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A Zariski-like topology on the 2-prime spectrum of commutative rings

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Abstract. A proper ideal P of a ring R is called *2-prime* if for all $x, y \in R$ such that $xy \in P$, then either $x^2 \in P$ or $y^2 \in P$. In this paper, we introduce a Zariski topology on the set of all 2-prime ideals of commutative rings. We investigate this topology and clarify the interplay between the properties of this topological space and the algebraic properties of the ring under consideration.

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1 Introduction

Throughout the paper, R is a commutative ring with the nonzero element 1_R and the set of all prime ideals of R is denoted by $\mathrm{Spec}(R)$.

The concept of the prime ideal is one of the significant concepts in the field of commutative algebra and algebraic geometry, and therefore it has been extensively studied by many researchers in recent decades. Some researchers have only investigated this concept from an algebraic point of view, and some have also investigated its application aspects in geometry. There are various generalizations of the concept of prime ideals in the articles, which somehow measure how far an ideal is from being prime. Among the existing research about the definitions related to the prime ideal or the generalizations of the prime ideal, we can mention the concepts of strongly prime ideals (see [15]), weakly prime ideals (see [5]), and almost prime ideals (see [9]) and 2-absorbing ideals (see [6]).

Also, some existing research has also extended the concept of prime ideals to modules, and sometimes they have defined a topology similar to the classic Zariski topology with their help. Among these, we can mention articles such as [1–4, 8, 10–14, 16–21]

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The concept of 2-prime ideals was introduced and studied by Beddani and Messirdi in [7] and then it was used in the characterization of valuation rings. Based on [7], a proper ideal P of a ring R is called *2-prime* if for all $x, y \in R$ such that $xy \in P$, then either $x^2 \in P$ or $y^2 \in P$.

Obviously, every prime ideal is 2-prime. However, the opposite is not true. For example, $4\mathbb{Z}$ is a 2-prime ideal of $\mathbb{Z}$ that is not prime. Recall that an element $e \in R$ is said to be *idempotent* if $e^2 = e$. If every element of R is idempotent, then R is called a *Boolean* ring. By this definition, the prime ideals and 2-prime ideals of a Boolean ring R are the same. For more information on 2-prime ideals see also [23].

In this paper, we are going to define a topology similar to the classic Zariski topology on the set of all 2-prime ideals of a commutative ring. We investigate this topology and clarify the interplay between the properties of this topological space, and the algebraic properties of the ring under consideration. In Section 2, we will introduce the notion of 2-Zariski topology and we will investigate the basic properties of this topology. In Lemma 1, we use the notion of 2-prime ideals to introduce a new expression for the radical of an ideal of R. Also, Sections 3, 4 and 5 are devoted to study of some properties of 2-Zariski topology, such as irreducibility, connectedness, and separation axioms, respectively.

2 Basic results

We define the *2-prime spectrum* of R to be the set of all 2-prime ideals of R and denote it by $\mathrm{Spec}_2(R)$. Note that if P is a 2-prime ideal, then $\sqrt{P}$ is a prime ideal. Using this fact, we provide the following definition.

Definition 1. *For a subset T of R we define 2-variety of T as follows:*

$$V_2(T) := \{P \in \mathrm{Spec}_2(R) \mid T \subseteq \sqrt{P}\}.$$

Clearly, for a subset T of R, the variety $V(T)$ of T in the prime spectrum $\mathrm{Spec}(R)$ is a subset of $V_2(T)$. But, in general these two concepts are different (see the following example).

Example 1. Since $\mathbb{Z}$ is a principal ideal domain, by [23, Theorem 2.3] we have $\mathrm{Spec}_2(\mathbb{Z}) = \{p^n\mathbb{Z} \mid p \text{ is a prime number and } n \in \mathbb{N}\} \cup \{0\}$, while $\mathrm{Spec}(\mathbb{Z}) = \{p\mathbb{Z} \mid p \text{ is a prime number}\} \cup \{0\}$. Now, consider $I = 27\mathbb{Z}$. Then $V(I) = \{3\mathbb{Z}\}$, while $V_2(I) = \{3\mathbb{Z}, 9\mathbb{Z}, 27\mathbb{Z}\}$.

Now, we give the following result, which will be used frequently throughout this article.

Theorem 1. *The following statements hold.*

1. *If I denotes the ideal generated by the subset $T \subseteq R$, then $V_2(T) = V_2(I) = V_2(\sqrt{I})$;*
2. *$V_2(R) = \emptyset$ and $V_2(0) = \mathrm{Spec}_2(R)$;*
3. *If $\{J_i\}_{i\in\Lambda}$ is any family of ideals of R, then $\cap_{i\in\Lambda} V_2(J_i) = V_2(\cup_{i\in\Lambda} J_i)$;*
4. *$V_2(I) \cup V_2(J) = V_2(I \cap J) = V_2(IJ)$ for any ideals I, J of R.*

Proof.

1. It is obvious that $V_2(T) = V_2(I)$. Since $I \subseteq \sqrt{I}$, we have $V_2(\sqrt{I}) \subseteq V_2(I)$. Now, let $P \in V_2(I)$. Then $I \subseteq \sqrt{P}$ and $\sqrt{P}$ is a prime ideal. Thus, $\sqrt{I} \subseteq \sqrt{\sqrt{P}} = \sqrt{P}$. Hence $V_2(I) \subseteq V_2(\sqrt{I})$.
2. Let $P \in \mathrm{Spec}_2(R)$. If $P \in V_2(R)$, then by definition we have $R \subseteq \sqrt{P}$. So, $1_R \in P$, a contradiction. Hence, we have $V_2(R) = \emptyset$. The claim $V_2(0) = \mathrm{Spec}_2(R)$ is clear.
3. It is easy to see that $P \in \cap_{i\in\Lambda} V_2(J_i)$ if and only if $J_i \subseteq \sqrt{P}$ for all $i \in \Lambda$ if and only if $\cup_{i\in\Lambda} J_i \subseteq \sqrt{P}$ if and only if $P \in V_2(\cup_{i\in\Lambda} J_i)$. Therefore, $\cap_{i\in\Lambda} V_2(J_i) = V_2(\cup_{i\in\Lambda} J_i)$.
4. By part (i), for any ideals I, J of R we have

   $$V_2(I \cap J) = V_2(\sqrt{I \cap J}) = V_2(\sqrt{IJ}) = V_2(IJ).$$

   Clearly, $I \cap J \subseteq \sqrt{P}$ whenever $I \subseteq \sqrt{P}$ or $J \subseteq \sqrt{P}$. The converse holds since $\sqrt{P}$ is a prime ideal. Therefore, $V_2(I) \cup V_2(J) = V_2(I \cap J)$.

□

Corollary 1. *Let I be an ideal of R. Then $V_2(I) = \emptyset$ if and only if $I = R$.*

Proof. If $I = R$, then $V_2(I) = \emptyset$. Conversely, there exists $P \in \mathrm{Spec}(R)$ such that $I \subseteq P = \sqrt{P}$. Thus, $P \in V_2(I)$ and so $V_2(I) \neq \emptyset$. □

From the above theorem, it is clear that the set $\{V_2(T) \mid T \subseteq R\}$ satisfies all axioms of closed sets for a topology on $\mathrm{Spec}_2(R)$. This topology is said to be a *2-Zariski topology* of R and denoted by $\mathrm{Spec}_2(R)$. Thus, any open set of $\mathrm{Spec}_2(R)$ has the form $X_T^2 = \mathrm{Spec}_2(R) \setminus V_2(T)$ for any subset $T \subseteq R$.

In the following theorem we determine the basis for 2-Zariski topology.

Theorem 2. *For $a \in R$, let $X_a^2 = \{P \in \mathrm{Spec}_2(R) \mid a \notin \sqrt{P}\}$. Then, the collection $\{X_a^2 \mid a \in R\}$ forms a basis for* $\mathrm{Spec}_2(R)$.

Proof. It is easy to see that $X_a^2 = \mathrm{Spec}_2(R) \setminus V_2((a))$. So, X_a^2 is open. Now if $\mathrm{Spec}_2(R) \setminus V_2(T)$ is a general open set, then by Theorem 1 we have

$$\mathrm{Spec}_2(R) \setminus V_2(T) = \mathrm{Spec}_2(R) \setminus V_2(\bigcup_{a\in T}\{a\}) = \mathrm{Spec}_2(R) \setminus \bigcap_{a\in T} V_2(a) = \bigcup_{a\in T} X_a^2.$$

So, we conclude that $\{X_a^2 \mid a \in R\}$ is a basis for $\mathrm{Spec}_2(R)$. □

In the next lemma, we introduce a new expression for radical of an ideal of R.

Lemma 1. *Let I be an ideal of R. Then, $\sqrt{I} = \bigcap_{P\in V_2(I)} \sqrt{P}$.*

Proof. Since $V(I) \subseteq V_2(I)$, we have

$$\bigcap_{P\in V_2(I)} \sqrt{P} \subseteq \bigcap_{P\in V(I)} \sqrt{P} = \sqrt{I}.$$

Conversely, let $x \in \sqrt{I}$ and $P \in V_2(I)$. Then, $x \in \sqrt{I} \subseteq \sqrt{\sqrt{P}} = \sqrt{P}$. So, $x \in \bigcap_{P\in V_2(I)} \sqrt{P}$. Therefore, $\sqrt{I} \subseteq \bigcap_{P\in V_2(I)} \sqrt{P}$. □

Corollary 2. *Let* $\mathrm{nil}(R)$ *denotes the set of all nilpotent elements of* R. *Then,* $\mathrm{nil}(R) = \bigcap_{P \in \mathrm{Spec}_2(R)} \sqrt{P}$.

Proof. Set $I = 0$ and apply Lemma 1. □

Proposition 1. *Let* $a, b \in R$. *The elements of the basis of* $\mathrm{Spec}_2(R)$ *satisfy the following properties:*

1. $X_a^2 \cap X_b^2 = X_{ab}^2$;
2. $X_a^2 = \emptyset$ *if and only if* a *is a nilpotent element of* R.
3. $X_a^2 = \mathrm{Spec}_2(R)$ *if and only if* a *is a unit element of* R.
4. $X_a^2 = X_b^2$ *if and only if* $\sqrt{(a)} = \sqrt{(b)}$.

Proof. 1.

$$\begin{aligned} X_{ab}^2 &= \mathrm{Spec}_2(R) \setminus V_2(ab) = \mathrm{Spec}_2(R) \setminus V_2((a)(b)) \\ &= \mathrm{Spec}_2(R) \setminus (V_2((a)) \cup V_2((b))) \\ &= (\mathrm{Spec}_2(R) \setminus V_2((a))) \cap (\mathrm{Spec}_2(R) \setminus V_2((b))) \\ &= X_a^2 \cap X_b^2. \end{aligned}$$

2. $X_a^2 = \emptyset$ precisely when $a \in \sqrt{P}$ for every 2-prime ideal of R. This occurs precisely when $a \in \bigcap_{P \in \mathrm{Spec}_2(R)} \sqrt{P}$ and by Corollary 2 precisely when a is nilpotent.

3. If a is a unit element of R, then there is no 2-prime ideal P of R such that $a \in \sqrt{P}$ and so $X_a^2 = \mathrm{Spec}_2(R)$. If a is nonunit, then (a) is contained in some maximal ideal, and hence $V_2((a)) \neq \emptyset$. Therefore, $X_a^2 \neq \mathrm{Spec}_2(R)$.

4. Suppose that $X_a^2 = X_b^2$. Then $V_2((a)) = V_2((b))$ and so $\sqrt{(a)} = \sqrt{(b)}$ by Lemma 1. On the other hand, if $\sqrt{(a)} = \sqrt{(b)}$, then $V_2((a)) = V_2(\sqrt{(a)}) = V_2(\sqrt{(b)}) = V_2((b))$. Thus, $X_a^2 = X_b^2$.

□

Proposition 2. $\mathrm{Spec}_2(R)$ *is a compact topological space.*

Proof. Let $\mathrm{Spec}_2(R) = \bigcup_{i \in \Lambda} X_{a_i}^2$ where $a_i \in R$, be an open cover of $\mathrm{Spec}_2(R)$. Then we obtain that

$$\begin{aligned} \mathrm{Spec}_2(R) &= \bigcup_{i \in \Lambda} (\mathrm{Spec}_2(R) \setminus V_2(a_i)) \\ &= \mathrm{Spec}_2(R) \setminus (\bigcap_{i \in \Lambda} V_2(a_i)) \\ &= \mathrm{Spec}_2(R) \setminus V_2(\bigcup_{i \in \Lambda} \{a_i\}_{i \in \Lambda}). \end{aligned}$$

Assume that I is the ideal of R generated by the set $\{a_i\}_{i\in\Lambda}$. Thus, $V_2(I)=\emptyset$ and so $I=R$ by Corollary 1. This means that $1_R=r_1a_{i_1}+\cdots r_na_{i_n}$ where $r_i\in R$ and $i_1,\cdots,i_n\in\Lambda$. Hence, $\emptyset=V_2(R)=V_2(I)=\bigcap_{j=1}^n V_2((a_{i_j}))$. Therefore,

$$\mathrm{Spec}_2(R)=\mathrm{Spec}_2(R)\setminus(\bigcap_{j=1}^n V_2((a_{i_j})))=\bigcup_{j=1}^n(\mathrm{Spec}_2(R)\setminus V_2(a_{i_j}))=\bigcup_{j=1}^n X^2_{a_{i_j}}.$$

Thus, $\mathrm{Spec}_2(R)$ is the union of finitely many $X^2_{a_i}$ and so is compact by [22, Definition, p.164]. □

3 Irreducibility

In the sequel, the closure of a subset T of $\mathrm{Spec}_2(R)$ is denoted by $\overline{T}$.

Lemma 2. *Let $Y\subseteq\mathrm{Spec}_2(R)$. Then $\overline{Y}=V_2(\bigcap_{P\in Y}\sqrt{P})$.*

Proof. Let $Q\in Y\subseteq\mathrm{Spec}_2(R)$. Then $\sqrt{Q}$ is a prime ideal such that $\bigcap_{P\in Y}\sqrt{P}\subseteq\sqrt{Q}$. So, $Q\in V_2(\bigcap_{P\in Y}\sqrt{P})$. Hence, $Y\subseteq V_2(\bigcap_{P\in Y}\sqrt{P})$. This implies that $\overline{Y}\subseteq V_2(\bigcap_{P\in Y}\sqrt{P})$. Now, suppose that $V_2(T)$ is an arbitrary closed subset of $\mathrm{Spec}_2(R)$ such that $Y\subseteq V_2(T)$. Then, for all $P\in Y$ we have $T\subseteq\sqrt{P}$. Hence, $T\subseteq\bigcap_{P\in Y}\sqrt{P}$. This yields that

$$V_2(\bigcap_{P\in Y}\sqrt{P})\subseteq V_2(T).$$

Therefore, $V_2(\bigcap_{P\in Y}\sqrt{P})\subseteq\overline{Y}$. □

Corollary 3. *Let $(0)\in Y\subseteq\mathrm{Spec}_2(R)$. Then Y is dense in $\mathrm{Spec}_2(R)$.*

Proof. For any $P\in\mathrm{Spec}_2(R)$ we have $0\in P$. So, by Lemma 2 and Theorem 1 we obtain that

$$\overline{Y}=V_2(\bigcap_{P\in Y}\sqrt{P})=V_2(\sqrt{(0)})=V_2(0)=\mathrm{Spec}_2(R).$$

□

Now, we characterize the closure $\overline{\{P\}}$ of the singleton $\{P\}$ for all $P\in\mathrm{Spec}_2(R)$.

Proposition 3. *Let $P\in\mathrm{Spec}_2(R)$.*

1. *$\overline{\{P\}}=V_2(P)$;*
2. *Let $Q\in\mathrm{Spec}_2(R)$. Then $\overline{\{P\}}=\overline{\{Q\}}$ if and only if $\sqrt{P}=\sqrt{Q}$.*

Proof.
1. By Lemma 2 and Theorem 1, $\overline{\{P\}}=V_2(\sqrt{P})=V_2(P)$.
2. Let $\overline{\{P\}}=\overline{\{Q\}}$. Then, by part (1) we have $V_2(P)=V_2(Q)$. Therefore, by Lemma 1 we obtain that

$$\sqrt{P}=\bigcap_{\mathfrak{p}\in V_2(P)}\sqrt{\mathfrak{p}}=\bigcap_{\mathfrak{p}\in V_2(Q)}\sqrt{\mathfrak{p}}=\sqrt{Q}.$$

On the other hand, if $\sqrt{P} = \sqrt{Q}$, then

$$\overline{\{P\}} = V_2(\sqrt{P}) = V_2(\sqrt{Q}) = \overline{\{Q\}}.$$

□

Recall that R is said to be a *von Neumann regular* ring if for each $x \in R$, there exists $a \in R$ such that $x = x^2 a$. It is clear that a ring R is a von Neumann regular ring if and only if for each $a \in R$, $(a) = (e)$ for some idempotent $e \in R$ (see [24]).

Lemma 3. *Let e be an idempotent element of R. Then $V_2(Re) = X^2_{1-e}$.*

Proof. Let $P \in V_2(Re)$. Then $P \in \mathrm{Spec}_2(R)$ and $e \in \sqrt{P}$. If $P \in V_2((1-e))$, then $1-e \in \sqrt{P}$ and so $e + (1-e) = 1 \in \sqrt{P}$, a contradiction. Thus, $P \in \mathrm{Spec}_2(R) \setminus V_2((1-e)) = X^2_{1-e}$. Now, suppose that $P \notin V_2((1-e))$. Hence, $1-e \notin \sqrt{P}$. Since $\sqrt{P}$ is a prime ideal of R, it follows from $e(1-e) = e - e^2 = 0 \in \sqrt{P}$ that $e \in \sqrt{P}$. Thus, $P \in V_2(Re)$. This completes the proof. □

Proposition 4. *Let R be a von Neumann regular ring and $a \in R$. Then, the open set X^2_a is also closed in* $\mathrm{Spec}_2(R)$*.*

Proof. Let $a \in R$. By hypotheses, there is an idempotent $e \in R$ such that $(a) = (e)$. Thus, $V_2((a)) = V_2((e))$ and so Lemma 3 implies that $V_2((a)) = X^2_{1-e}$. Hence, $X^2_a = V_2((1-e))$ which completes the proof. □

Recall that a topological space X is said to be *irreducible* if $X \neq \emptyset$ and it cannot be written as a union of two nonempty closed sets, i.e. for any decomposition $X = A \cup B$ with nonempty closed subsets A, B of X, we have $A = X$ or $B = X$.

Proposition 5. *Let $P \in \mathrm{Spec}_2(R)$. Then $V_2(P)$ is an irreducible set in* $\mathrm{Spec}_2(R)$*.*

Proof. Let $V_2(P) = V_2(I) \cup V_2(J)$ where I, J are proper ideals of R. Clearly, $V_2(I) \subseteq V_2(P)$. Also, $P \in V_2(P)$ implies that $P \in V_2(I)$ or $P \in V_2(J)$. Assume that $P \in V_2(I)$ and $Q \in V_2(P)$. So, $I \subseteq \sqrt{P}$ and $P \subseteq \sqrt{Q}$. Thus, $I \subseteq \sqrt{P} \subseteq \sqrt{\sqrt{Q}} = \sqrt{Q}$ yields that $Q \in V_2(I)$ and so $V_2(P) \subseteq V_2(I)$. This completes the proof. □

Proposition 6. *Let I be an ideal of R such that $V_2(I)$ is irreducible. Then $\sqrt{I}$ is a prime ideal.*

Proof. Let $a, b \in R$ such that $ab \in \sqrt{I}$, but $a \notin \sqrt{I}$ and $b \notin \sqrt{I}$. So, by Lemma 1, there is $P, Q \in V_2(I)$ such that $a \notin \sqrt{P}$ and $b \notin \sqrt{Q}$. Thus, $P \in V_2(I) \cap X^2_a$ and $Q \in V_2(I) \cap X^2_b$. Hence, $V_2(I) \cap X^2_a$ and $V_2(I) \cap X^2_b$ are non-empty open subsets in $V_2(I)$ with respect to subspace topology. Since $V_2(I)$ is irreducible, we obtain that $(V_2(I) \cap X^2_a) \cap (V_2(I) \cap X^2_b) \neq \emptyset$. Let $P' \in V_2(I) \cap X^2_a \cap X^2_b$. Then $P' \in X^2_{ab} \cap V_2(\sqrt{I})$. This implies that $ab \notin \sqrt{P'}$ and $\sqrt{I} \subseteq \sqrt{P'}$. Therefore, $ab \notin \sqrt{I}$, a contradiction. Note that since $V_2(\sqrt{I}) = V_2(I) \neq \emptyset$, by Corollary 1 we obtain that $\sqrt{I} \neq R$. Therefore, $\sqrt{I}$ is a prime ideal of R. □

Theorem 3. $\mathrm{Spec}_2(R)$ *is an irreducible space if and only if* $\mathrm{nil}(R)$ *is a prime ideal of R.*

Proof. Let $I = 0$. Then, by Theorem 1 we have $V_2(I) = \mathrm{Spec}_2(R)$ and $\sqrt{I} = \mathrm{nil}(R)$. Therefore, the assertion follows from Propositions 5 and 6. □

Definition 2. *A ring R is said to be* 2-integral domain *if whenever $ab = 0$ for some $a, b \in R$, then either $a^2 = 0$ or $b^2 = 0$. Thus, a ring R is a 2-integral domain if and only if the zero ideal is a 2-prime ideal of R.*

Example 2. Note that every integral domain is a 2-integral domain, but the converse is not true. For example, the ring $\mathbb{Z}_{27}$ is a 2-integral domain which is not an integral domain.

Corollary 4. *Let R be a 2-integral domain. Then, $\mathrm{Spec}_2(R)$ is an irreducible space.*

Proof. Since R is a 2-integral domain, the zero ideal is a 2-prime ideal of R. Thus, $\sqrt{0} = \mathrm{nil}(R)$ is a prime ideal. So, the assertion follows from Theorem 3. □

4 Connectedness

A topological space X is *connected* if it cannot be written as a union of two disjoint proper open sets.

Theorem 4. $\mathrm{Spec}_2(R)$ *is connected if and only if R contains no idempotents $\neq 0$ or 1.*

Proof. ($\Rightarrow$) Let $\mathrm{Spec}_2(R)$ be connected and e be an idempotent of R. We claim that $V_2((e)) \cap V_2((1-e)) = \emptyset$ and $V_2((e)) \cup V_2((1-e)) = \mathrm{Spec}_2(R)$. For do this, consider $P \in V_2((e)) \cap V_2((1-e))$. Then $P \in \mathrm{Spec}_2(R)$ such that $e \in \sqrt{P}$ and $1 - e \in \sqrt{P}$. Thus, $1 \in \sqrt{P}$ which is a contradiction since $\sqrt{P}$ is a prime ideal of R. Also, we have

$$V_2((e)) \cup V_2((1-e)) = V_2((e)(1-e)) = V_2(e(1-e)) = V_2(0) = \mathrm{Spec}_2(R).$$

Thus, $X_e^2 \cap X_{1-e}^2 = \emptyset$ and $X_e^2 \cup X_{1-e}^2 = \mathrm{Spec}_2(R)$. So, by assumption, $X_e^2 = \mathrm{Spec}_2(R)$ or $X_{1-e}^2 = \mathrm{Spec}_2(R)$. This implies that $V_2((e)) = \emptyset$ or $V_2((1-e)) = \emptyset$ and so $R = Re$ or $R = R(1-e)$. If $R = Re$, then there is $r \in R$ such that $1 = re$. Therefore, $1 - e = re(1-e) = 0$ and so $e = 1$. Similarly, if $R = R(1-e)$, then $e = 0$.

($\Leftarrow$) Suppose that U and V are open sets in $\mathrm{Spec}_2(R)$ such that $\mathrm{Spec}_2(R) = U \cup V$ and $U \cap V = \emptyset$. Then there exist ideals I and J of R such that $\mathrm{Spec}_2(R) \setminus U = V_2(I)$, $\mathrm{Spec}_2(R) \setminus V = V_2(J)$, $V_2(I) \cap V_2(J) = \emptyset$ and $V_2(I) \cup V_2(J) = \mathrm{Spec}_2(R)$. Thus, $V_2(I \cup J) = V_2(I + J) = \emptyset$ and $V_2(I \cap J) = \mathrm{Spec}_2(R)$ and so $X_{I \cap J}^2 = \emptyset$. In the light of Corollary 1 and Proposition 1 we obtain that $I + J = R$ and $IJ \subseteq I \cap J \subseteq \mathrm{nil}(R)$. Hence, there exist $a \in I$, $b \in J$ and $n \in \mathbb{N}$ such that $a + b = 1$ and $(ab)^n = 0$. Thus, $1 = (a+b)^n = a^n + b^n + z$ where $z \in I \cap J$. If $(a^n) + (b^n) \neq R$, then $V_2((a^n)) \cap V_2((b^n)) = V_2((a^n) + (b^n)) \neq \emptyset$. So, there is $P \in \mathrm{Spec}_2(R)$ such that $a^n \in \sqrt{P}$ and $b^n \in \sqrt{P}$. This implies that $a, b \in \sqrt{P}$ and so $1 = a + b \in \sqrt{P}$ which is a contradiction. Therefore, $(a^n) + (b^n) = R$. Thus, there exist $x \in (a^n) \subseteq I$ and $y \in (b^n) \subseteq J$ with $x + y = 1$. Now, we conclude that $xy = x(1-x) \in (a^n)(b^n) = 0$ which yields that $x^2 = x$. So by assumption, $x = 1$ or $x = 0$, i.e. $x = 1$ or $y = 1$. Hence, $I = R$ or $J = R$. This implies that $U = \mathrm{Spec}_2(R)$ or $V = \mathrm{Spec}_2(R)$. Therefore, $\mathrm{Spec}_2(R)$ is a connected space. □

Example 3. By Theorem 4, $\mathrm{Spec}_2(\mathbb{Z})$ is a connected topological space.

Corollary 5. *If $\mathrm{Spec}_2(R)$ is a disconnected topological space, then $R \cong R_1 \oplus R_2$ where R_1 and R_2 are nonzero rings.*

Proof. By assumption and Theorem 4, R has an idempotent $e \neq 0, 1$. Define non-zero subrings of R by $R_1 = (e)$ and $R_2 = (1-e)$. Then, for any $x \in R$ we have $x = xe + x(1-e) \in R_1 + R_2$ which implies that $R = R_1 + R_2$. If $y \in R_1 \cap R_2$, then $y = re$ and $y = s(1-e)$ for some $r, s \in R$. But $re = ree = s(1-e)e = 0$, and so $y = 0$. Therefore, $R_1 \cap R_2 = \{0\}$. Therefore, $R \cong R_1 \oplus R_2$ □

5 Separation axioms

A topological space X is called a T_0-space if for each distinct $x, y \in X$, there exists an open set U such that $x \in U$, $y \notin U$ or $y \in U$, $x \notin U$. It is easy to see that a topological space X is a T_0-space if and only if for each $x, y \in X$, $\overline{\{x\}} = \overline{\{y\}}$ implies that $x = y$. It is known that the prime spectrum $\mathrm{Spec}(R)$ of a commutative ring R is always a T_0-space while this rule may not hold for the 2-prime spectrum $\mathrm{Spec}_2(R)$. The following example shows that $\mathrm{Spec}_2(R)$ is not a T_0-space in general.

Example 4. It is known that $\mathrm{Spec}(\mathbb{Z}) = \{p\mathbb{Z} \mid p \text{ is a prime number}\} \cup \{0\}$, while $\mathrm{Spec}_2(\mathbb{Z}) = \{p^n\mathbb{Z} \mid p \text{ is a prime number and } n \in \mathbb{N}\} \cup \{0\}$. Let p be a prime integer. Then $\sqrt{p\mathbb{Z}} = \sqrt{p^2\mathbb{Z}}$. So, $\overline{\{p\mathbb{Z}\}} = \overline{\{p^2\mathbb{Z}\}}$ by Proposition 3, while $p\mathbb{Z} \neq p^2\mathbb{Z}$. Therefore, $\mathrm{Spec}_2(R)$ is not a T_0-space.

Theorem 5. $\mathrm{Spec}_2(R)$ *is a T_0-space if and only if every 2-prime ideal of R is prime.*

Proof. Let $\mathrm{Spec}_2(R)$ is T_0-space and $P \in \mathrm{Spec}_2(R)$. By definition, $\sqrt{P}$ is a prime ideal of R. So, $\sqrt{\sqrt{P}} = \sqrt{P}$. Thus, in the light of Proposition 3 we have $\overline{\{\sqrt{P}\}} = \overline{\{P\}}$. Now by assumption, $P = \sqrt{P}$ which implies that P is a prime ideal of R. Conversely, assume that $P, Q \in \mathrm{Spec}_2(R)$ such that $\overline{\{P\}} = \overline{\{Q\}}$. So, $\sqrt{P} = \sqrt{Q}$ by Proposition 3. Therefore, $P = Q$ since $P, Q \in \mathrm{Spec}(R)$ by assumption. This completes the proof. □

We recall that a topological space X is a T_1-space if for each distinct elements $x, y \in X$ there exist open sets U and V such that $x \in U, y \notin U$ and $x \notin V, y \in V$. Clearly, every T_1-space is a T_0-space and a topological space is a T_1-space if and only if every singleton subset is closed.

Theorem 6. $\mathrm{Spec}_2(R)$ *is a T_1-space if and only if every 2-prime ideal of R is a maximal ideal.*

Proof. Let $\mathrm{Spec}_2(R)$ is T_1-space and $P \in \mathrm{Spec}_2(R)$. Then, by Proposition 3 we obtain that $\{P\} = \overline{\{P\}} = V_2(P)$. Since every T_1-space is a T_0-space, P is a prime ideal by Theorem 5. Let $Q \in \mathrm{Spec}(R)$ such that $P \subseteq Q \subsetneqq R$. Then $P = \sqrt{P} \subseteq \sqrt{Q}$, that is, $Q \in V_2(P) = \{P\}$. Therefore, $Q = P$ which implies that P is a maximal ideal of R. Conversely, suppose that $P \in \mathrm{Spec}_2(R)$ and $Q \in \overline{\{P\}} = V_2(P)$. Thus, $P \subseteq \sqrt{Q}$. Hence, it follows from the assumption that $P = \sqrt{Q} = Q$, that is, $\overline{\{P\}} = \{P\}$. So, $\mathrm{Spec}_2(R)$ is a T_1-space □

A topological space X is called a *Hausdorff* space if for each distinct $x, y \in X$, there exist two disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 7. *If R is a 2-integral domain which is not a field, then* $\mathrm{Spec}_2(R)$ *cannot be Hausdorff.*

Proof. Let $\mathrm{Spec}_2(R)$ is Hausdorf. Then by definition $\mathrm{Spec}_2(R)$ is a T_1-space and so by Theorem 6, every 2-prime ideal of R is a maximal ideal. Therefore, by assumption (0) is a maximal ideal of R. This implies that R is a field, a contradiction. □

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