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Orthogonality in the category of N-complexes

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Abstract. Let $\mathcal{A}$ be an exact category and $\mathbf{C}_N(\mathcal{A})$ be the category of all N-complexes in $\mathcal{A}$. If $\mathbb{X}$ is a sufficiently nice class of objects in $\mathbf{C}_N(\mathcal{A})$, then, we give a characterization of elements in the right orthogonal $\mathbb{X}^\perp$ of $\mathbb{X}$ in $\mathbf{C}_N(\mathcal{A})$ with respect to the induced exact structure.

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1 Introduction

The theory of complexes in a given abelian category $\mathcal{K}$ is a well-studied topic in Homological algebra. Recall that a chain $\mathbf{X} = (X^i, \partial^i_{\mathbf{X}})$ of objects and morphisms in $\mathcal{K}$ is said to be a complex if $\partial^2_{\mathbf{X}} = 0$. As a natural generalization, an N-complex has the same structure as $\mathbf{X} = (X^i, \partial^i_{\mathbf{X}})$ but we have $\partial^N = 0$ instead of $\partial^2 = 0$ where $N \geq 2$ is a fix positive integer. We know that complexes are used to study representations of groups and N-complexes are used to study quantum groups (see [3] and [8]).

Let $N \geq 2$ be a fix positive integer. N-complexes were first introduced by Mayer [13]. He studied this objects as an alternative homology theory on topological spaces. But, his approach did not receive significant attention because it is equivalent to the ordinary homology theory. Later, Kapranov was studied algebraic properties of N-complexes in [10] and Dubois-Violette developed the homological properties of N-complexes in [4] (see also [5]). Recently, N-complexes have been studied by several authors in, [7], [12], [14], [2], [15], [16] [1], ... In particular, cotorsion theories in the category of N-complexes was completely studied by Bahiraei in [1]. It is known that orthogonal pairs are primary tools in cotorsion theories and cotorsion theories have an essential role in the relative homological algebra. In this article, if $\mathcal{A}$ is an exact category and $\mathbf{X}$ is a nice class of N-complexes in $\mathcal{A}$, we characterize elements in the right orthogonal of $\mathbf{X}$ with respect to the induced exact structure.

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Recall from [11] that an additive category $\mathcal{A}$ is said to be exact if it has a class of conflations

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

(f (resp. g) is called an *inflation* (resp. *deflation*)) such that the following axioms hold.

(i) For any object $A \in \mathcal{A}$, the identity morphism 1_A is an inflation.

(ii) For any object $A \in \mathcal{A}$, the identity morphism 1_A is a deflation.

(iii) Deflations (resp. Inflations) are closed under composition.

(iv) The pullback (resp. pushout) of a deflation (resp. inflation) along an arbitrary morphism exists and yields a deflation (resp. inflation).

A class $\mathbb{X}$ of objects in $\mathcal{A}$ is called *coresolving* if for any conflation $X \longrightarrow Y \xrightarrow{u} Z$ with $X, Y \in \mathbb{X}$ then so is Z. The *right orthogonal* of $\mathbb{X}$ in $\mathcal{A}$ is defined by

$$\mathbb{X}^{\perp} := \{B \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(X, B) = 0, \text{ for all } X \in \mathbb{X}\}$$

where $\mathrm{Ext}^1_{\mathcal{A}}(X, B)$ is the group of all equivalence classes of conflations $B \to T \to X$ in $\mathcal{A}$. An object $A \in \mathcal{A}$ has a *special* $\mathbb{X}^{\perp}$*-preenvelope* if there exists a conflation $A \longrightarrow C \longrightarrow X$ such that $C \in \mathbb{X}^{\perp}$ and $X \in \mathbb{X}$. The class $\mathbb{X}^{\perp}$ is called *preenveloping* if any object in $\mathcal{A}$ has a special $\mathbb{X}^{\perp}$-preenvelope.

Throughout this paper, we assume that $\mathcal{A}$ is a fixed exact category. Before starting, let us recall some notations and definitions that we need in the whole of the paper.

1.1 N-Complexes

A cochain

$$\cdots \xrightarrow{\partial^{n-1}_{\mathbf{X}}} X^i \xrightarrow{\partial^{i}_{\mathbf{X}}} X^{i+1} \xrightarrow{\partial^{i+1}_{\mathbf{X}}} \cdots$$

of objects and morphisms in $\mathcal{A}$ is said to be an N*-complex* if composing of any N-consecutive maps gives 0, i.e. $\partial^N = 0$. Let $\mathbf{X} = (X^i, \partial^i)$ and $\mathbf{Y} = (Y^i, \delta^i)$ be N-complexes in $\mathcal{A}$. A morphism $f : \mathbf{X} \to \mathbf{Y}$ of N-complexes is a collection of morphisms $f^i : X^i \to Y^i$ in $\mathcal{A}$ making all the rectangles commute. The category of N-complexes in $\mathcal{A}$ is denoted by $\mathbf{C}_N(\mathcal{A})$. Deflations (resp. inflations), conflations, pullbacks (resp. pushouts) are defined componentwise. So, $\mathbf{C}_N(\mathcal{A})$ admits an exact structure in the sense of [11]. An N-complex $\mathbf{X} = (X^i, \partial^i)$ is called N*-acyclic* if $Z^i_r(\mathbf{X}) = B^i_r(\mathbf{X})$ for all $i \in \mathbb{Z}$ and all $r = 1, 2, \ldots, N-1$ where

$$Z^i_r(\mathbf{X}) := \mathrm{Ker}(\partial^{i+r-1}_{\mathbf{X}} \cdots \partial^{i}_{\mathbf{X}}), \quad B^i_r(\mathbf{X}) := \mathrm{Im}(\partial^{i-1}_{\mathbf{X}} \cdots \partial^{i-r}_{\mathbf{X}}).$$

For a given class $\mathcal{S}$ in $\mathcal{A}$, $\widetilde{\mathcal{S}}_N$ is the class of all N-acyclic N-complexes $\mathbf{S}$ in $\mathcal{S}$ such that for all i and $r = 1, \ldots, N-1$, $Z^i_r(\mathbf{S}) \in \mathcal{S}$. Let M be an object in $\mathcal{A}$, n be an integer and $1 \leq r \leq N$. The N-acyclic N-complex $D^n_r(M)$ is defined by

$$\cdots \longrightarrow 0 \longrightarrow M^{n-r+1} \xrightarrow{\partial^{n-r+1}} \cdots \xrightarrow{\partial^{n-2}} M^{n-2} \xrightarrow{\partial^{n-1}} M^n \longrightarrow 0 \longrightarrow \cdots$$

such that $M^{n-i} = M$ $(0 \leq i \leq r-1)$ and $\partial^{n-i} = 1_M$ $(0 < i \leq r-1)$.

1.2 The homotopy category of N-Complexes

A morphism $f : \mathbf{X} = (X^i, \partial^i_{\mathbf{X}}) \longrightarrow \mathbf{Y} = (Y^i, \partial^i_{\mathbf{Y}})$ of N-complexes is called *null-homotopic* if there exists a set $\{s^i : X^i \to Y^{i-N+1}\}_{i\in\mathbb{Z}}$ such that for any $i \in \mathbb{Z}$,

$$f^i = \sum_{j=0}^{N-1} \partial_{\mathbf{Y}}^{i-1} \cdots \partial_{\mathbf{Y}}^{i-(N-j)} s^{i+j-1} \partial_{\mathbf{X}}^{i+j-2} \cdots \partial_{\mathbf{X}}^{i}.$$

A given morphism $g : \mathbf{X} \to \mathbf{Y}$ of N-complexes is *homotopic* to f, denoted by $g \sim f$, if $g - f$ is null-homotopic. The homotopy catgeory of N-complexes in $\mathcal{A}$, denoted by $\mathbb{K}_N(\mathcal{A})$, is defined as follows. Objects in $\mathbb{K}_N(\mathcal{A})$ are N-complexes in $\mathcal{A}$ and morphisms are maps of N-complexes modulo homotopy equivalence, i.e. for each pair $\mathbf{X}$, $\mathbf{Y}$ of N-complexes,

$$\mathrm{Hom}_{\mathbb{K}_N(\mathcal{A})}(\mathbf{X}, \mathbf{Y}) = \mathrm{Hom}_{\mathbf{C}_N(\mathcal{A})}(\mathbf{X}, \mathbf{Y})/\mathrm{Ht}(\mathbf{X}, \mathbf{Y})$$

where $\mathrm{Ht}(\mathbf{X}, \mathbf{Y})$ is the subgroup of $\mathrm{Hom}_{\mathbf{C}_N(\mathcal{A})}(\mathbf{X}, \mathbf{Y})$ consisting of homotopic morphisms. Recall from [9], the *suspension* functor $\Sigma : \mathbb{K}_N(\mathcal{A}) \longrightarrow \mathbb{K}_N(\mathcal{A})$ is defined as follows. For a given N-complex $\mathbf{Y}$ in $\mathcal{A}$, $\Sigma\mathbf{Y}$ is defined by

$$(\Sigma\mathbf{Y})^m = \bigoplus_{i=m+1}^{m+N-1} \mathbf{Y}^i, \qquad \partial^m_{\Sigma\mathbf{Y}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ -\partial^{\{N-1\}} & -\partial^{\{N-2\}} & \cdots & \cdots & -\partial^2 & -\partial \end{bmatrix}$$

and $\Sigma^{-1}\mathbf{Y}$ is defined by

$$(\Sigma^{-1}\mathbf{Y})^m = \bigoplus_{i=m-N+1}^{m-1} \mathbf{Y}^i, \qquad \partial^m_{\Sigma^{-1}\mathbf{Y}} = \begin{bmatrix} -\partial & 1 & 0 & \cdots & \cdots & 0 \\ -\partial^{\{2\}} & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & 0 \\ -\partial^{\{N-2\}} & 0 & \cdots & \cdots & 0 & 1 \\ -\partial^{\{N-1\}} & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

Let $f : \mathbf{X} = (X^i, \partial^i) \longrightarrow \mathbf{Y} = (Y^i, \delta^i)$ be a morphism of N-complexes in $\mathcal{A}$. The *mapping cone* of f, denoted by $C(f)$, is defined by

$$C(f)^m = Y^m \oplus \coprod_{i=m+1}^{m+N-1} X^i,$$

$$\partial^m_{C(f)} = \begin{bmatrix} \delta & f^{m+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ & 0 & \cdots & & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & -\partial^{\{N-1\}} & -\partial^{\{N-2\}} & \cdots & -\partial^2 & -\partial^{m+N-1} \end{bmatrix}$$

Thus we have a conflation $\mathbf{Y} \to C(f) \to \Sigma\mathbf{X}$ in $\mathbf{C}_N(\mathcal{A})$ and a triangle $\mathbf{X} \to \mathbf{Y} \to C(f) \to \Sigma\mathbf{X}$ in $\mathbb{K}_N(\mathcal{A})$.

2 Right orthogonality

This section is devoted to the orthogonal pairs in the category of N-complexes in $\mathcal{A}$. Recall that, for a given class $\mathbb{X}$ in $\mathbf{C}_N(\mathcal{A})$,

$$\mathbb{X}^{\perp} := \{\mathbf{B} \in \mathbf{C}_N(\mathcal{A}) \mid \mathrm{Ext}^1_{\mathbf{C}_N(\mathcal{A})}(\mathbf{X}, \mathbf{B}) = 0, \text{ for all } \mathbf{X} \in \mathbb{X}\}$$

where $\mathrm{Ext}^1_{\mathbf{C}_N(\mathcal{A})}(\mathbf{X}, \mathbf{B})$ is the group of equivalence classes of conflations $\mathbf{B} \to \mathbf{Z} \to \mathbf{X}$ in $\mathbf{C}_N(\mathcal{A})$. Also, $\mathrm{Ext}^1_{dw}(\mathbf{X}, \mathbf{Y})$ is the subgroup of $\mathrm{Ext}^1_{\mathbf{C}_N(\mathcal{A})}(\mathbf{X}, \mathbf{Y})$ consisting of all conflations which are split in each degree.

Lemma 1. *Let $\mathbf{X}$ and $\mathbf{Y}$ be N-complexes in $\mathcal{A}$. Then, there exists an isomorphism*

$$\mathrm{Ext}^1_{dw}(\mathbf{X}, \mathbf{Y}) \cong \mathrm{Hom}_{\mathcal{K}_N(\mathcal{A})}(\mathbf{X}, \Sigma\mathbf{Y})$$

of abelian groups.

Proof. See ([1, Lemma 2.4]). □

Proposition 1. *Let $f \in \mathrm{Hom}_{\mathbf{C}_N(\mathcal{A})}(\mathbf{X}, \mathbf{Y})$. Then the conflation,canonical exact sequence,*

$$\mathbf{Y} \xrightarrow{\nu} C(f) \xrightarrow{u} \Sigma\mathbf{X}$$

splits if and only if f is nullhomotopic.

Proof. [16, Proposition 2.14] (3). □

Now, we prove the main result of this article. Let $\mathcal{S}$ be a coresolving class of objects in $\mathcal{A}$ such that any $A \in \mathcal{A}$ has a special $\mathcal{S}^{\perp}$-preenvelope, i.e. there exists a conflation $A \longrightarrow S \longrightarrow C$ such that $S \in \mathcal{S}$ and $C \in \mathcal{S}^{\perp}$. Indeed, it is a generalization of the [6, Proposition 3.4] in the category of N-complexes in $\mathcal{A}$.

Theorem 1. *Let $\mathbf{X}$ be an N-complex in $\mathcal{A}$. Then, $\mathbf{X} \in \widetilde{\mathcal{S}}^{\perp}_N$ if and only if*

(i) $\forall i, X^i \in \mathcal{S}^{\perp}$

(ii) $\forall \mathbf{F} \in \widetilde{\mathcal{S}}_N, \mathrm{Hom}_{\mathbb{K}_N(\mathcal{A})}(\mathbf{F}, \mathbf{X}) = 0$

Proof. Let $\mathbf{X} \in \widetilde{\mathcal{S}}_N^{\perp}$. The proof contains two steps.

Step1. For $i \in \mathbb{Z}$ and $N = 3$, consider a special $\mathcal{S}^{\perp}$-preenvelope

$$X^i \xrightarrow{s^i} C^i \longrightarrow F$$

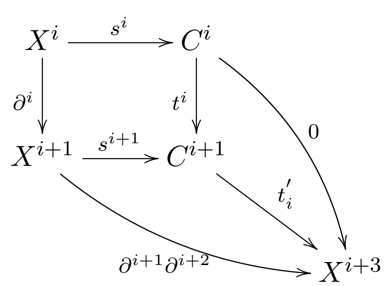
of X^i. Let

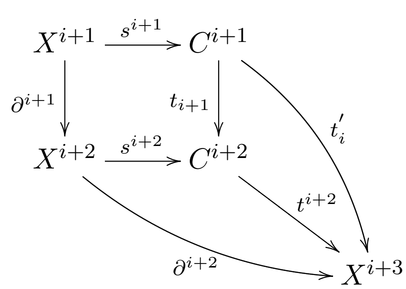
$$\begin{CD} X^i @>{s^i}>> C^i \\ @V{\partial^i}VV @VV{t^i}V \\ X^{i+1} @>{s^{i+1}}>> C^{i+1} \end{CD}$$

be the pushout of s^i and ∂^i and

$$\begin{CD} X^{i+1} @>{s^{i+1}}>> C^{i+1} \\ @V{\partial^{i+1}}VV @VV{t^{i+1}}V \\ X^{i+2} @>{s^{i+2}}>> C^{i+2} \end{CD}$$

be the pushout of s^{i+1} and ∂^{i+1}. By the universal property of pushouts, we have the following commutative diagram

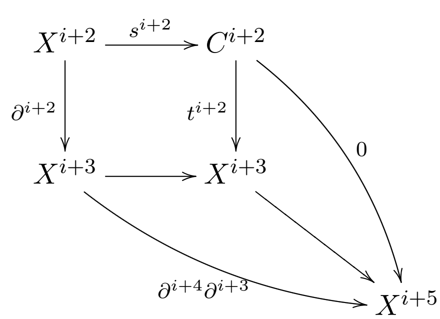




which are induced the following commutative diagrams

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
X^{i-1} & = & X^{i-1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \partial^{i-1}} & & \downarrow{\scriptstyle s^i\circ\partial^{i-1}} & & \downarrow \\
X^{i} & \xrightarrow{s^i} & C^{i} & \longrightarrow & F \\
\downarrow{\scriptstyle \partial^{i}} & & \downarrow{\scriptstyle t^{i}} & & \| \\
X^{i+1} & \xrightarrow{s^{i+1}} & C^{i+1} & \longrightarrow & F \\
\downarrow{\scriptstyle \partial^{i-1}} & & \downarrow{\scriptstyle t^{i+1}} & & \| \\
X^{i+2} & \xrightarrow{s^{i+2}} & C^{i+2} & \longrightarrow & F \\
\downarrow{\scriptstyle \partial^{i+2}} & & \downarrow{\scriptstyle t^{i+2}} & & \downarrow \\
X^{i+3} & = & X^{i+3} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \partial^{i+3}} & & \downarrow{\scriptstyle \partial^{i+3}} & & \downarrow \\
X^{i+4} & = & X^{i+4} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}
\qquad (1)
$$

of cochains in $\mathcal{A}$. The first and the last co-chains are 3-complexes. It is enough to show that the middle co-chain is an 3-complex. Due to the universal property of pushout, according to previous pushout diagrams, we have $t'_i s^{i+1}\partial^i = 0, t'_i t^i s^i = 0, t'_i t_i = 0$,so $t^{i+2}t^{i+1}t^i = 0$ and $\partial^{i+2}\partial i+1 = t'_i s^{i+1} = t^{i+2}t^{i+1}s^{i+1}$, $t^{i+2}s^{i+2}\partial^{i+1} = t^{i+2}t^{i+1}$,so $\partial^{i+3}t^{i+2}t^{i+1} = 0$. Morever, by the universal property of pushouts, we have the following commutative diagram



such that $\partial^{i+4}\partial^{i+3}t^{i+2} = 0$. Now, we deduce the following conflation

$$\mathbf{X} \longrightarrow \mathbf{C} \longrightarrow D_3^{i+2}(F)$$

of 3-complex which is split by assumption. In particular, $X^i \to C^i \to F$ splits. Hence, for each integer i, $X^i \in \mathcal{S}^{\perp}$.

Step 2. Let $\mathbf{X} \in \widetilde{\mathcal{S}}_N^{\perp}$. Then,by the same argument that used in the Step 1, we construct the following pushout diagrams

$$\begin{array}{ccc} X^i & \xrightarrow{s^i} & C^i \\ {\scriptstyle \partial^i}\downarrow & & \downarrow{\scriptstyle t^i} \\ X^{i+1} & \xrightarrow{s^{i+1}} & C^{i+1} \end{array} \quad , \quad \begin{array}{ccc} X^{i+1} & \xrightarrow{s^{i+1}} & C^{i+1} \\ {\scriptstyle \partial^{i+1}}\downarrow & & \downarrow{\scriptstyle t^{i+1}} \\ X^{i+2} & \xrightarrow{s^{i+2}} & C^{i+2} \end{array}$$

$$\vdots$$

$$\begin{array}{ccc} X^{i+N-2} & \xrightarrow{s^{i+N-2}} & C^{i+N-2} \\ {\scriptstyle \partial^{i+N-2}}\downarrow & & \downarrow{\scriptstyle t^{i+N-2}} \\ X^{i+N-1} & \xrightarrow{s^{i+N-1}} & C^{i+N-1} \end{array}$$

where $X^i \longrightarrow C^i \longrightarrow F$ is the special $\mathcal{S}^{\perp}$-preenvelope of X^i. So, we deduce the following commutative diagram

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ X^{i-1} & = & X^{i-1} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \partial^{i-1}} & & \downarrow{\scriptstyle s^i \circ \partial^{i-1}} & & \downarrow \\ X^i & \xrightarrow{s^i} & C_i & \longrightarrow & F \\ \downarrow{\scriptstyle \partial^i} & & \downarrow{\scriptstyle t^i} & & \| \\ X^{i+1} & \xrightarrow{s^{i+1}} & C_{i+1} & \longrightarrow & F \\ \downarrow{\scriptstyle \partial^{i+1}} & & \downarrow{\scriptstyle t^{i+1}} & & \| \\ \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \| \\ X^{i+N-1} & \longrightarrow & C^{i+N-1} & \longrightarrow & F \\ \downarrow{\scriptstyle \partial^{i+N-1}} & & \downarrow{\scriptstyle t^{i+N-1}} & & \downarrow \\ X^{i+N} & = & X^{i+N} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \partial^{i+N}} & & \downarrow & & \downarrow \\ X^{i+N-1} & = & X^{i+N-1} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots \end{array}$$

of cochains in $\mathcal{A}$ where the first and the last cochains are N-complexes $(D_N^{i+N-1}(\mathcal{S}) \in \widetilde{\mathcal{S}})$. By the same argument used in the proof of Step 1 and the universally property of pushouts, the middle sequence is an N-complex. Therefore, we deduce a conflation of N-complexes which is split by assumption. Hence, $X^i \in \mathcal{S}^\perp$. Morever, by [15, Lemma 3.9], the condition (ii) obtained.

Conversely, let $\mathbf{X} \longrightarrow \mathbb{K} \longrightarrow \mathbf{F}$ be a given conflation. By assumption, it is degree-wise pure and so is isomorphic to a conflation $\mathbf{X} \longrightarrow \mathrm{C}(f) \longrightarrow \mathbf{F}$ where f is a morphism in $\mathrm{Hom}_{\mathbb{K}_N(\mathcal{A})}(\Sigma^{-1}\mathbf{F}, \mathbf{X})$. But, by [16, Lemma 2.17],

$$\mathrm{Hom}_{\mathbb{K}_N(\mathcal{A})}(\mathbf{F}, \Sigma\mathbf{X}) \cong \mathrm{Hom}_{\mathbb{K}_N(\mathcal{A})}(\Sigma^{-1}\mathbf{F}, \mathbf{X})$$

where is trivial by assumption by Lemma 1. Consequently, by Lemma 1, the conflation $\mathbf{X} \longrightarrow \mathrm{C}(f) \longrightarrow \mathbf{F}$ splits and hence $\mathrm{Ext}^1_{\mathbf{C}_N(\mathcal{A})}(\mathbf{F}, \mathbf{X}) = 0$. □

Acknowledgments

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