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# The domination uniform subdivision number of $G^{++-}$

**T. Berjin Magizha**[†], **M. K. Angel Jebitha**[‡*], **S. Sujitha**[§]

[†]*St. Xaviers Catholic College of Engineering (Autonomous), Nagercoil, Tamilnadu, India*
[‡ §]*Department of Mathematics, Holy Cross College (Autonomous) Nagercoil, Tamilnadu, India*
*Emails:* *berjin@sxcce.edu.in, angeljebitha@holycrossngl.edu.in, sujitha.s@holycrossngl.edu.in*

**Abstract.** Theory of domination plays a vital role in network communications. Different domination parameters have been studied by various mathematicians. In this paper, the exact value of domination uniform subdivision number of transformation graphs $G^{++-}$ of some standard graphs are obtained. Furthermore, the bounds of $usd_\gamma(G^{++-})$ for any graph $G$ are obtained. Finally, $sd_\gamma-$critical graph on $G^{++-}$ are characterized.

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## 1 Introduction

Let $G = (V, E)$ be a simple undirected graph of order $n$ and size $m$. If $v \in V(G)$, then the neighborhood of $v$ is the set $N_G(v)$ $(orN(v))$ consisting of all vertices $u$ which are adjacent to $v$. The closed neighborhood is $N_G[v] = N_G(v) \cup \{v\}$. The degree of $v$ in $G$ is $|N(v)|$ and is denoted by $deg(v)$. The minimum degree of $G$ is $min\{deg_G(v) : v \in V(G)\}$ and is denoted by $\delta(G)$. A vertex $v$ is said to be pendant vertex if $deg(v) = 1$. A path, a cycle and a complete graph on $n$ vertices are denoted by $P_n$, $C_n$ and $K_n$ respectively. A complete bipartite graph is denoted by $K_{m,n}$. A graph is said to be connected if there exists a path between any pair of vertices. Otherwise it is said to be disconnected. The distance $d(u, v)$ between two vertices $u$ and $v$ of a connected graph $G$ is defined to be the length of any shortest path joining $u$ and $v$. A shortest $u-v$ path is often called as geodesic. The diameter of a connected graph $G$ is the length of any longest geodesic and is denoted by $diam(G)$. Two graphs $G$ and $H$ are disjoint if they have no vertex in common, and their union is denoted by $G + H$. The disjoint union of $k$ copies of $G$ is written as $kG$.

[*]Corresponding author
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A subset $D$ of $V(G)$ is said to be dominating set if every vertex of $V(G)-D$ is adjacent to at least one vertex in $D$. The minimum cardinality taken over all minimal dominating sets of $G$ is the domination number of $G$ and is denoted by $\gamma(G)$. The domination subdivision number was introduced by Arumugam, Velammal in [10]. Its bound was obtained in [10] and several authors characterized trees according to their domination subdivision number. Also, many results have been reached on the parameters $sd_{dd}$, $sd_{\gamma c}$ and $sd_{\gamma t}$. An edge $e = uv$ is said to be subdivided when it is deleted and replaced by a $u-v$ path of length two with a new internal vertex $w$ (subdividing vertex). $G \wedge \{e\}$ is the graph obtained by subdividing the edge $e$. The domination subdivision number of a graph $G$ is the minimum number of edges whose subdivision increases the domination number. It can also be defined as $sd_\gamma(G) = min\{|E^{'}| : \gamma(G \wedge E^{'}) > \gamma(G)\}$.

A domination uniform subdivision number of $G$ is the least positive integer $k$ such that the subdivision of any $k$ edges from $G$ results in a graph having domination number greater than that of $G$ and is denoted by $usd_\gamma(G)$. If it does not exist, then $usd_\gamma(G) = 0$. This number was introduced and studied in [4]. Any graph $G$ is called $sd_\gamma$-critical if $usd_\gamma(G) = 1$.

A subset $S \subseteq E(G)$ is said to be domination subdivision stable set if $\gamma(G \wedge S) = \gamma(G)$. A domination subdivision stable set $S$ is said to be maximum domination subdivision stable set if there is no domination subdivision stable set $S^{'}$ such as $|S^{'}| > |S|$. For any graph $usd\gamma(G) = |S| + 1$, $S$ is the maximum domination subdivision stable set of $G$.

Wu and Meng [13] generalized the concept of total graphs to a total transformation graph $G^{xyz}$ with $x, y, z \in \{+, -\}$ where $G^{+++}$ is the complement of $G^{---}$. Each of these eight kinds of transformation graph $G^{xyz}$ appears to have some nice properties. For instance, their diameters are small in most cases [13], and their edge connectives are equal to their minimum degree etc. [5]. Several authors have discussed various concepts on transformation graphs [1, 2, 6, 9–11, 13].

The transformation graph $G^{++-}$ of $G$ is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows: (a) two elements in $V(G)$ are adjacent if and only if they are adjacent in $G$, (b) two elements in $E(G)$ are adjacent if and only if they are adjacent in $G$ and (c) an element of $V(G)$ and an element of $E(G)$ are adjacent if and only if they are not incident in $G$. The domination subdivision number of the transformation graph $G^{-+-}$ was studied in [3]. In [2], the domination uniform subdivision number of $G^{---}$ has been investigated. In this paper, we investigate the domination uniform subdivision number of $G^{++-}$ and we provide some bounds for $usd_\gamma(G^{++-})$. Further we characterize $sd_\gamma$-critical graphs. Terms not defined here are used in the sense of [8].

## 2 Basic results on $G^{++-}$

Let $G$ be a graph of order $n$ and size $m$. A graph $G^{++-}$ is a derived graph. The order of $G^{++-}$ is $m + n$, $d_{G^{++-}}(x) = m$ for $x \in V(G)$ and $d_{G^{++-}}(e) = n - 4 + d_G(u) + d_G(v)$ for any $e = uv \in E(G)$. So, $\delta(G^{++-}) = min\{m, n - 4 + min_{uv \in E(G)}\{d(u) + d(v)\}\}$.

**Remark 1** ([9]). *Let $G$ be a $r$-regular graph with $n$ vertices and $m$ edges. Let $u \in V(G)$ and $e \in E(G)$, then $d_{G^{++-}}(u) = m$ and $d_{G^{++-}}(e) = 2r + n - 4$.*

**Remark 2** ([12]). *Let $G$ be a graph of order $n \geq 6$ and size $m$. If $m \geq n$, $G^{++-}$ is Hamiltonian.*

By the definition of $G^{++-}$, any vertex $v$ of $G$ is adjacent to all the adjacent vertex of $v$ in $G$ and all the non incident edges of $v$ in $G$. Also any incident edge of $v$ in $G$ is adjacent all the remaining non-adjacent vertex of $v$ and incident edge $v$. Hence we have the following observation.

**Remark 3.** *For any graph $G$, $\gamma(G^{++-}) = 2$.*

**Remark 4** ([14]). *For two graphs $G_1$ and $G_2$, $G_1^{++-} \cong G_2^{++-}$ if and only if $G_1 \cong G_2$.*

Wu and Meng [12] determined the independence number of $G^{++-}$ and obtained a lower bound for the connectivity of $G^{++-}$. Also they provided a simple sufficient condition for $G^{++-}$ to be hamiltonian. Furthermore in [11], b-chromatic number of $G^{++-}$ was studied. In [7], clique covering of $G^{++-}$ was discussed. The object of this paper is to study the domination uniform subdivision number of $G^{++-}$ and also its bounds.

## 3 Results on standard graphs

**Theorem 1.** *For $n \geq 8$, $usd_\gamma(P_n^{++-}) = 3$.*

*Proof.* Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{e_1, e_2, \ldots, e_{n-1}\}$. Then $V(P_n^{++-}) = V(P_n) \cup E(P_n)$. Also $d_{P_n^{++-}}(e_i) = n$, $(2 \leq i \leq n-2)$, $d_{P_n^{++-}}(e_1) = d_{P_n^{++-}}(e_{n-1}) = n-1$ and $d_{P_n^{++-}}(v_i) = n-1$, $(1 \leq i \leq n)$. Let $D$ be a minimum dominating set of $P_n^{++-}$. Then $D = \{v_i, e_i\}$ where $v_i$ is incident with $e_i$ in $P_n$. The vertex $e_i(2 \leq i \leq n-2)$ is of maximum degree in $P_n^{++-}$ and is adjacent to $e_{i-1}$, $e_{i+1}, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_n$. But $v_i$ is adjacent only two vertices $e_{i-1}$, $e_{i+1}$ among those vertices. Hence, any subdivision stable set has atmost two edges of $P_n^{++-} S_1 = \{e_i v_{i-1}, e_i e_{i+1}\}$. Therefore, $S_2 = \{e_i v_{i+2}, e_i e_{i-1}\}$ are maximum domination subdivision stable sets of $P_n^{++-}$. Also $|S_1| = |S_2| = 2$. Therefore $usd_\gamma(P_n^{++-}) = 3$. □

**Theorem 2.** *For any cycle $C_n (n \geq 4)$, $usd_\gamma(C_n^{++-}) = 3$.*

*Proof.* Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$. Then $V(C_n^{++-}) = V(C_n) \cup E(C_n)$ and $\gamma(C_n^{++-}) = 2$. All the vertices of $C_n^{++-}$ are of same degree. Let $D$ be a minimum dominating set of $C_n^{++-}$. Then $D = \{v_i, e_j\}$, where $v_i$ is incident with $e_j$ in $C_n$. Therefore, there are two vertices of $C_n^{++-}$ which are common adjacent vertices of $v_i$ and $e_j$. Hence any maximum domination subdivision stable set consists of only two edges. Therefore $usd_\gamma(C_n^{++-}) = 3$. □

**Theorem 3.** *For any $r \geq 3$, $usd_\gamma(K_{1,r}^{++-}) = r$.*

*Proof.* Let $V(K_{1,r}) = \{v, v_1, v_2, \ldots, v_r\}$, $E(K_{1,r}) = \{e_1 = vv_1, e_2 = vv_2, \ldots, e_r = vv_r\}$ and $V(K_{1,r}^{++-}) = V(K_{1,r}) \cup E(K_{1,r})$. In a graph $K_{1,r}^{++-}$, $d_{K_{1,r}}^{++-}(e) = 2r - 1$, $e \in E(G)$ and $d_{K_{1,r}}^{++-}(v) = r, v \in V(G)$. We have $\gamma(K_{1,r}^{++-}) = 2$ and the minimum dominating set $D = \{e_i, v\}$ or $D = \{e_i, v_i\}$ of $K_{1,r}^{++-}$. Now $N[v_i] = \{v, e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, v_r\}$ and $N[e_i] =$

$\{e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_r, v_1, v_2, \ldots, v_{i-1}, \ldots, v_r\}$ . Thus, $N[v_i] \cap N[e_i] = \{e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_r\}$. Hence the maximum domination subdivision stable sets are $S_1 = \{v_ie_1, v_ie_2, \ldots, v_ie_{i-1}, v_ie_{i+1}, \ldots, v_ie_r\}$ and $S_2 = \{e_ie_1, e_ie_2, \ldots, ee_{i-1}, e_ie_{i+1}, \ldots, e_ie_r\}$. Therefore, $|S_1| = |S_2| = r - 1$. Now $N[v] = \{v_1, v_2, \ldots, v_r, e_1, e_2, \ldots, e_r\}$, $N[v] \cap N[e_i] = \{e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_r, v_1, v_2, \ldots, v_{i-1}, \ldots, v_r\}$. Hence maximum domination subdivision stable sets are $S_3 = \{vv_1, vv_2, \ldots, vv_{i-1}, vv_{i+1}, \ldots, vv_r\}$ and $S_4 = \{e_ie_1, e_ie_2, \ldots, e_ie_{i-1}, e_ie_{i+1}, \ldots, e_ie_r\}$ Therefore $|S_3| = |S_4| = r - 1$. Since these four are the only maximum domination subdivision stable sets, $usd_\gamma(K_{1,r}^{++-}) = r - 1 + 1 = r$. □

**Theorem 4.** *For all $r, s \geq 2$, $usd_\gamma(K_{r,s}^{++-}) = r + s - 1$.*

*Proof.* Let the vertex set of $K_{r,s}^{++-}$ is partitioned into two sets $V = \{v_1, v_2, \ldots, v_r\}$ and $U = \{u_1, u_2, \ldots, u_s\}$. Thus $V(K_{r,s}^{++-}) = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_s\}$. Take $e_{ij} = v_iu_j$ and $E(K_{r,s}^{++-}) = \{e_{11}, e_{12}, \ldots, e_{1s}, e_{21}, e_{22}, \ldots, e_{2s}, \ldots, e_{r1}, e_{r2}, \ldots, e_{rs}\}$. Since $\gamma(K_{r,s}^{++-}) = 2$, a minimum dominating set of $K_{r,s}^{++-}$ is either $D = \{v_i, e_{ij}\}$ or $D = \{u_i, e_{ij}\}$.
Case (i): $D = \{v_i, e_{ij}\}$.
If $D = \{v_i, e_{ij}\}$, the maximum domination subdivision stable sets are
$S_1 = \{v_iu_1, v_iu_2, \ldots, v_iu_{j-1}, v_iu_{j+1}, \ldots, v_iu_s\} \cup \{v_1e_{1j}, v_2e_{2j}, \ldots, v_{j-1}e_{(j-1)j}, v_{j+1}e_{(j+1)}j, \ldots, v_re_{rj}\}$ and $S_2 = \{e_{ij}u_1, e_{ij}u_2, \ldots, e_{ij}u_{j-1}, e_{ij}u_{j+1}, \ldots, e_{ij}u_s\} \cup \{e_{ij}e_{1j}, e_{ij}e_{2j}, \ldots, , e_{ij}e_{(j+1)j}, \ldots, e_{ij}e_{rj}\}$.
Then $|S_1| = |S_2| = r + s - 2$.
Case (ii): $D = \{u_i, e_ij\}$.
Similar to case(i), there are $r + s - 2$ vertices are common to $u_i$ and $e_{ij}$ in $K_{r,s}^{++-}$ and hence any maximum domination subdivision stable set has $r + s - 2$ edges of $K_{r,s}^{++-}$. Therefore, $usd_\gamma(K_{r,s}^{++-}) = r + s - 2 + 1 = r + s - 1$. □

**Theorem 5.** *For all $n \geq 4$, $usd_\gamma(K_n^{++-}) = 2n - 3$.*

*Proof.* Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Thus $V(K_n^{++-}) = V(K_n) \cup E(K_n)$. In $K_n^{++-}$, $d_{K_n^{++-}}(v_i) = \frac{n(n-1)}{2}$ and $d_{K_n{}^{++-}}(e) = 3(n-2), e \in E(K_n)$. We have $\gamma(K_n^{-+-}) = 2$. Let $D$ be a minimum dominating set of $K_n^{++-}$. Then $D$ consists of two vertices $D = \{v_i, e_j\}$, where $e_j$ is an edge that is incident with $v_i$ in $K_n$. Let $e_j = v_iv_j$. Then $S = \{v_iv_1, v_iv_2, \ldots, v_iv_{j-1}, v_iv_{j+1}, \ldots, v_iv_n\} \cup \{v_if | f \text{is incident with } v_j \text{in} K_n\}$ is a maximum domination subdivision stable set of $K_n^{++-}$ since $v_i$ and $e_j$ are adjacent to $\{v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\} \cup \{f | f \text{is incident with} v_j \text{in} K_n\}$ . Therefore $|S| = n - 2 + n - 2 = 2n - 4$. Hence $usd_\gamma(K_n^{++-}) = 2n - 4 + 1 = 2n - 3$. □

## 4 Lower and upper bounds of $usd_\gamma(G^{++-})$

In this section, we obtained the bounds for $usd_\gamma(G^{++-})$ for connected graphs and derived some results on disconnected graphs.

**Theorem 6.** *If $G \cong K_n \cup K_{1,r}$ then $usd_\gamma(G^{++-}) = \frac{(n+1)(n-4)}{2}$ for all $n \geq 4$ and $r \geq 3$.*

*Proof.* Let $V(K_{1,r}) = \cup[u, u_1, u_2, \ldots, u_r\}V(K_n) = \{v_1, v_2, \ldots, v_n\}, E(K_{1,r}) = \{e_i = uu_i | 1 \leq i \leq n\}$ and $V(K_n) = \{f_1, f_2, \ldots, f_k\}$ where $k = \frac{n(n-1)}{2}$. Therefore $v_1, v_2, \ldots, v_n, u, u_1, u_2, \ldots, u_r \in V(G^{++-})$. Since $K_{1,r}$ is a star, there is a vertex $u$ which is adjacent to all other vertices

$u_j(1 \leq j \leq r)$ of $K_{1,r}$ and all the edges $f_1, f_2, \ldots, f_k$ of $K_n$ in $G^{++-}$. Moreover since $K_n$ is a complete graph, any vertex $v_i \in K_n$ is adjacent to all other vertices $v_i$ of $K_n$, all the edges $e_i(1 \leq i \leq r)$ of $K_{1,r}$ and some edges $f_j(1 \leq j \leq k)$ of $K_n$ also. Therefore $\gamma(G^{++-}) = 2$ and $\{u, v_i\}, 1 \leq i \leq n$ is a dominating set of $G^{++-}$. In $G^{++-}$, any vertex $v_i(\in K_n)$ is adjacent to $\frac{n(n-1)}{2} - d_{K_n}(v_i) = \frac{n(n-1)}{2} - (n-1) = \frac{(n^2-n-2n+2)}{2} = \frac{(n^2-3n+2)}{2}$ edges of $G$. Also these $\frac{(n^2-3n+2)}{2}$ edges of $G$ are common adjacent vertices of $u$ and $v_i, 1 \leq i \leq n$ in $G^{++-}$. Thus any maximum domination subdivision stable set $S$ of $G^{++-}$ has at most $\frac{(n^2-3n+2)}{2}$ edges. Hence $usd_\gamma(G^{++-}) = \frac{n^2-3n+2}{2} + 1 = \frac{n^2-3n+2+2}{2} = \frac{(n-1)(n+4)}{2}$. □

**Theorem 7.** *If $G \cong K_1 \cup K_n$, then $usd_\gamma(G^{++-}) = \frac{(n-1)(n+4)}{2}$ for all $n \geq 4$.*

*Proof.* Let $V(K_1) = \{v\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Then $v$ is adjacent to all the edges of $G$. Further, $v_i \in V(K_n)$ is adjacent to all other vertices $v_j$ of $K_n$ and hence $\{v, v_i\}$ is a minimum dominating set of $G^{++-}$. Further, $v_i$ is adjacent to $\frac{n(n-1)}{2} - (n-1)$ edges of $K_n$ in $G^{++-}$. Hence any domination subdivision stable set $S$ has at most $\frac{n(n-1)}{2} - (n-1)$ edges. Therefore any maximum domination subdivision stable set $S$ has at most $\frac{n(n-1)}{2} - (n-1)$ edges. That is $|S| = \frac{n(n-1)}{2} - (n-1) = \frac{n^2-3n+2}{2}$ and hence $usd_\gamma(G^{++-}) = \frac{n^2-3n+2}{2} + 1 = \frac{(n-1)(n+4)}{2}$. □

**Theorem 8.** *For any connected graph $G$ with $n \geq 3$, $usd_\gamma(G^{++-}) > 1$.*

*Proof.* Suppose $usd_\gamma(G^{++-}) = 1$.
Then there is no maximum domination subdivision stable set. Since $\gamma(G^{++-}) = 2$, let $D = \{x, y\}$ be a minimum dominating set of $G^{++-}$. Then $N_{G^{++-}}[x] \cap N_{G^{++-}}[y] = \emptyset$.
Case(i): $x, y \in V(G)$.
Then $x$ and $y$ have no common adjacent vertices of $G^{++-}$. That is $x$ and $y$ have no common adjacent vertices of $G$ and have no common non-incident edges of $G$ in $G^{++-}$. Since $D$ is a minimum dominating set of $G^{++-}$, there is no edge between $x$ and $y$. Hence $G$ is disconnected. We get a contradiction.
Case(ii): $x, y \in E(G)$.
Then $x$ and $y$ are non-adjacent edges of $G$. In $G^{++-}$, each edge of $G$ is adjacent to all the vertices except end vertices of $e$. Since $N_{G^{++-}}[x] \cap N_{G^{++-}}[y] = \emptyset$, $G \cong K_2 \cup K_2$ which is disconnected. Hence we get contradiction that $G$ is connected.
Case(iii): $x \in V(G)$ and $y \in E(G)$.
Subcase(i) : $x$ is incident with $y$.
Then $y = xz$, $z \in V(G)$. If $d_G(x) = 1$, then the edges incident with $z$ in $G$ are common adjacent vertices of $x$ and $y$ in $G^{++-}$. If $d_G(z) = 1$, then the edges incident with $x$ in $G$ are common adjacent vertices of $x$ and $y$ in $G^{++-}$. Hence we get a contradiction to $N_{G^{++-}}[x] \cap N_{G^{++-}}[y] = \emptyset$.
Subcase(ii): $x$ is not incident with $y$.
Then $x$ and $y$ are adjacent in $G^{++-}$ and both are in $D$. Hence $N_{G^{++-}}[x] \cap N_{G^{++-}}[y] \neq \emptyset$. Which is a contradiction. Hence $usd_\gamma(G^{++-}) \neq 1$ for connected graph with $n \geq 3$. □

**Theorem 9.** *For any connected graph $G$ with $n \geq 6$, $2 \leq usd_\gamma(G^{++-}) \leq \frac{(n+1)(n-4)}{2}$.*

*Proof.* We have $\gamma(G^{++-}) = 2$. Let $D = \{x, y\}$ be the minimum dominating set of $G^{++-}$. Let $S$ be the maximum domination subdivision stable set of $G^{++-}$.
Case(i) $x, y \in V(G)$.
Then $x$ and $y$ are not incident vertices in $G$ and each vertex in $V(G) \setminus \{x, y\}$ adjacent to $x$ and / or $y$ in $G$. There are atmost $n-2$ vertices which are adjacent to both $x$ and $y$ in $G^{++-}$. There exists atmost $\frac{(n-2)(n-3)}{2}$ edges in $< V(G) \setminus \{x, y\} >$. Then these edges are adjacent to both $x$ and $y$ in $G^{++-}$. Therefore $|S| \leq n-2+\frac{(n-2)(n-3)}{2} = \frac{(n-2)(n-1)}{2}$ Hence $usd_\gamma(G^{++-}) \leq \frac{(n-2)(n-1)}{2} + 1 = \frac{(n+1)(n-4)}{2}$.
Case(ii): $x, y \in E(G)$.
Then $x$ and $y$ are not adjacent in $G$. There are $(n-4)$ vertices which are not incident to both $x$ and $y$ in $G$. Therefore these $n-4$ vertices are adjacent to both $x$ and $y$ in $G^{++-}$. Also each edge other than $x$ and $y$ should adjacent to $x$ and / or $y$ in $G$. From these edges , atmost 4 edges are adjacent to both $x$ and $y$ in $G^{++-}$. Hence $|S| \leq n-4+4 = n$. Therefore $usd_\gamma(G^{++-}) \leq n+1$
Case(iii): $x \in V(G)$ and $y \in E(G)$.
Subcase(i): $x$ is incident with $y$.
Let $y = xz$. Then $x$ is adjacent to atmost $n-2$ vertices other than $z$ in $G$. Then both $x$ and $y$ are adjacent to atmost $n-2$ vertices in $G^{++-}$. Also $z$ is adjacent to atmost $n-2$ vertices except $x$ in $G$. Then there exists atmost $n-2$ edges whose end vertex is $z$ in $G$. Therefore these $n-2$ edges are adjacent to both $x$ and $y$ in $G^{++-}$. Hence $|S| \leq n-2+n-2 = 2(n-2)$. Thus $usd_\gamma(G^{++-}) \leq 2(n-2)+1$.
Subcase(ii): $x$ is not incident with $y$.
Then $x$ and $y$ are adjacent in $G^{++-}$. Let $y = ab$. Then $x$ is adjacent to $a$ and $b$ in $G^{++-}$. Also $x$ is adjacent to atmost $n-3$ vertices except $a$ and $b$ in $G$. Therefore both $x$ and $y$ are adjacent to atmost $n-3$ vertices in $G^{++-}$. There are atmost $n-3$ edges (except $y$ and $ax$) whose end vertex is $a$ and there are atmost $n-3$ edges (except $y$ and $bx$) whose end vertex is $b$ in $G$. That is these $2(n-3)$ edges are adjacent to both $x$ and $y$ in $G^{++-}$. Hence $|S| \leq 2(n-3)$. Thus $usd_\gamma(G^{++-}) \leq 2(n-3)+1$.
Case(iv): $x \in E(G)$ and $y \in V(G)$.
This case is similar to case (iii). Hence $usd_\gamma(G^{++-}) \leq 2(n-3)+1$. From all the above three cases, $usd_\gamma(G^{++-}) \leq \frac{(n+1)(n-4)}{2}$. By Theorem 8, $usd_\gamma(G^{++-}) > 1$. Thus $2 \leq usd_\gamma(G^{++-}) \leq \frac{(n+1)(n-4)}{2}$. □

## 5 $sd_\gamma$-critical graphs

In this section we characterize $sd_\gamma$-critical graphs and discuss some of its properties. Also we derive some important results on disconnected graphs.

**Theorem 10.** *If $G \cong K_1 \cup H$, where $H = K_{1,r_1} \cup K_{1,r_2} \cup \ldots \cup K_{1,r_t}$, then $usd_\gamma(G^{++-}) = 1$ .*

*Proof.* Let us consider a graph $G$ and $G \cong K_1 \cup H$, where $H = K_{1,r_1} \cup K_{1,r_2} \cup \ldots \cup K_{1,r_t}$. Let $x \in V(K_1)$. Since $x$ is an isolated vertex of $G$, $x$ is adjacent to all the edges of $G$ in $G^{++-}$. Since each component is a star, for each component there exists exactly one vertex $y_i$ (say) which is adjacent to all other vertices of that component. Therefore $\gamma(G^{++-}) = t+1$ and no two vertices in the dominating set has common adjacent vertices of $G^{++-}$. Hence there exists no domination subdivision stable set. Thus $usd_\gamma(G^{++-}) = 1$. □

**Theorem 11.** *Let $H$ be any graph. If $G \cong K_2 \cup H$ then $usd_\gamma(G^{++-}) = 1$.*

*Proof.* Let $v_1, v_2 \in V(K_2)$ and $e_1 \in E(K_2)$. Then $e_1$ is adjacent to all the vertices of H in $G^{++-}$. Also $v_1$ and $v_2$ are adjacent to all the edges of $H$ in $G^{++-}$. Thus a minimum dominating set $D$ of $G^{++-}$ is either $\{v_1, e_1\}$ or $\{v_2, e_1\}$. For both the pairs $\{v_1, e_1\}$ and $\{v_2, e_1\}$, there is no common adjacent vertices in $G^{++-}$. Therefore there is no domination subdivision stable set. Hence $usd_\gamma(G^{++-}) = 1$. □

**Theorem 12.** *For any graph $G$, if $G \cong K_{1,r} \cup K_{1,s}$, then $usd_\gamma(G^{++-}) = 1$.*

*Proof.* Let us consider a graph $G$ and $G \cong K_{1,r} \cup K_{1,s}$. Let $K_{1,r}$ and $K_{1,s}$ be star graphs on $r+1$ and $s+1$ vertices respectively. Let $v, v_1, v_2, \ldots, v_r \in V(K_{1,r}), u_1, u_2, \ldots, u_s) \in V(K_{1,s})$, $e_1, e_2, \ldots, e_r \in E(K_{1,r})$ and $f_1, f_2, \ldots, f_s \in E(K_{1,s})$. Then $\gamma(G^{++-}) = 2$, because the minimum dominating set $D$ of $G^{++-}$ is $D = \{u, v\}$. Here the vertex $u$ is adjacent to $u_1, u_2, \ldots, u_s$ and $e_1, e_2, \ldots, e_r$. Similarly the vertex $v$ is adjacent to $v_1, v_2, \ldots v_r$ and $f_1, f_2, \ldots f_s$. Hence there is no common adjacent vertex of $u$ and $v$ in $G^{++-}$. Thus domination subdivision stable set does not exist. Hence $usd_\gamma(G^{++-}) = 1$. □

**Theorem 13.** *Any graph $G$ is $sd_\gamma$-critical if and only if $G$ is any one of the following:*

1. *$G \cong K_2 \cup H$, where $H$ is any graph.*
2. *$G \cong K_{1,r} \cup K_{1,s}$.*
3. *$G \cong K_1 \cup K_{1,r_1} \cup K_{1,r_2} \cup \ldots \cup K_{1,r_t}$.*

*Proof.* Assume that $G^{++-}$ is $sd_\gamma-$critical graph. Then $usd_\gamma(G^{++-}) = 1$. Suppose $G$ is not isomorphic to the graphs given in (i) , (ii) or (iii). Then we consider the following cases.
Case(i) : $G$ is connected.
By Theorem 9, $usd_\gamma(G^{++-}) \geq 2$, which contradicts our assumption.
Case(ii): $G$ has exactly two components.
Let $G \cong G_1 \cup G_2$. Since $G \ncong K_2 \cup H$, $G \ncong K_{1,r} \cup K_{1,s}$ and $G \ncong K_1 \cup K_{1,r}$, the following subcases are discussed.
Subcase(i) : $G$ has $K_1$.
Let $G \cong K_1$ and $v \in V(G_1)$. Then $G_2 \ncong K_{1,r}, r \geq 2$. Therefore there exists two non-adjacent edges $e_1$ and $e_2$ in $G_2$. Since $v$ is an isolated vertex in $G$, $v$ belongs to all the dominating sets of $G^{++-}$. Also $v$ is adjacent to all the edges of $G$ and adjacent to none of the vertices of $G_2$. If $u \in V(G_2)$ belongs to any dominating set of $G^{++-}$, then there exists an edge of $G_2$ that is adjacent to both $u$ and $v$ in $G^{++-}$. Hence $usd_\gamma(G^{++-}) \neq 1$. If $e \in V(G_2)$ belongs to any dominating set of $G^{++-}$, then $v$ and $e$ are adjacent in $G^{++-}$. Hence $usd_\gamma(G^{++-}) \neq 1$.
Subcase(ii): $G$ does not contain $K_2$.
Since $G \ncong K_2 \cup H$, each component of $G$ has atleast three vertices. Since $G \ncong K_{1,r} \cup K_{1,s}$, each component of $G$ contains atleast two non-adjacent edges. Let $D = \{x, y\}$ be a minimum dominating set of $G^{++-}$. If $x, y \in V(G)$, then $x \in V_1(G)$ and $y \in V_2(G)$. Since

$G \ncong K_{1,r} \cup K_{1,s}, N_{G^{++-}}[x] \cap N_{G^{++-}}[y] \neq \emptyset$. If $x \in V(G)$ and $y \in E(G)$, then both belong to same component. Since $G$ has atleast two non-adjacent edges $N_{G^{++-}}[x] \cap N_{G^{++-}}[y] \neq \emptyset$. Also there is no possibility that $D$ contains two edges of $G$.
Case(iii): $G$ has more than two components.
If $e \in V(G_i)$, there is an edge in $G_i$ which is adjacent to both $u$ and $v$ in $G^{++-}$. Therefore $N_{G^{++-}}[u] \cap N_{G^{++-}}[v] \neq \emptyset$.
Subcase(i): $G$ contains only one $K_1$.
Then $G_i \ncong K_{1,r}, r \geq 2$. Let $v \in V(K_1)$. Then $v$ is in any minimum dominating set $D$ of $G^{++-}$. If $u \in V(G_i)$, $G_i \ncong K_1$ , there is an edge in $G_i$ which is adjacent to both $u$ and $v$ in $G^{++-}$. Therefore $N_{G^{++-}}[u] \cap N_{G^{++-}}[v] \neq \emptyset$.
Subcase(ii): $G$ contains more than one $K_1$.
Let $G_1 \cong K_1$ and $G_2 \cong K_2$ and let $v \in V(G_1)$, $u \in V(G_2)$. Since $u$ and $v$ are isolated vertices, $u$ and $v$ belong to any minimum dominating set of $G^{++-}$. Both are adjacent to all the edges of $G$ , hence $usd_\gamma(G^{++-}) \neq 1$.
Subcase(iii): $G$ has no $K_1$.
Then every component has atleast two edges since $G \ncong K_2 \cup H$. Let $D = \{x, y\}$ be the minimum dominating set of $G^{++-}$. Then $x \in V(G_i)$ , $y \in E(G_i)$ and $x$ is incident with $y$. Therefore, $N_{G^{++-}}[x] \cap N_{G^{++-}}[y] \neq \emptyset$. Hence, from all the above cases we get $usd_\gamma(G^{++-}) \neq 1$. Therefore $G$ is isomorphic to the graphs given in (i) , (ii) or (iii). Conversely assume that $G$ is any one of the above given graphs. Then by Theorems 11, 12 and 10 the graph $G$ is $sd_\gamma$-critical.

□

## 6 Concluding remarks

In this paper, we have determined the lower bound and upper bound for $usd_\gamma(G^{++-})$ on connected graph $G$ and the extremal graphs for lower bound is characterized. This study will be extended by characterizing the extremal graphs for upper bound.

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