Journal of Algebra and Related Topics
Vol. 13, No. 02, 2025, pp. 1-13. Research Article

Ordered BCI-algebras, Y-kernels and (ordered) functions

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Abstract. The concept of kernels in ordered BCI-algebras was first introduced by Yang-Roh-Jun. This paper extends the concept to specific kernels, called here Y-kernels. To be more precise, two sorts of Y-kernels related to function were first introduced, and the relations between them and between these Y-kernels and kernels were studied. Next, related to ordered function (and (ordered) homomorphism), the same relations are investigated.

Keywords: Kernel, Y-kernel, Ordered BCI-algebra, Ordered function.
AMS Subject Classification 2010: 03B05, 03G25, 06F35.

1 Introduction

One of the important motivations of universal algebra and universal logic is to investigate the most basic properties of algebra and logic. Following this motivation, many algebraic and logical systems based on more general structures have been introduced. Especially, algebraic and logical systems with underlying partial orders were introduced: Tonoids and partial gaggles [1, 6, 7], implicational tonoids [10], partially ordered algebras [9] and ordered algebras [8] are algebraic examples. Tonoid and partial gaggle logics [1, 6], implicational tonoid, and partial Galois logics [10, 11], weakly implicative logics [2, 4], and implicational logics [3, 5] are logical examples.

Following this research trend, the notion of ordered BCI-algebras was recently introduced by Yang, Roh and Jun in [12]. This notion is a partially ordered generalization of BCI-algebras. In ordered BCI-algebras fiters, subalgebras, and specific filters such as Y-filters, R-filters, and

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Received: 27 January 2024/ Revised: 02 March 2024/ Accepted: 25 May 2024
DOI: 10.22124/JART.2024.26608.1626

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J-filters were also introduced, and related properties were investigated in it. In [13, 14], they further introduced the notion of "kernels" of ordered homomorphisms and ordered functions in ordered BCI-algebras and addressed the relations of kernels to subalgebras and filters. However, they did not introduce specific kernels corresponding to Y-filters, R-filters and J-filters.

We here introduce Y-kernels, which correspond to Y-filters, in ordered BCI-algebras. More precisely, two sorts of Y-kernels related to function are first introduced and their relations and the relations between those Y-kernels and kernels are then dealt with. Similar relations associated with an ordered function (and (ordered) homomorphism) are next addressed.

2 Preliminaries

Definition 1 ([12]). *Let X be a set with a binary operation "$\to$", a constant "e" and a binary relation "$\leq_e$". Then $\boldsymbol{X} := (X, \to, e, \leq_e)$ is called an ordered BCI-algebra (briefly, OBCI-algebra) if it satisfies the following conditions:*

$$(\textit{For all} \quad x, y, z \in X)(e \leq_e (x \to y) \to ((y \to z) \to (x \to z))), \tag{1}$$
$$(\textit{for all} \quad x, y \in X)(e \leq_e x \to ((x \to y) \to y)), \tag{2}$$
$$(\textit{for all} \quad x \in X)(e \leq_e x \to x), \tag{3}$$
$$(\textit{for all} \quad x, y \in X)(e \leq_e x \to y,\ e \leq_e y \to x \ \Rightarrow\ x = y), \tag{4}$$
$$(\textit{for all} \quad x, y \in X)(x \leq_e y \ \Leftrightarrow\ e \leq_e x \to y), \tag{5}$$
$$(\textit{for all} \quad x, y \in X)(e \leq_e x,\ x \leq_e y \ \Rightarrow\ e \leq_e y). \tag{6}$$

Proposition 1 ([12]). *Every OBCI-algebra $\boldsymbol{X} := (X, \to, e, \leq_e)$ satisfies:*

$$(\textit{For all} \quad x \in X)(e \to x = x). \tag{7}$$
$$(\textit{For all} \quad x, y, z \in X)(z \to (y \to x) = y \to (z \to x)). \tag{8}$$
$$(\textit{For all} \quad x, y, z \in X)(e \leq_e x \to y \ \Rightarrow\ e \leq_e (z \to x) \to (z \to y)). \tag{9}$$

For future convenience, $\mathbf{X} := (X, \to, e, \leq_e)$ represents the OBCI-algebra unless otherwise specified.

Definition 2 ([13]). *Let $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ and $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ be OBCI-algebras. A function $f : X \to Y$ is called an ordered function (briefly, O-function) if it satisfies*

$$(\textit{for all} \quad x, y \in X)(e_X \leq_X x \to_X y \ \Rightarrow\ e_Y \leq_Y f(x) \to_Y f(y)); \tag{10}$$

a homomorphism if it satisfies

$$(\textit{for all} \quad x, y \in X)(f(x \to_X y) = f(x) \to_Y f(y)); \tag{11}$$

an ordered homomorphism if it satisfies both (10) and (11).

Proposition 2 ([13]). *Let f be a function from an OBCI-algebra $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to an OBCI-algebra $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$.*

(i) *If f is an O-function, then it satisfies:*

$$e_Y \leq_Y f(e_X) \to_Y f(e_X), \tag{12}$$
$$(\textit{for all} \quad x, y \in X)(x \leq_X y \ \Rightarrow f(x) \leq_Y f(y)). \tag{13}$$

(ii) *If f is a homomorphism, then it satisfies* (12) *and:*

$$e_Y \leq_Y f(e_X). \tag{14}$$

(iii) *If f is an O-homomorphism, then it satisfies* (12), (13) *and* (14).

Definition 3 ([13, 14])**.** *Let $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ and $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ be OBCI-algebras. Given a function $f : X \to Y$, a subset A of X is said to be the kernel of f if it satisfies:*

$$e_X \in A, \tag{15}$$
$$(\textit{for all} \quad x \in X)(x \in A \ \Leftrightarrow \ e_Y \leq_Y f(x)). \tag{16}$$

3 Functions and Y-kernels

For convenience, $\mathbf{X} := (X, \to_X, e_X, \leq_X)$ and $\mathbf{Y} := (Y, \to_Y, e_Y, \leq_Y)$ henceforth denote OBCI-algebras unless otherwise specified. Here we introduce two sorts of Y-kernels related to function and deal with the relations between two sorts of Y-kernels and kernels.

Definition 4. *Let f be a function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$.*

(i) *A subset A of X is said to be a Y1-kernel of f if it satisfies* (15) *and:*

$$(\textit{for all} \quad x, y, z \in X)\left(\begin{array}{c} y \in A,\, (x \to_X y) \to_X z \in A \\ \Rightarrow \ e_Y \leq_Y f(x \to_X z) \end{array}\right). \tag{17}$$

(ii) *A subset A of X is said to be a Y2-kernel of f if it satisfies* (15) *and:*

$$(\textit{for all} \quad x, y, z \in X)\left(\begin{array}{c} y \in A,\, (x \to_X y) \to_X z \in A \\ \Rightarrow \ e_Y \leq_Y f(x) \to_Y f(z) \end{array}\right). \tag{18}$$

Two sorts of Y-kernels related to function in Definition 4 are independent of each other as seen in the following two examples.

Example 1. Let $X := \{1, \frac{2}{3}, \frac{1}{3}, 0\}$ be a set with the binary operation " $\to$ " given by Table 1. Let $\leq_e$ be the natural order in X. Then $\mathbf{X} := (X, \to, \frac{2}{3}, \leq_e)$ is an OBCI-algebra. Define a function f from X to X as follows:

$$f : X \to X, \ q \mapsto \begin{cases} 1 & \text{if } q = 0, \\ \frac{2}{3} & \text{if } q \in \{1, \frac{2}{3}\}, \\ \frac{1}{3} & \text{if } q = \frac{1}{3}. \end{cases} \tag{19}$$

Table 1: Cayley table for the binary operation "→"

$\to$	1	$\frac{2}{3}$	$\frac{1}{3}$	0
1	1	0	0	0
$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0
$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	0
0	1	1	1	1

Take the set $A := \{1, \frac{2}{3}\}$. Clearly A is a Y1-kernel of f. However, it is not a Y2-kernel of f. To verify this, consider $(1 \to \frac{2}{3}) \to \frac{1}{3}$ and $\frac{2}{3}$. Then

$$(1 \to \tfrac{2}{3}) \to \tfrac{1}{3} = 0 \to \tfrac{1}{3} = 1 \in A$$

and $\frac{2}{3} \in A$. Morevoer,

$$\tfrac{2}{3} \leq_e f(1 \to \tfrac{1}{3}) = f(0) = 1.$$

However,

$$\tfrac{2}{3} \not\leq_e f(1) \to f(\tfrac{1}{3}) = \tfrac{2}{3} \to \tfrac{1}{3} = \tfrac{1}{3}.$$

Therefore, A does not form a Y2-kernel of f.

Example 2. Take the OBCI-algebra $\mathbf{X} := (X, \to, \frac{2}{3}, \leq_e)$ introduced in Example 1. We define a function f from X to X as follows:

$$f : X \to X, \ q \mapsto \begin{cases} \frac{2}{3} & \text{if } q \in \{1, \frac{2}{3}, \frac{1}{3}\}, \\ 0 & \text{if } q = 0. \end{cases} \tag{20}$$

Take also the set $A := \{1, \frac{2}{3}\}$. Certainly, A is a Y2-kernel of f. But it is not a Y1-kernel of f. To verify this, consider $(1 \to \frac{2}{3}) \to \frac{1}{3}$ and $\frac{2}{3}$. Then $(1 \to \frac{2}{3}) \to \frac{1}{3} = 1 \in A$. Morevoer,

$$\tfrac{2}{3} \leq_e f(1) \to f(\tfrac{2}{3}) = \tfrac{2}{3} \to \tfrac{2}{3} = \tfrac{2}{3}.$$

However,

$$\tfrac{2}{3} \not\leq_e f(1 \to \tfrac{2}{3}) = f(0) = 0.$$

Therefore, A does not form a Y1-kernel of f.

We then address the relations between two sorts of Y-kernels and kernels. First of all, the following example shows that the kernel of f may not be the two sorts of Y-kernels.

Example 3. Let $X = \{0, 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\}$ be a set with a binary operation " $\to$ " given by Table 2 and let $\leq_e$ be the natural order in X. Then $\mathbf{X} := (X, \to, e, \leq_e)$, where $e = \frac{3}{4}$, is an OBCI-algebra. Let $f : X \to X$ be an automorphim as the identity function. The kernel A of f is the set $\{1, \frac{3}{4}\}$ (see [12]). Then $\frac{3}{4} \in A$ and $(1 \to \frac{3}{4}) \to \frac{3}{4} = 1$. However, $0 = 1 \to \frac{3}{4} \notin A$. Hence, the kernel of f satisfies neither (17) nor (18). Therefore, it is neither a Y1-kernel of f nor a Y2-kernel of f.

Instead Y1-kernels can be kernels as verified in the following theorem.

Table 2: Cayley table for the binary operation "$\to$"

$\to$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
1	1	0	0	0	0
$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	0
$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0
0	1	1	1	1	1

Theorem 1. *For a function f from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$, every Y1-kernel of f is a kernel of f.*

Proof. Let A be a Y1-kernel of f and $x \in A$. Since $(e_X \to_X e_X) \to_X x = x$ by (7), we have $(e_X \to_X e_X) \to_X x \in A$, and so

$$e_Y \leq_Y f(e_X \to_X x) = f(x)$$

by (17). Hence $e_Y \leq_Y f(x)$. Therefore A be a kernel of f. □

Note that Example 2 shows that the Y2-kernel A of f is not the kernel of f since $\frac{2}{3} \not\leq_X f(\frac{2}{3}) = \frac{1}{3}$. Hence, the concepts of Y2-kernel and kernel are independent of each other. We introduce the necessary condition for a Y2-kernel to be a kernel.

Theorem 2. *Let f be a function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ that satisfies $e_Y = f(e_X)$. Every Y2-kernel of f is a kernel of f.*

Proof. Let $A \subseteq X$ be a Y2-kernel of f, $e_Y = f(e_X)$ and $x \in A$. As in the proof of Theorem 1, we have $(e_X \to_X e_X) \to_X x \in A$ and so $e_Y \leq_Y f(e_X) \to_Y f(x)$ by (18). Then $e_Y \leq_Y e_Y \to_Y f(x)$ by $e_Y = f(e_X)$, and so $e_Y \leq_Y f(x)$ by (7). Therefore A be a kernel of f. □

We finally introduce necessary conditions for kernels to be two sorts of Y-kernels.

Theorem 3. *Let f be a function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$.*

(i) *Suppose that $\boldsymbol{X}$ satisfies:*

$$(\textit{For all} \quad x, y, z \in X)(e \leq_e (x \to y) \to z \;\Rightarrow\; e \leq_e x \to (y \to z). \tag{21}$$

If A is a kernel of f and satisfies:

$$(\textit{for all} \quad x \in X)(x \in A \;\Rightarrow\; e \leq_e x), \tag{22}$$
$$(\textit{for all} \quad x, y \in X)(x \leq_e y,\, x \in A \;\Rightarrow\; y \in A), \tag{23}$$

then it is a Y1-kernel of f.

(ii) *Suppose that f satisfies:*

$$(\textit{for all} \quad x, y \in X)(f(x \to_X y) \leq_Y f(x) \to_Y f(y)). \tag{24}$$

If $\boldsymbol{X}$ satisfies (21), and A is a kernel of f and satisfies (22) and (23), then it is a Y2-kernel of f.

Proof. (i) Suppose that $\mathbf{X}$ satisfies (21), and A is a kernel of f and satisfies (22) and (23). Let $x, y, z \in X$ be such that $y \in A$ and $(x \to_X y) \to_X z \in A$. We have $e_X \leq_X y$ and $e_X \leq_X (x \to_X y) \to_X z$ by (22), and so

$$e_X \leq_X x \to_X (y \to_X z) \overset{(8)}{=} y \to_X (x \to_X z)$$

by (21). Then $e_X \leq_X x \to_X z$ by (5) and (6). Since $e_X \in A$, we further obtain $x \to_X z \in A$ by (23). Hence,

$$e_Y \leq_Y f(x \to_X z)$$

by (16). Therefore A is a Y1-kernel of f.

(ii) Suppose that f satisfies (24), $\mathbf{X}$ satisfies (21), and A is a kernel of f and satisfies (22) and (23). Let $x, y, z \in X$ be such that $y \in A$ and $(x \to_X y) \to_X z \in A$. Then as in the proof of (i), we have $e_Y \leq_Y f(x \to_X z)$. Since $f(x \to_X y) \leq_Y f(x) \to_Y f(y)$ by (24), we further obtain $e_Y \leq_Y f(x) \to_Y f(z)$ by (6). Therefore A is a Y2-kernel of f. □

Theorem 4. *Let f be a function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$.*

(i) *If A is a kernel of f and satisfies (22), (23) and*

$$(\textit{for all} \quad x, y \in X)(x \in A \ \Rightarrow \ y \to x \in A), \tag{25}$$

then it is a Y1-kernel of f.

(ii) *Suppose that f satisfies (24). If A is a kernel of f and satisfies (22), (23) and (25), then it is a Y2-kernel of f.*

Proof. (i) Suppose that A is a kernel of f and satisfies (22), (23) and (25). Let $x, y, z \in X$ be such that $y \in A$ and $(x \to_X y) \to_X z \in A$. We have $x \to_X y \in A$ by (25), and so $e_X \leq_X x \to_X y$ and $e_X \leq_X (x \to_X y) \to_X z$ by (22). Then $e_X \leq_X z$ by (5) and (6), and so $z \in A$ by (23). Hence, $x \to_X z \in A$ by (25), and so

$$e_Y \leq_Y f(x \to_X z)$$

by (16). Therefore A is a Y1-kernel of f.

(ii) Suppose that f satisfies (24) and A is a kernel of f and satisfies (22), (23) and (25). Let $x, y, z \in X$ be such that $y \in A$ and $(x \to_X y) \to_X z \in A$. Then as in the proof of (i), we have $e_Y \leq_Y f(x \to_X z)$. Since $f(x \to_X y) \leq_Y f(x) \to_Y f(y)$ by (24), we further obtain $e_Y \leq_Y f(x) \to_Y f(z)$ by (6). Therefore A is a Y2-kernel of f. □

As corollaries, we can introduce necessary conditions for a Y1-kernel to be a Y2-kernel and vice versa.

Corollary 1. *Let f be a function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$.*

(i) *Suppose that f satisfies (24) and $\boldsymbol{X}$ satisfies (21). If A is a Y1-kernel of f and satisfies (22) and (23), then it is a Y2-kernel of f.*

(ii) *Suppose that f satisfies $e_Y = f(e_X)$ and $\boldsymbol{X}$ satisfies (21). If A is a Y2-kernel of f and satisfies (22) and (23), then it is a Y1-kernel of f.*

Proof. The claim (i) follows from Theorems 1 and 3(ii). The claim (ii) follows from Theorems 2 and 3(i). □

Corollary 2. *Let f be a function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$.*

(i) *Suppose that f satisfies (24). If A is a Y1-kernel of f and satisfies (22), (23) and (25), then it is a Y2-kernel of f.*

(ii) *Suppose that f satisfies $e_Y = f(e_X)$. If A is a Y2-kernel of f and satisfies (22), (23) and (25), then it is a Y1-kernel of f.*

Proof. The claim (i) follows from Theorems 1 and 4(ii). The claim (ii) follows from Theorems 2 and 4(i). □

4 Ordered functions and Y-kernels

In this section, let f be an O-function from $\mathbf{X} := (X, \to_X, e_X, \leq_X)$ to $\mathbf{Y} := (Y, \to_Y, e_Y, \leq_Y)$ unless otherwise specified. We first introduce the necessary condition for a Y1-kernel to be a Y2-kernel and vice versa. Associated with this, we note that Corollaries 1 and 2 hold in O-function (see Corollary 3 below) and they are provided based on Theorems 1 and 2. Here we deal with the necessary conditions without relying on Theorems 1 and 2. The following example illustrates that Y1-kernels need not satisfy the assertion

$$\text{(for all} \quad x \in X)(e_Y \leq_Y f(x) \ \Rightarrow \ e_X \leq_X x). \tag{26}$$

Example 4. Let $X := \{1, \frac{2}{3}, \frac{1}{3}, 0\}$ be a set with the binary operation " $\to$ " given by Table 1, and $\leq_e$ be the natural order in X. Then $\mathbf{X} := (X, \to, \frac{2}{3}, \leq_e)$ is an OBCI-algebra. Define a function f from X to X as follows:

$$f : X \to X, \ q \mapsto \begin{cases} 1 & \text{if } q = 1, \\ \frac{2}{3} & \text{if } q \in \{\frac{2}{3}, \frac{1}{3}\}, \\ 0 & \text{if } q = 0. \end{cases} \tag{27}$$

Clearly f is an O-function and the set $A := \{1, \frac{2}{3}, \frac{1}{3}\}$ is a Y1-kernel of f. However, it does not satisfy (26) since $\frac{2}{3} \leq_e f(\frac{1}{3})$ but $\frac{2}{3} \not\leq_e \frac{1}{3}$.

We can introduce the assertion (26) as the necessary condition to derive a Y2-kernel from a Y1-kernel.

Theorem 5. *Let f be an O-function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$ satisfy (26). If A is a Y1-kernel of f, then it is a Y2-kernel of f.*

Proof. Let f satisfy (26) and A be a Y1-kernel of f. Let $x, y, z \in X$ be such that $(x \to_X y) \to_X z \in A$ and $y \in A$. Then $e_Y \leq_Y f(x \to_X z)$ by (17) and so $e_X \leq_X x \to_X z$ by (26). Then it follows from (10) that $e_Y \leq_Y f(x) \to_Y f(z)$. Therefore A is a Y2-kernel of f. □

Note that the function f defined in Example 2 is order-preserving and so it is an O-function. This function does not satisfy the assertion

$$(\text{for all} \quad x, y \in X)(f(x) \leq_Y f(y) \;\Rightarrow\; x \leq_X y) \tag{28}$$

since $\frac{1}{3} = f(\frac{2}{3}) \leq_Y f(\frac{1}{3}) = \frac{1}{3}$ but $\frac{2}{3} \not\leq_X \frac{1}{3}$. The assertion (28) and $e_Y = f(e_X)$ are the necessary conditions to derive a Y1-kernel from a Y2-kernel.

Theorem 6. *Let f be an O-function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$, and satisfy (28) and $e_Y = f(e_X)$. If A is a Y2-kernel of f, then it is a Y1-kernel of f.*

Proof. Let f satisfy (28) and $e_Y = f(e_X)$ and A be a Y2-kernel of f. Let $(x \to_X y) \to_X z \in A$ and $y \in A$. Then $e_Y \leq_Y f(x) \to_Y f(z)$ by (18) and so $f(x) \leq_Y f(z)$ by (5). Thus $x \leq_X z$ by (28), and so $e_X \leq_X x \to_X z$ by (5). Then we have

$$e_X \leq_X x \to_X z = e_X \to_X (x \to_X z)$$

by (7), and so $e_Y \leq_Y f(e_X) \to_Y f(x \to_X z)$ by (10). Hence

$$\begin{aligned} e_Y &\leq_Y f(e_X) \to_Y f(x \to_X z) \\ &\overset{e_Y = f(e_X)}{=} e_Y \to_Y f(x \to_X z) \\ &\overset{(7)}{=} f(x \to_X z). \end{aligned}$$

Therefore A is a Y1-kernel of f. □

Now we investigate relations between two sorts of Y-kernels and kernels. First we have the following as a corollary of Theorems 1 and 2.

Corollary 3. *Let f be an O-function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$.*

(i) *Every Y1-kernel of f is a kernel of f.*

(ii) *Suppose $e_Y = f(e_X)$. Every Y2-kernel of f is a kernel of f.*

We then introduce several necessary conditions to derive two sorts of Y-kernels from kernels.

Theorem 7. *Let f be an O-function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$.*

(i) *If the kernel A of f satisfies (22) and*

$$(\textit{for all} \quad x, y \in X)((y \to e) \to x \in A \;\Rightarrow\; y \to x \in A), \tag{29}$$

then it is both a Y1-kernel and a Y2-kernel of f.

(ii) *If the kernel A of f satisfies (22) and*

$$(\textit{for all} \quad x, y, z \in X)((x \to y) \to z \in A \;\Rightarrow\; x \to (y \to z) \in A), \tag{30}$$

then it is both a Y1-kernel and a Y2-kernel of f.

(iii) *If the kernel of f satisfies (22) and (25), then it is both a Y1-kernel and a Y2-kernel of f.*

Proof. (i): Let A be the kernel of f and satisfy (22) and (29). Let $x, y, z \in X$ be such that $(x \to_X y) \to_X z \in A$ and $y \in A$. We first show that A is a Y1-kernel of f. Since $e_X \leq_X y \to_X y$ by (3), we have

$$x \to_X e_X \leq_X x \to_X (y \to_X y) \overset{(8)}{=} y \to_X (x \to_X y)$$

by (9) and (5), and

$$y \to_X (x \to_X y) \leq_X ((x \to_X y) \to_X z) \to_X (y \to_X z)$$

by (1) and (5). Then since $x \to_X e_X \leq_X ((x \to_X y) \to_X z) \to_X (y \to_X z)$, we obtain

$$\begin{aligned} e_X &\overset{(5)}{\leq_X} (x \to_X e_X) \to_X (((x \to_X y) \to_X z) \to_X (y \to_X z)) \\ &\overset{(8)}{=} ((x \to_X y) \to_X z) \to_X ((x \to_X e_X) \to_X (y \to_X z)). \end{aligned}$$

Moreover, we get

$$e_Y \leq_Y f((x \to_X y) \to_X z)$$

by (16) and

$$e_Y \leq_Y f((x \to_X y) \to_X z) \to_Y f((x \to_X e_X) \to_X (y \to_X z))$$

by (10). Thus

$$e_Y \leq_Y f((x \to_X e_X) \to_X (y \to_X z))$$

by (5) and (6), and so $(x \to_X e_X) \to_X (y \to_X z) \in A$ by (16). Then $x \to_X (y \to_X z) \in A$ by (29), and so $e_X \leq_X x \to_X (y \to_X z)$ by (22). Thus we have

$$e_X \leq_X x \to_X (y \to_X z) \overset{(8)}{=} y \to_X (x \to_X z)$$

and so $e_Y \leq_Y f(y) \to_Y f(x \to_X z))$ by (10). Note that $e_Y \leq_Y f(y)$ by (16). Hence we get $e_Y \leq_Y f(x \to_X z)$ by (5) and (6). Therefore A is a Y1-kernel of f.

We next show that A is a Y2-kernel of f. As above, one obtains $e_X \leq_X y \to_X (x \to_X z)$. Then since $e_X \leq_X y$ by (22), we further have $e_X \leq_X x \to_X z$ by (5) and (6), and so

$$e_Y \leq_Y f(x) \to_Y f(z)$$

by (10). Therefore A is a Y2-kernel of f.

(ii): Let A be the kernel of f and satisfy (22) and (30). Let $x, y, z \in X$ be such that $(x \to_X y) \to_X z \in A$ and $y \in A$. We first show that A is a Y1-kernel of f. Note that $x \to_X (y \to_X z) \in A$ by (30), and so

$$y \to_X (x \to_X z) \overset{(8)}{=} x \to_X (y \to_X z) \in A.$$

Then $e_X \leq_X y \to_X (x \to_X z)$ by (22), and so $e_Y \leq_Y f(y) \to_Y f(x \to_X z)$ by (10). Notice also that $e_Y \leq_Y f(y)$ by (16). Hence, $e_Y \leq_Y f(x \to_X z)$ by (5) and (6). Therefore A is a Y1-kernel of f.

Next we verify that A is a Y2-kernel of f. As above, one has $e_X \leq_X y \to_X (x \to_X z)$. Then since $e_X \leq_X y$ by (22), we obtain $e_X \leq_X x \to_X z$ by (5) and (6), and so

$$e_Y \leq_Y f(x) \to_Y f(z)$$

by (10). Therefore A is a Y2-kernel of f.

(iii): Let A be the kernel of f and satisfy (22) and (25). Let $x, y, z \in X$ be such that $(x \to_X y) \to_X z \in A$ and $y \in A$. We first show that A is a Y1-kernel of f. Note that $x \to_X y \in A$ by (25) and

$$(x \to_X y) \to_X (x \to_X z) \overset{(8)}{=} x \to ((x \to_X y) \to_X z) \in A$$

by (25). Then

$$e_X \leq_X (x \to_X y) \to_X (x \to_X z)$$

by (22), and so

$$e_Y \leq_Y f(x \to_X y) \to_Y f(x \to_X z)$$

by (10). Thus since $e_Y \leq_Y f(x \to_X y)$ by (16), we get $e_Y \leq_Y f(x \to_X z)$ by (5) and (6). Therefore A is a Y1-kernel of f.

We next verify that A is a Y2-kernel of f. As above, $x \to_X y \in A$ and $(x \to_X y) \to_X (x \to_X z) \in A$. Then $e_X \leq_X x \to_X y$ and

$$e_X \leq_X (x \to_X y) \to_X (x \to_X z)$$

by (22). Hence, $e_X \leq_X x \to_X z$ by (5) and (6), and so

$$e_Y \leq_Y f(x) \to_Y f(z)$$

by (10). Therefore A is Y2-kernel of f. □

As corollaries, by means of the kernel, we can introduce several necessary conditions to derive a Y2-kernel from a Y1-kernel and vice versa.

Corollary 4. *Let f be an O-function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$.*

(i) *If A is a Y1-kernel of f and satisfies* (22) *and* (29)*, then it is a Y2-kernel of f.*

(ii) *If A is a Y1-kernel of f and satisfies* (22) *and* (30)*, then it is a Y2-kernel of f.*

(iii) *If A is a Y1-kernel of f and satisfies* (22) *and* (25)*, then it is a Y2-kernel of f.*

Corollary 5. *Let f be an O-function from $\boldsymbol{X} := (X, \to_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \to_Y, e_Y, \leq_Y)$, and satisfy $e_Y = f(e_X)$.*

(i) *If A is a Y2-kernel of f and satisfies* (22) *and* (29)*, then it is a Y1-kernel of f.*

(ii) *If A is a Y2-kernel of f and satisfies* (22) *and* (30)*, then it is a Y1-kernel of f.*

(iii) *If A is a Y2-kernel of f and satisfies* (22) *and* (25)*, then it is a Y1-kernel of f.*

If f is an (ordered) homomorphism from $\mathbf{X} := (X, \rightarrow_X, e_X, \leq_X)$ to $\mathbf{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$, then for all $x, y \in X$,

$$f(x \rightarrow_X y) = f(x) \rightarrow_Y f(y)$$

by (11). This implies that associated with (ordered) homomorphism f, the notions "Y1-kernel," "Y2-kernel" are equivalent to each other. Thus we call these kernels simply *Y-kernels* in (ordered) homomorphism. It is certain that if f is an (ordered) homomorphism from $\mathbf{X} := (X, \rightarrow_X, e_X, \leq_X)$ to $\mathbf{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$, then every Y-kernel of f is a kernel of f.

We finally address necessary conditions for a kernel to be a Y-kernel in (ordered) homomorphism.

Theorem 8. *Let f be an (ordered) homomorphism from $\boldsymbol{X} := (X, \rightarrow_X, e_X, \leq_X)$ to $\boldsymbol{Y} := (Y, \rightarrow_Y, e_Y, \leq_Y)$.*

(i) *If the kernel of f satisfies (30), then it is a Y-kernel of f.*

(ii) *If the kernel of f satisfies (25), then it is a Y-kernel of f.*

(iii) *If the kernel of f satisfies (29) and (23), then it is a Y-kernel of f.*

Proof. (i): Let the kernel A of f satisfy (30) and $x, y, z \in X$ be such that $(x \rightarrow_X y) \rightarrow_X z \in A$ and $y \in A$. Then

$$y \rightarrow_X (x \rightarrow_X z) \overset{(8)}{=} x \rightarrow_X (y \rightarrow_X z) \in A$$

by (30), and so $e_Y \leq_Y f(y)$ and $e_Y \leq_Y f(y \rightarrow_X (x \rightarrow_X z))$ by (16). Hence,

$$e_Y \leq_Y f(y \rightarrow_X (x \rightarrow_X z)) = f(y) \rightarrow_Y f(x \rightarrow_X z)$$

by (11), and thus $e_Y \leq_Y f(x \rightarrow_X z)$ by (5) and (6). Therefore A is a Y1-kernel of f and so it is a Y2-kernel of f by (11).

(ii): Let the kernel A of f satisfy (25) and $x, y, z \in X$ be such that $(x \rightarrow_X y) \rightarrow_X z \in A$ and $y \in A$. As in the proof of Theorem 7(iii), $x \rightarrow_X y \in A$ and $(x \rightarrow_X y) \rightarrow_X (x \rightarrow_X z) \in A$. Then $e_Y \leq_Y f(x \rightarrow_X y)$ and $e_Y \leq_Y f((x \rightarrow_X y) \rightarrow_X (x \rightarrow_X z))$ by (16). Hence,

$$e_Y \leq_Y f((x \rightarrow_X y) \rightarrow_X (x \rightarrow_X z)) = f(x \rightarrow_X y) \rightarrow_Y f(x \rightarrow_X z)$$

by (11), and thus $e_Y \leq_Y f(x \rightarrow_X z)$ by (5) and (6). Therefore A is a Y1-kernel of f and so it is a Y2-kernel of f by (11).

(iii): Let the kernel A of f satisfy (29) and (23). Let $x, y, z \in X$ be such that $(x \rightarrow_X y) \rightarrow_X z \in A$ and $y \in A$. As in the proof of Theorem 7(i), we have

$$e_X \leq_X ((x \rightarrow_X y) \rightarrow_X z) \rightarrow_X ((x \rightarrow_X e_X) \rightarrow_X (y \rightarrow_X z)).$$

Then

$$(x \rightarrow_X y) \rightarrow_X z \leq_X (x \rightarrow_X e_X) \rightarrow_X (y \rightarrow_X z)$$

by (5), and so $(x \rightarrow e) \rightarrow (y \rightarrow z) \in A$ by (23). Hence,

$$y \rightarrow (x \rightarrow z) \overset{(8)}{=} x \rightarrow (y \rightarrow z) \in A$$

by (29), and so

$$e_Y \leq_Y f(y \rightarrow_X (x \rightarrow_X z)) \overset{(11)}{=} f(y) \rightarrow_Y f(x \rightarrow_X z)$$

by (16). Then since $e_Y \leq_Y f(y)$ by (16), we obtain $e_Y \leq_Y f(x \rightarrow_X z)$ by (5) and (6). Therefore A is a Y1-kernel of f and so it is a Y2-kernel of f by (11). □

5 Conclusion

We introduced two sorts of Y-kernels in ordered BCI-algebras. To be more exact, we first introduced those Y-kernels related to function and investigated relations between them and their relations to kernels. We next studied the same relations associated with an ordered function (and (ordered) homomorphism).

Some problems or future works remain. First of all, we need to extend our study to more specific kernels, such as R-kernels and J-kernels. We have to introduce related notions. Moreover, we need to deal with relations of those Y-kernels to subalgebras and filters.

Acknowledgments

The authors would like to thank the referee for careful reading.

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