 + 10b^2\alpha^4\beta^4 - 11\alpha^2\beta^6 + 4b^2\alpha^2\beta^6 - 17\alpha^6\beta^2 - 36b^2\alpha^6\beta^2 - 2\alpha^8 - 6\beta^8\}(s^i_0y^j - s^j_0y^i) \\
&- \alpha^2\beta\{r_{00}(-8\alpha^2\beta^4 + 10\alpha^4\beta^2 - \beta^6 + 3\alpha^6) - 2s_0\alpha^2(-18\alpha^4\beta + 5\alpha^2\beta^3 + 2\beta^5)\}(b^iy^j - b^jy^i) = 0,
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\{27\alpha^4\beta^4 + 2b^2\alpha^4\beta^4 + 4\alpha^2\beta^6 + 10b^2\alpha^2\beta^6 - 10\alpha^6\beta^2 - 18b^2\alpha^6\beta^2 - 15\beta^8 - \alpha^8\}B^{ij} \\
&- \alpha^2\{-41\alpha^2\beta^5 - 14b^2\alpha^2\beta^5 - 7\alpha^4\beta^3 + 22b^2\alpha^4\beta^3 + 21\beta^7 + 10\alpha^6\beta + 12b^2\alpha^6\beta\}(s^i_0y^j - s^j_0y^i) \\
&- \alpha^2\beta\{r_{00}(\alpha^2\beta^3 + 5\beta^5 - 9\alpha^4\beta) - 2s_0\alpha^2(11\alpha^2\beta^2 + 6\alpha^4 - 7\beta^4)\}(b^iy^j - b^jy^i) = 0.
\end{aligned} \tag{14}$$

Eliminating B^{ij} from (13) and (14), we obtain

$$A(s^i_0y^j - s^j_0y^i) + B(b^iy^j - b^jy^i) = 0, \tag{15}$$

where

$$\begin{aligned} A =& \{-37\alpha^4\beta^5 - 16b^2\alpha^4\beta^5 + 20b^2\alpha^6\beta^3 + 22\alpha^2\beta^7 - 2b^2\alpha^2\beta^7 + 5\alpha^8\beta + 6b^2\alpha^8\beta + 3\beta^9\} \\ &\times \{-41\alpha^2\beta^5 - 14b^2\alpha^2\beta^5 - 7\alpha^4\beta^3 + 22b^2\alpha^4\beta^3 + 21\beta^7 + 10\alpha^6\beta + 12b^2\alpha^6\beta\} \\ &- \{27\alpha^4\beta^4 + 2b^2\alpha^4\beta^4 + 4\alpha^2\beta^6 + 10b^2\alpha^2\beta^6 - 10\alpha^6\beta^2 - 18b^2\alpha^6\beta^2 - 15\beta^8 - \alpha^8\} \\ &\times \{55\alpha^4\beta^4 + 10b^2\alpha^4\beta^4 - 11\alpha^2\beta^6 + 4b^2\alpha^2\beta^6 - 17\alpha^6\beta^2 - 36b^2\alpha^6\beta^2 - 2\alpha^8 - 6\beta^8\}, \end{aligned} \tag{16}$$

$$\begin{aligned} B =& \beta[\{-37\alpha^4\beta^5 - 16b^2\alpha^4\beta^5 + 20b^2\alpha^6\beta^3 + 22\alpha^2\beta^7 - 2b^2\alpha^2\beta^7 + 5\alpha^8\beta + 6b^2\alpha^8\beta + 3\beta^9\} \\ &\times \{r_{00}(\alpha^2\beta^3 + 5\beta^5 - 9\alpha^4\beta) - 2s_0\alpha^2(11\alpha^2\beta^2 + 6\alpha^4 - 7\beta^4)\} \\ &- \{27\alpha^4\beta^4 + 2b^2\alpha^4\beta^4 + 4\alpha^2\beta^6 + 10b^2\alpha^2\beta^6 - 10\alpha^6\beta^2 - 18b^2\alpha^6\beta^2 - 15\beta^8 - \alpha^8\} \\ &\times \{r_{00}(-8\alpha^2\beta^4 + 10\alpha^4\beta^2 - \beta^6 + 3\alpha^6) - 2s_0\alpha^2(-18\alpha^4\beta + 5\alpha^2\beta^3 + 2\beta^5)\}]. \end{aligned}$$

Transvection of (15) by b_iy_j leads to

$$As_0 + \beta B_1(b^2\alpha^2 - \beta^2) = 0. \tag{17}$$

where

$$\begin{aligned} B_1 =& \{r_{00}(-24\alpha^4\beta^8 - 5\alpha^6\beta^6 + 19\alpha^2\beta^{10} - 3\beta^{12} - 5\alpha^{10}\beta^2 + 16\alpha^8\beta^4 + 3\alpha^{12}) \\ &+ \beta s_0(636\alpha^6\beta^6 + 244b^2\alpha^6\beta^6 + 240\alpha^{10}\beta^2 + 276b^2\alpha^{10}\beta^2 - 574\alpha^4\beta^8 - 562\alpha^8\beta^4 \\ &- 72b^2\alpha^4\beta^8 - 416b^2\alpha^8\beta^4 - 24\alpha^{12} + 108\alpha^2\beta^{10} - 72b^2\alpha^{12} + 12b^2\alpha^2\beta^{10} - 18\beta^{12})\}. \end{aligned}$$

The term of (17) which does not contain α^2 is found in $3\beta^{15}(r_{00} - 3\beta s_0)$. Hence, there exists $hp(15) : V_{15}$ such that

$$\beta^{15}(r_{00} - 3\beta s_0) = \alpha^2 V_{15}. \tag{18}$$

Then it will be better to divide our consideration into three cases as follows:

1. $V_{15} = 0$,
2. $V_{15} \neq 0, \alpha^2 \not\equiv 0(mod\beta)$,
3. $V_{15} \neq 0, \alpha^2 \equiv 0(mod\beta)$.

Case (1) :

In case of $V_{15} = 0$: from (18), $r_{00} = 3\beta s_0$, that is, $2r_{ij} = 3(b_is_j + b_js_i)$. Put $r_{00} = 3\beta s_0$ into (17), we get

$$s_0\{A + \beta^2 B_1^{'}(b^2\alpha^2 - \beta^2)\} = 0, \tag{19}$$

where

$$B_1^{'} = 621\alpha^6\beta^6 + 244b^2\alpha^6\beta^6 + 225\alpha^{10}\beta^2 + 276b^2\alpha^{10}\beta^2 \tag{20}$$

$$+165\alpha^2\beta^{10}+12b^2\alpha^2\beta^{10}-646\alpha^4\beta^8-72b^2\alpha^4\beta^8$$
$$-514\alpha^8\beta^4-416b^2\alpha^8\beta^4-15\alpha^{12}-72b^2\alpha^{12}-27\beta^{12}.$$

If $A+\beta^2B_1^{'}(b^2\alpha^2-\beta^2)=0$ in (19), then we get

$$A+\beta^2B_1^{'}(b^2\alpha^2-\beta^2)=\alpha^2A_1,$$

where

$$\begin{aligned}A_1=&-465\alpha^6\beta^8-339b^2\alpha^6\beta^8+89\alpha^{10}\beta^4-87b^2\alpha^{10}\beta^4\\&-23\alpha^2\beta^{12}-3b^2\alpha^2\beta^{12}+534\alpha^4\beta^{10}-86b^2\alpha^4\beta^{10}\\&+206\alpha^8\beta^6+270b^2\alpha^8\beta^6-90\beta^{14}-3b^2\beta^{14}\\&+13\alpha^{12}\beta^2+33b^2\alpha^{12}\beta^2-2\alpha^{14}.\end{aligned}$$

Thus, the term $A_1=0$, which does not contain α^2 is $-3(b^2+30)\beta^{14}$. Therefore, there exixt $hp(12):V_{12}$ such that

$$(b^2+30)\beta^{14}=\alpha^2V_{12},$$

where we assume $b^2\neq-30$. Hence, we have $V_{12}=0$, it leads to a contradiction. Therefore, $A+\beta^2B_1^{'}(b^2\alpha^2-\beta^2)\neq0$. Then we have $s_0=0$ from (19) and we obtained $r_{00}=0$. Substituting $s_0=0$ and $r_{00}=0$ into equation (15). We have,

$$A(s^i_0y^j-s^j_0y^i)=0. \tag{21}$$

If $A=0$, then we have from (16)

$$\begin{aligned}A=&\{-37\alpha^4\beta^5-16b^2\alpha^4\beta^5+20b^2\alpha^6\beta^3+22\alpha^2\beta^7-2b^2\alpha^2\beta^7+5\alpha^8\beta+6b^2\alpha^8\beta+3\beta^9\}\\&\times\{-41\alpha^2\beta^5-14b^2\alpha^2\beta^5-7\alpha^4\beta^3+22b^2\alpha^4\beta^3+21\beta^7+10\alpha^6\beta+12b^2\alpha^6\beta\}\\&-\{27\alpha^4\beta^4+2b^2\alpha^4\beta^4+4\alpha^2\beta^6+10b^2\alpha^2\beta^6-10\alpha^6\beta^2-18b^2\alpha^6\beta^2-15\beta^8-\alpha^8\}\\&\times\{55\alpha^4\beta^4+10b^2\alpha^4\beta^4-11\alpha^2\beta^6+4b^2\alpha^2\beta^6-17\alpha^6\beta^2-36b^2\alpha^6\beta^2-2\alpha^8-6\beta^8\}\\=&\,0.\end{aligned} \tag{22}$$

The term of equation (22) which seemingly does not contain α^2 is $-27\beta^{16}$. Thus, there exists $hp(14):V_{14}$ such that $-27\beta^{16}=\alpha^2V_{14}$. From this equation, we have $V_{14}=0$, it leads to a contradiction. Therefore, $A\neq0$, Thus we have from (21)

$$s^i_0y^j-s^j_0y^i=0. \tag{23}$$

Transvection of equation (23) by y_j gives $s^i_0=0$. Finally $r_{ij}=s_{ij}=0$ are concluded, that is, $b_{i;j}=0$.

Case (2) :

In case of $V_{15} \neq 0; \alpha^2 \not\equiv 0(mod\beta)$: In this case, equation (18) shows that there exists a function $h = h(x)$ satisfying

$$r_{00} - 3\beta s_0 = h(x)\alpha^2. \tag{24}$$

Substituting (24) in (17), we have

$$\begin{aligned} s_0\{ &- 465\alpha^6\beta^8 - 339b^2\alpha^6\beta^8 - 23\alpha^2\beta^{12} - 3b^2\alpha^2\beta^{12} \\ &+ 89\alpha^{10}\beta^4 - 159b^2\alpha^{10}\beta^4 + 534\alpha^4\beta^{10} - 86b^2\alpha^4\beta^{10} \\ &+ 20\alpha^8\beta^6 - 270b^2\alpha^8\beta^6 - 90\beta^{14} - 111b^2\beta^{14} \\ &+ 13\alpha^{12}\beta^2 + 33b^2\alpha^{12}\beta^2 + 72b^4\alpha^{12}\beta^2 - 72b^4\alpha^{13}\beta \\ &+ 72b^2\alpha^{11}\beta^3 + 2\alpha^{14}\} + h\beta\{-24b^2\alpha^6\beta^8 + 5\alpha^6\beta^8 \\ &- 5b^2\alpha^8\beta^6 - 16\alpha^8\beta^6 + 19b^2\alpha^4\beta^{10} + 24\alpha^4\beta^{10} \\ &- 3b^2\alpha^2\beta^{12} - 19\alpha^2\beta^{12} - 5b^2\alpha^{12}\beta^2 - 3\alpha^{12}\beta^2 \\ &+ 16b^2\alpha^{10}\beta^4 + 5\alpha^{10}\beta^4 + 3b^2\alpha^{14} + 3\beta^{14}\} = 0. \end{aligned} \tag{25}$$

The term of equation (25) which seemingly does not contain α^2 is $3\{-(30 + 37b^2)s_0 + h\beta\}\beta^{14}$. Hence, there exists $hp(13)$: V_{13} such that $\{-(30 + 37b^2)s_0 + h\beta\}\beta^{14} = \alpha^2 V_{13}$. Since $\alpha^2 \not\equiv 0(mod\beta)$, we must have $V_{13} = 0$. Thus we have,

$$\left\{-(30 + 37b^2)s_0 + h\beta\right\}\beta^{14} = 0,$$

which implies

$$s_0 = \frac{h(x)}{(30 + 37b^2)}\beta. \tag{26}$$

From equation (26), we have $s_i = \frac{h(x)b_i}{(30+37b^2)}$. Transvecting by b^i, we obtain $h(x)b^2 = 0$. Hence $h(x) = 0$. Substitute $h(x) = 0$ into (24) and (26), we obtain $s_0 = 0$ and $r_{00} = 0$. Therefore (15) is reduced to $A(s^i_0 y^j - s^j_0 y^i) = 0$. Since $A \neq 0$, we have $s^i_0 y^j - s^j_0 y^i = 0$. Transvection of this equation by y_j gives $s^i_0 = 0$. Finally, $r_{ij} = s_{ij} = 0$ are concluded, that is, $b_{i;j} = 0$.

Case (3) :

In case of $V_{17} \neq 0; \alpha^2 \equiv 0(mod\beta)$: In this case, lemma 1 shows that $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, where $\delta = d_i(x)y^i$. From (18) we have $\beta^{14}(r_{00} - 3\beta s_0) = \delta V_{15}$, which must be reduced to $r_{00} - 3\beta s_0 = \delta V$, where $V = V_i(x)y^i$. From equation (26), we obtain $s_0 = \frac{h(x)}{30}\beta$ easily.

Transvection of $r_{00} - 3\beta s_0 = \delta V$ by b_i we have

$$r_{00}b^i = 2Vy^i. \tag{27}$$

Again, Transvection of equation (27), by b_i leads to

$$r_{00}b^2 = 2\beta V. \tag{28}$$

Hence $V = 0$, it leads to a contradiction for $V = V_i(x)y^i$. It is possible only if $s_0 = 0$ and $r_{00} = 0$. Substituting $s_0 = 0$ and $r_{00} = 0$ in equation (15), we get $A(s^i_0 y^j - s^j_0 y^i) = 0$. Since $A \neq 0$, we have $s^i_0 y^j - s^j_0 y^i = 0$. Transvection of this equation by y_j gives $s^i_0 = 0$. Thus $r_{ij} = s_{ij} = 0$ are concluded, that is, $b_{i;j} = 0$.
Conversely, if $b_{i;j} = 0$, then we obtain $B^{ij} = 0$ from (11). Hence F^n is a Douglas space. Consequently, we have

Theorem 3. *An n-dimensional Finsler space F^n with the deformed Berwald-Infinite series metric is a Douglas space, if and only if*

1. $\alpha^2 \not\equiv 0(mod\beta)$: $b_{j;i} = 0$.
2. $\alpha^2 \equiv 0(mod\beta)$: $n = 2$, $b^2 = 0$ *and* $b_{j;i} = 0$, *where* $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$ *and* $h = h(x)$.

From Theorem 1 and Theorem 3, we have

Theorem 4. *If an n-dimensional Finsler space F^n with the deformed Berwald-Infinite series metric is a Douglas space, then F^n is a Berwald space.*

5 Conclusion

In the present paper, we have studied the Finsler space with Berwald Infinite series metric and obtained the condition under which the Finsler space F^n will be a Berwald and Douglas space and the conditions are described in theorems 1, 2, 3 and 4, respectively. Since this is an important combination of two special (α, β)-metric, so in our future work, we study other important Finsler properties such as reducibility, main scalars in two and three dimensions, Landsberg space etc, with this metric.

Acknowledgments

The authors would like to thank the referee for careful reading.

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