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Some properties of FP-injective modules over group rings

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Abstract. FP-injective modules, which are also called absolutely pure modules, play an important role in characterizing some classical rings such as semihereditary, Noetherian, von-Neumann regular, and coherent rings. These modules have excellent properties over coherent rings similar to injective modules over Noetherian rings. In the present article, we study this class of modules over the group ring $R\Gamma$ of a group Γ, concerning a commutative ring R. We show that if Γ' is a finite index subgroup of Γ, then the restriction of scalars along the natural ring homomorphism $R\Gamma' \to R\Gamma$ and its right adjoint $R\Gamma \otimes_{R\Gamma'} -$ preserve FP-injective modules. We will also examine the properties of FP-injective modules over the group ring of $\mathbf{LH}\mathfrak{F}$-groups. Next, we will switch to the so-called Ding-Chen rings. These rings are coherent versions of Iwanaga-Gorenstein rings, where Noetherian and self-injectivity are replaced by coherence and self-FP-injectivity, respectively. In particular, we have investigated the ascent and descent of the Ding-Chen property between the rings $R\Gamma$ and $R\Gamma'$.

Keywords: Group ring, FP-injective module, Ding-Chen ring.
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1 Introduction

The concept of FP-injective modules was introduced by Stenström in [30]. The exact definition of this class of modules is that if M is a left R-module (R is an associative ring with unity), then M is called FP-injective if $\mathrm{Ext}^1_R(F, M) = 0$, for every finitely presented left R-module F. The philosophy of the emergence of FP-injective modules, as stated in [30], is that to generalize some results of a homological nature, it is sometimes possible to transform Noetherian rings into coherent rings. In this way, finitely generated modules should be replaced with finitely presented modules. It can be said that FP-injective modules over coherent rings play a role

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similar to injective modules over Noetherian rings. It is noted that FP-injective modules were first introduced under the title of absolutely pure modules by Maddox in [24] and his study was continued by other authors such as Megibben [26], Jain [21], Adams [1], Pinzon [28] and also Stenström [30]. As mentioned, FP-injective modules are a generalization of injective modules. Although this is a compelling reason to study this class of modules, a stronger motivation for choosing such modules for research is that they can be used as an efficient tool to describe some classical rings. For example, Megibben shown in [26, Theorem 2] that a ring R is left semihereditary if and only if the homomorphic image of an FP-injective left R-module is FP-injective. He also proved that the necessary and sufficient condition for a ring R to be left Noetherian is that every FP-injective left R-module is injective, see [26, Theorem 3]. Moreover, it was shown in [26, Theorem 5] that the left von-Neumann regularity of the ring R is equivalent to the fact that every left R-module is FP-injective. In addition, Stenström used FP-injective modules in [30] to characterize coherent rings. More precisely, he proved that R is a left coherent ring if and only if every direct limit of FP-injective left R-modules is FP-injective. On the other hand, by a result of Pinzon in [28], if R is a left coherent ring, then the class of FP-injective left modules is (pre)covering. Of course, it has been proved that the converse of Pinzon's result is also true, i.e., R is a left coherent ring if and only if the class of FP-injective left modules is (pre)covering, see [11, Corollary 3.5]. Another valuable point about FP-injective modules is that they can be thought of as a dual version of flat modules because by Proposition 2.6 of [30], a left R-module E is FP-injective if and only if every short exact sequence

$$0 \longrightarrow E \longrightarrow M \longrightarrow N \longrightarrow 0$$

of left R-modules is pure, while a left R-module F is flat if and only if every short exact sequence of left R-modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow F \longrightarrow 0$$

is pure, see for example [23, Theorem 4.85].

In this note, we intend to study FP-injective modules and some of their related properties over group rings. Throughout, R is a commutative ring with unity element, and for a multiplicative group Γ, $R\Gamma$ denotes the group ring formed by Γ with coefficients in R. Because it is well-known that there is no distinction between the class of left and right modules over group rings, henceforth the use of left or right attributes for modules is avoided. For basic properties of group rings, the reader is referred to [7] and [10].

For a subgroup Γ' of Γ, a ring homomorphism $R\Gamma' \longrightarrow R\Gamma$ is induced by the inclusion $\Gamma' \hookrightarrow \Gamma$. One of the problems that one usually encounters while working in the category of $R\Gamma$-modules is related to the transfer of properties of modules along this ring homomorphism. It is pointed out that in solving such problems, the restriction of scalars and the extension of scalars via the functor $R\Gamma \otimes_{R\Gamma'} -$ play an essential role. Inspired by this, in the second section, we first investigate the behavior of recent functors on the class of FP-injective modules, especially when Γ' is a finite index subgroup of Γ. On the other hand, based on a result of Benson and Goodearl, the projective and flat modules over group rings are closely related, see [6, Corollary 4.8.]. Therefore, it is natural to ask whether the dual version of Benson and Goodearl's result holds. Here we show that the answer is positive provided that the ring $R\Gamma$ is coherent. In fact, what we will prove is that if $\Gamma' \leqslant \Gamma$ is a finite- index subgroup and $R\Gamma$ is a coherent ring, then an

FP-injective $R\Gamma$-module that is injective over $R\Gamma'$ is injective over $R\Gamma$, see Theorem 1. Then we will discuss FP-injective modules over group rings of $\mathbf{LH}\mathfrak{F}$-groups. It is reminded that this class of groups, which is a hierarchical system of groups, was introduced by Kropholler in [22], and the hierarchy in its definition is based on the acceptance of certain geometric properties by the groups. This hierarchy, which started with the class of finite groups as the base class, creates an infinite family of infinite groups that has fascinating properties. During the investigation of the properties of the group ring $R\Gamma$, there is a problem related to the ascent and descent of the properties between the ring $R\Gamma$ and R. To see a brief history of solving such problems, which has been of interest for decades, the reader can refer to [20]. A type of ring whose definition is based on FP-injectivity is the Ding-Chen ring, renamed by Gillespie in [18]. Ding-Chen rings are a generalization of Iwanaga-Gorenstein rings where the Noetherian condition is reduced to coherence and the injective dimension is replaced by the FP-injective dimension. In the third section of the article, we will first discuss FP-injective dimensions over group rings. It is shown in [18] that if R is a Ding-Chen ring and Γ is a finite group, then $R\Gamma$ is Ding-Chen. We are interested to know whether $R\Gamma$ being Ding-Chen also implies that R is Ding-Chen. What we will get here is that this ascending and descending problem is generally valid between the rings $R\Gamma'$ and $R\Gamma$, where Γ' is a finite-index subgroup of Γ (Corollary 5).

2 FP-injective modules over group rings

The main theme of this section is the study of FP-injective modules over group rings, but because this class of modules and the class of finitely presented modules are closely related to each other, it is necessary to briefly discuss finitely presented modules over group rings. So we start our work by proving the following lemmas.

Lemma 1. *If Γ' is a subgroup of the group Γ and M is a finitely presented $R\Gamma'$-module, then $R\Gamma \otimes_{R\Gamma'} M$ is a finitely presented $R\Gamma$-module.*

Proof. Since M is a finitely presented $R\Gamma'$-module, there exists an exact sequence

$$\bigoplus^{m} R\Gamma' \longrightarrow \bigoplus^{n} R\Gamma' \longrightarrow M \longrightarrow 0$$

of $R\Gamma'$-modules, where m and n are non-negative integers. By applying the functor $R\Gamma \otimes_{R\Gamma'} -$ on it and using that $R\Gamma$ is $R\Gamma'$-free, the commutative diagram

$$\begin{array}{ccccccc}
R\Gamma \otimes_{R\Gamma'} (\bigoplus^{m} R\Gamma') & \longrightarrow & R\Gamma \otimes_{R\Gamma'} (\bigoplus^{n} R\Gamma') & \longrightarrow & R\Gamma \otimes_{R\Gamma'} M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus^{m}(R\Gamma \otimes_{R\Gamma'} R\Gamma') & \longrightarrow & \bigoplus^{n}(R\Gamma \otimes_{R\Gamma'} R\Gamma') & \longrightarrow & R\Gamma \otimes_{R\Gamma'} M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus^{m} R\Gamma & \longrightarrow & \bigoplus^{n} R\Gamma & \longrightarrow & R\Gamma \otimes_{R\Gamma'} M & \longrightarrow & 0
\end{array}$$

with exact rows is obtained, where the vertical arrows are natural isomorphisms. This shows the lemma. □

The following lemma is obvious.

Lemma 2. *Let Γ' be a subgroup of Γ of finite index. If M is a finitely presented $R\Gamma$-module, then M is finitely presented as an $R\Gamma'$-module.*

Lemma 3. *Let Γ' be a subgroup of the group Γ. If $[\Gamma : \Gamma']$ is finite, then $R\Gamma$ is finitely presented over $R\Gamma'$.*

Proof. When $[\Gamma : \Gamma'] < \infty$, then $R\Gamma$ is a finitely generated $R\Gamma'$-module. Also, $R\Gamma$ is a projective $R\Gamma'$-module. Therefore, by [29, Theorem 3.63], the result is obtained. □

Definition 1. *Suppose Γ is a group and n is a non-negative integer. A finite n-presentation of an $R\Gamma$-module M is an exact sequence of $R\Gamma$-modules*

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

in which each P_i is a finitely generated projective (or free) $R\Gamma$-module. The $R\Gamma$-module M is said to be finitely n-presented (or of type FP_n) if it has a finite n-presentation.

The class of finitely n-presented $R\Gamma$-modules is denoted by $\mathcal{FP}_n(R\Gamma)$ and it can be shown that

$$\mathcal{FP}_0(R\Gamma) \supseteq \mathcal{FP}_1(R\Gamma) \supseteq \cdots \supseteq \mathcal{FP}_n(R\Gamma) \supseteq \cdots \supseteq \mathcal{FP}_\infty(R\Gamma),$$

where $\mathcal{FP}_\infty(R\Gamma) = \bigcap_{n\geqslant 0} \mathcal{FP}_n(R\Gamma)$. According to the above definition, $\mathcal{FP}_0(R\Gamma)$ is the class of finitely generated $R\Gamma$-modules and $\mathcal{FP}_1(R\Gamma)$ is exactly the class of finitely presented $R\Gamma$-modules. Finitely n-presented modules are defined over any associative ring (not necessarily group rings) in [7] and [10] and have been extensively studied in [8] and [9]. The following properties can be proved for these types of modules over group rings similar to Lemmas 1 and 2.

Corollary 1. *Let Γ' be a subgroup of Γ and n be a non-negative integer. The following statements hold.*

(i) If $N \in \mathcal{FP}_n(R\Gamma')$, then $R\Gamma \otimes_{R\Gamma'} N \in \mathcal{FP}_n(R\Gamma)$.

(ii) If the index of Γ' in Γ is finite, then $M \in \mathcal{FP}_n(R\Gamma)$ yields $M \in \mathcal{FP}_n(R\Gamma')$.

Due to the effective role of coherent rings in the study of FP-injective modules, we prove the following proposition. It is recalled that when Γ is an Abelian group, the existence of several statements equivalent to the coherence of the ring $R\Gamma$ has been studied by Glaz in [19]. For example, she proved that, if Γ is a torsion Abelian group, then $R\Gamma$ is coherent if and only if R is a coherent ring, see [19, Theorem 1]. What we will deal with here is the ascent and descent of coherence between the rings $R\Gamma$ and $R\Gamma'$, where Γ' is a finite-index subgroup of the (not necessarily Abelian) group Γ. An immediate consequence of the next proposition is that if Γ is a finite group, then the ring $R\Gamma$ is coherent if and only if the ring R is.

Proposition 1. *If Γ' is a subgroup of Γ of finite index, then $R\Gamma$ is a coherent ring if and only if $R\Gamma'$ is so.*

Proof. To prove the sufficiency, i.e. that the coherence property is descending, we assume that $R\Gamma$ is a coherent ring and choose a member of $\mathcal{FP}_1(R\Gamma')$ such as M.

Because by Lemma 1, $R\Gamma \otimes_{R\Gamma'} M$ belongs to $\mathcal{FP}_1(R\Gamma)$, then by [9, Proposition 2.1], $R\Gamma \otimes_{R\Gamma'} M$ belongs to $\mathcal{FP}_2(R\Gamma)$. Therefore, by Corollary 1, $R\Gamma \otimes_{R\Gamma'} M \in \mathcal{FP}_2(R\Gamma')$. Since M, as an $R\Gamma'$-module, is a direct summand of $R\Gamma \otimes_{R\Gamma'} M$, so $M \in \mathcal{FP}_2(R\Gamma')$ by [9, Proposition 1.7]. Therefore, another use of [9, Proposition 2.1] implies that $R\Gamma'$ is a coherent ring. Conversely, if $R\Gamma'$ is a coherent ring, then by Chase's theorem, $\prod R\Gamma'$ is flat, see [29, Theorem 3.66]. On the other hand, by [23, Proposition 4.44] we have $R\Gamma$-isomorphisms

$$\prod R\Gamma \cong \prod (R\Gamma \otimes_{R\Gamma'} R\Gamma') \cong R\Gamma \otimes_{R\Gamma'} (\prod R\Gamma'),$$

because by Lemma 3, $R\Gamma$ is finitely presented as an $R\Gamma'$-module. Now, since the right side is a flat $R\Gamma$-module, therefore $\prod R\Gamma$ is also a flat $R\Gamma$-module, so by using Chase's theorem again, we can conclude that $R\Gamma$ is a coherent ring, which means that the coherence property is ascending. □

Now it is time to investigate the properties of FP-injective modules over group rings. In the first step, we examine the behavior of this class of modules along the ring homomorphism $R\Gamma' \longrightarrow R\Gamma$, where Γ' is a subgroup of Γ.

Proposition 2. *Suppose Γ' is a subgroup of the group Γ and M is an $R\Gamma$-module. If M is FP-injective as an $R\Gamma$-module, then it is FP-injective as an $R\Gamma'$-module as well.*

Proof. Consider a finitely presented $R\Gamma'$-module N. According to Lemma 1, $R\Gamma \otimes_{R\Gamma'} N$ is a finitely presented $R\Gamma$-module. By the Eckmann-Shapiro Lemma, [5, Corollary 2.8.4], we have

$$\mathrm{Ext}^1_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} N, M) \cong \mathrm{Ext}^1_{R\Gamma'}(N, M).$$

Since M is FP-injective over $R\Gamma$, the left hand side vanished. So $\mathrm{Ext}^1_{R\Gamma'}(N, M) = 0$. Therefore, M is an FP-injective $R\Gamma'$-module. □

Proposition 3. *Let Γ' be a subgroup of Γ of finite index and M be an $R\Gamma'$-module. If M is an FP-injective $R\Gamma'$-module, then the $R\Gamma$-module $R\Gamma \otimes_{R\Gamma'} M$ is FP-injective.*

Proof. If N is a finitely presented $R\Gamma$-module, then it can be concluded from Lemma 2 that N is also finitely presented as an $R\Gamma'$-module. Now since $[\Gamma : \Gamma'] < \infty$, we have $R\Gamma \otimes_{R\Gamma'} M \cong \mathrm{Hom}_{R\Gamma'}(R\Gamma, M)$. Applying [29, Theorem 10.75] besides the assumption, gives the isomorphisms:

$$\begin{aligned} \mathrm{Ext}^1_{R\Gamma}(N, R\Gamma \otimes_{R\Gamma'} M) &\cong \mathrm{Ext}^1_{R\Gamma}(N, \mathrm{Hom}_{R\Gamma'}(R\Gamma, M)) \\ &\cong \mathrm{Ext}^1_{R\Gamma'}(N, M) = 0. \end{aligned}$$

Therefore $R\Gamma \otimes_{R\Gamma'} M$ is an FP-injective $R\Gamma$-module. □

Some properties of FP-injective modules, especially their relationship with injective modules, have been deeply studied in [24], [26], and [30]. What is certain, in general, is that FP-injective modules are not necessarily injective. The next theorem proves that these two classes of modules over group rings are closely related. It is worth mentioning that this theorem is somehow a dual version of Benson and Goodearl's result. Their result states that flat $R\Gamma$-modules are projective provided that they are projective as $R\Gamma'$-modules, where $\Gamma' \leqslant \Gamma$ is a finite-index subgroup, see [6, Corollary 4.8.]. Let M be an $R\Gamma$-module. Recall that FP-id$_{R\Gamma}M$, which indicates the FP-injective dimension of M, is by definition the smallest non-negative integer n such that for every finitely presented $R\Gamma$-module F, $\mathrm{Ext}^{n+1}_{R\Gamma}(F, M) = 0$.

Theorem 1. *Let $\Gamma' \leqslant \Gamma$ be a finite-index subgroup and suppose $R\Gamma$ is coherent. Let M be an FP-injective $R\Gamma$-module. If M is injective over $R\Gamma'$, then it is also injective over $R\Gamma$.*

Proof. Consider the exact sequence

$$0 \longrightarrow K_0 \longrightarrow R\Gamma \otimes_{R\Gamma'} M \xrightarrow{\pi} M \longrightarrow 0$$

of $R\Gamma$-modules where π is the canonical epimorphism. It is known that this sequence is split when considered over $R\Gamma'$. Since M is injective as $R\Gamma'$-module, by [16, Theorem 3.2.9] $R\Gamma \otimes_{R\Gamma'} M \cong \mathrm{Hom}_{R\Gamma'}(R\Gamma, M)$ is injective as an $R\Gamma$-module and hence it is injective as an $R\Gamma'$-module. So K_0 is also an injective $R\Gamma'$-module. By repeating this process for K_0 instead of M, an exact sequence like

$$0 \longrightarrow K_1 \longrightarrow I_1 \longrightarrow K_0 \longrightarrow 0$$

of $R\Gamma$-modules is obtained, where I_1 and K_1 are $R\Gamma$-injective and $R\Gamma'$-injective, respectively. Therefore, it is possible to construct the exact sequence

$$\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow M \longrightarrow 0 \qquad (\dagger)$$

of $R\Gamma$-modules such that the I_i are injective and it splits over $R\Gamma'$. On the other hand, we have the exact sequence of $R\Gamma$-modules

$$0 \longrightarrow M \xrightarrow{\iota} \mathrm{Hom}_{R\Gamma'}(R\Gamma, M) \longrightarrow K_{-2} \longrightarrow 0,$$

where ι is the canonical monomorphism and this sequence is split over $R\Gamma'$. Since $\mathrm{Hom}_{R\Gamma'}(R\Gamma, M)$ is an injective $R\Gamma$-module, K_{-2} is injective when viewed as an $R\Gamma'$-module. Here again, if we repeat the above process for K_{-2} instead of M, we can construct the exact sequence

$$0 \longrightarrow K_{-2} \longrightarrow I_{-2} \longrightarrow K_{-3} \longrightarrow 0$$

of $R\Gamma$-modules such that it splits over $R\Gamma'$ and I_{-2} and K_{-3} are injective over $R\Gamma$ and $R\Gamma'$, respectively. In this way, the exact sequence

$$0 \longrightarrow M \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots \qquad (\dagger\dagger)$$

of $R\Gamma$-modules is obtained, which is split over $R\Gamma'$ and I_i's are injective. Now, if we glue the sequences (†) and (†) together, the exact sequence

$$\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots \qquad (\ddagger)$$

of injective $R\Gamma$-modules is made, which is split over $R\Gamma'$, the cycles are all injective $R\Gamma'$-modules and $M \cong \mathrm{Ker}(I_{-1} \longrightarrow I_{-2})$. We claim that the cycles are FP-injective $R\Gamma$-modules. For this purpose, we assume that for every $i \in \mathbb{Z}$, $K_i = \mathrm{Ker}(I_i \longrightarrow I_{i-1})$ and $M = K_{-1}$. By assumption

$$0 \longrightarrow M \longrightarrow I_{-1} \longrightarrow K_{-2} \longrightarrow 0$$

is a pure exact sequence of $R\Gamma$-modules, see [30, Proposition 2.6]. So for every finitely presented $R\Gamma$-module N,

$$0 \longrightarrow \mathrm{Ext}^1_{R\Gamma}(N, M) \longrightarrow \mathrm{Ext}^1_{R\Gamma}(N, I_{-1}) \longrightarrow \mathrm{Ext}^1_{R\Gamma}(N, K_{-2}) \longrightarrow 0$$

is exact. Hence $\mathrm{Ext}^1_{R\Gamma}(N, K_{-2}) = 0$ and therefore K_{-2} is FP-injective. This shows that for every $i \leq -1$, K_i is FP-injective. Now assume $i \geq 0$ and then consider the exact sequence

$$0 \longrightarrow K_0 \longrightarrow I_0 \longrightarrow M \longrightarrow 0.$$

We prove that for every finitely presented $R\Gamma$-module N, $\mathrm{Ext}^1_{R\Gamma}(N, K_0)$ is vanish. To this end, take the $R\Gamma'$-split exact sequence

$$0 \longrightarrow N \overset{\iota}{\longrightarrow} \mathrm{Hom}_{R\Gamma'}(R\Gamma, N) \longrightarrow L \longrightarrow 0 \qquad (\ddagger\ddagger)$$

of $R\Gamma$-modules. Since N is finitely presented, by Proposition 3, $\mathrm{Hom}_{R\Gamma'}(R\Gamma, N) \cong R\Gamma \otimes_{R\Gamma'} N$ is finitely presented as an $R\Gamma$-module, so according to [23, Lemma 4. 54], the $R\Gamma$-module L is also finitely presented. But by applying the functor $\mathrm{Hom}_{R\Gamma}(-, K_0)$ on the exact sequence $(\ddagger\ddagger)$, the exact sequence

$$\mathrm{Ext}^1_{R\Gamma}(\mathrm{Hom}_{R\Gamma'}(R\Gamma, N), K_0) \longrightarrow \mathrm{Ext}^1_{R\Gamma}(N, K_0) \longrightarrow \mathrm{Ext}^2_{R\Gamma}(L, K_0)$$

is obtained, and because $R\Gamma$ is coherent and $\text{FP-id}_{R\Gamma} K_0 \leq 1$, $\mathrm{Ext}^2_{R\Gamma}(L, K_0) = 0$. On the other hand, by [29, Theorem 10.74] and the injectivity of K_0 as an $R\Gamma'$-module,

$$\begin{aligned} \mathrm{Ext}^1_{R\Gamma}(\mathrm{Hom}_{R\Gamma'}(R\Gamma, N), K_0) &\cong \mathrm{Ext}^1_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} N, K_0) \\ &\cong \mathrm{Ext}^1_{R\Gamma'}(N, K_0) = 0. \end{aligned}$$

Then $\mathrm{Ext}^1_{R\Gamma}(N, K_0) = 0$ and this means that K_0 is an FP-injective $R\Gamma$-module. In the same way, it can be argued that for every $i \geq 0$, all K_i's are FP-injective. So by [30, Proposition 2.6], $(\ddagger)$ is a pure exact sequence of injective $R\Gamma$-modules. In addition, by applying [4, Theorem 5.1], it is concluded that $(\ddagger)$ is also contractible, which means that M is injective as an $R\Gamma$-module. □

Considering the role of groups known as Kropholler's hierarchy in the next theorem, for the convenience of the reader, we first remind the definition of these types of groups from [22]. Let us start with an arbitrary class $\mathfrak{X}$ of groups. We say that the group Γ is in $\mathbf{H}_1\mathfrak{X}$ if Γ acts on a finite-dimensional contractible CW-complex such that the setwise stabilizer of each cell is the same as its pointwise stabilizer and it is a subgroup of Γ, usually called an isotropy group of Γ, and is included in $\mathfrak{X}$. Now, $\mathbf{H}\mathfrak{X}$ is defined by using the transfinite induction. For this, let

$\mathbf{H}_0\mathfrak{X} = \mathfrak{X}$ and if $\beta = \alpha+1$ is a successor ordinal, put $\mathbf{H}_\beta\mathfrak{X} = \mathbf{H}_1(\mathbf{H}_{\alpha+1}\mathfrak{X})$ and set $\mathbf{H}_\beta\mathfrak{X} = \bigcup_{\alpha<\beta} \mathbf{H}_\alpha\mathfrak{X}$, provided that β is a limit ordinal. Finally, define $\mathbf{H}\mathfrak{X} = \bigcup_{\alpha} \mathbf{H}_\alpha\mathfrak{X}$ where the union is taken over all ordinal numbers α. A locally $\mathbf{H}\mathfrak{X}$-group, which is also called $\mathbf{LH}\mathfrak{X}$-group for short, is a group Γ such that every finitely generated subgroup of it is contained in $\mathbf{H}\mathfrak{X}$. It is noted that the (locally) Kropholler's hierarchy arises when $\mathfrak{X}$ is the class of finite groups $\mathfrak{F}$.

The next lemma, which is a special case of Lemma 3.1 of [22], is needed in the process of proving Theorem 2.

Lemma 4. *Suppose that Γ is a group,*

$$0 \longrightarrow X_r \longrightarrow X_{r-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow N \longrightarrow 0$$

is an exact sequence of $R\Gamma$-modules, and M is an $R\Gamma$-module. If i is an integer with $\mathrm{Ext}^i_{R\Gamma}(M,N) \neq 0$, then there exists an integer j, $0 \le j \le r$, such that $\mathrm{Ext}^{i+j}_{R\Gamma}(M, X_j) \neq 0$.

Also, the proof of the following lemma can be seen in Matthews's Ph.D. thesis, see [25, Theorem 1.6.3].

Lemma 5. *Let $\{R_j, j \in I\}$ be a direct system of rings and $R = \varinjlim_j R_j$. If M is a left R-module and N is a right R-module, then $M \otimes_R N = \varinjlim_j (M \otimes_{R_j} N)$, where M and N are right and left R_j-modules, respectively, via the canonical map $\varphi_j : R_j \longrightarrow R$.*

Now we are ready to prove the following theorem.

Theorem 2. *Suppose Γ is a group, $R\Gamma$ is coherent, and N is an $R\Gamma$-module such that for every finite subgroup Γ' of Γ, $R\Gamma \otimes_{R\Gamma'} N$ is an FP-injective $R\Gamma$-module. Then, for every $\mathbf{LH}\mathfrak{F}$-subgroup Λ of Γ, $R\Gamma \otimes_{R\Lambda} N$ is an FP-injective $R\Gamma$-module.*

Proof. To start the proof, we assume $\Lambda \in \mathbf{H}\mathfrak{F}$ and then we use induction on the ordinal number α where $\Lambda \in \mathbf{H}_\alpha\mathfrak{F}$. Since $\mathbf{H}_0\mathfrak{F}$ is the class of finite groups, according to the assumption, it is enough to prove the result in the case where $\Lambda \in \mathbf{H}_\beta\mathfrak{F}$ for $\beta < \alpha$ with $\alpha > 0$. It is known that under these conditions there exists an exact sequence of $R\Lambda$-modules

$$0 \longrightarrow C_r \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow R \longrightarrow 0, \qquad (\star)$$

where every C_i is a direct sum of permutation $R\Lambda$-modules of the form $R[\frac{\Lambda}{\Lambda'}]$, with Λ' being an $\mathbf{H}_\beta\mathfrak{F}$-subgroup of Λ for $\beta < \alpha$. It is clear that when we consider the sequence $(\star)$ as a sequence of R-modules, it is pure. Therefore, by applying the functor $- \otimes_R N$ to it, an exact sequence of $R\Lambda$-modules is obtained, and the result of the application of the functor $R\Gamma \otimes_{R\Lambda} -$ on such a sequence is the exact sequence

$$0 \longrightarrow R\Gamma \otimes_{R\Lambda} (C_r \otimes_R N) \longrightarrow \cdots \longrightarrow R\Gamma \otimes_{R\Lambda} (C_1 \otimes_R N)$$

$$\longrightarrow R\Gamma \otimes_{R\Lambda} (C_0 \otimes_R N) \longrightarrow R\Gamma \otimes_{R\Lambda} N \longrightarrow 0 \qquad (\star\star)$$

of $R\Gamma$-modules. Note that we have the sequence of isomorphisms

$$\begin{aligned} R\Gamma \otimes_{R\Lambda} (C_i \otimes_R N) &\cong R\Gamma \otimes_{R\Lambda} \big(\oplus R[\frac{\Lambda}{\Lambda'}] \otimes_R N\big) \\ &\cong \oplus\Big(R\Gamma \otimes_{R\Lambda} \big(R[\frac{\Lambda}{\Lambda'}] \otimes_R N\big)\Big) \\ &\cong \oplus\Big(R\Gamma \otimes_{R\Lambda} \big((R\Lambda \otimes_{R\Lambda'} R) \otimes_R N\big)\Big) \\ &\cong \oplus\big(R\Gamma \otimes_{R\Lambda'} N\big). \end{aligned}$$

Therefore, taking the finitely presented $R\Gamma$-module M, for each i, we have:

$$\begin{aligned} \operatorname{Ext}^k_{R\Gamma}\Big(M, R\Gamma \otimes_{R\Lambda} \big(C_i \otimes_R N\big)\Big) &\cong \operatorname{Ext}^k_{R\Gamma}\Big(M, \oplus\big(R\Gamma \otimes_{R\Lambda'} N\big)\Big) \\ &\cong \oplus \operatorname{Ext}^k_{R\Gamma}\big(M, R\Gamma \otimes_{R\Lambda'} N\big), \end{aligned}$$

where $k > 0$. But note that by the induction hypothesis, the $R\Gamma$-module $R\Gamma \otimes_{R\Lambda'} N$ is FP-injective. So in light of the above isomorphisms and the coherence of $R\Gamma$, we can conclude that for every i, $\operatorname{Ext}^k_{R\Gamma}\Big(M, R\Gamma \otimes_{R\Lambda} \big(C_i \otimes_R N\big)\Big) = 0$. Now, using Lemma 4 for the exact sequence $(\star\star)$, it follows that $\operatorname{Ext}^1_{R\Gamma}(M, R\Gamma \otimes_{R\Lambda} N)$ becomes zero, which means that $R\Gamma \otimes_{R\Lambda} N$ is FP-injective. To complete the proof, we assume $\Lambda \in \mathbf{LH}\mathfrak{F}$. By [27, Lemma 2.1] there exists a family of finitely generated subgroups $\{\Lambda_s\}$ of Λ ordered by inclusion, such that $\Lambda = \varinjlim_s \Lambda_s$. Since every finitely generated $\mathbf{LH}\mathfrak{F}$-group is an $\mathbf{H}\mathfrak{F}$-group, then based on the first part of the proof, for every s, $\operatorname{Ext}^1_{R\Gamma}(M, R\Gamma \otimes_{R\Lambda_s} N) = 0$. On the other hand, due to the fact that $R\Gamma$ is coherent, M is FP-injective, and that $R\Lambda = \varinjlim_s R\Lambda_s$, Lemma 5 provides the following sequence of isomorphisms

$$\begin{aligned} \operatorname{Ext}^1_{R\Gamma}\big(M, R\Gamma \otimes_{R\Lambda} N\big) &\cong \operatorname{Ext}^1_{R\Gamma}\Big(M, \big(R\Gamma \otimes_{\varinjlim_s R\Lambda_s} N\big)\Big) \\ &\cong \operatorname{Ext}^1_{R\Gamma}\Big(M, \varinjlim_s \big(R\Gamma \otimes_{R\Lambda_s} N\big)\Big) \\ &\cong \varinjlim_s \operatorname{Ext}^1_{R\Gamma}\Big(M, \big(R\Gamma \otimes_{R\Lambda_s} N\big)\Big) \\ &= 0. \end{aligned}$$

Hence $R\Gamma \otimes_{R\Lambda} N$ is FP-injective when considered as an $R\Gamma$-module. □

We end this section with the following corollary which follows immediately from the previous theorem.

Corollary 2. *Assume that Γ is a group in $\mathbf{LH}\mathfrak{F}$ such that $R\Gamma$ is a coherent ring. Let N be an $R\Gamma$-module. If $R\Gamma \otimes_{R\Gamma'} N$ is an FP-injective $R\Gamma$-module for every finite subgroup Γ' of Γ, then N is also an FP-injective module.*

3 Ding-Chen property and group rings

In this section, we are going to examine the Ding-Chen property for group rings. But the reliance of the definition of Ding-Chen rings on the concept of FP-injective dimension convinces us to first discuss these dimensions and some related properties over group rings. In the beginning, we prove the following lemmas.

Lemma 6. *If Γ' is a subgroup of group Γ, then for each $R\Gamma$-module M, $\text{FP-id}_{R\Gamma'}M \leq \text{FP-id}_{R\Gamma}M$.*

Proof. If $\text{FP-id}_{R\Gamma}M = \infty$, then there is nothing to prove. Suppose $\text{FP-id}_{R\Gamma}M = n < \infty$ and then take a finitely presented $R\Gamma'$-module F. Since by Lemma 1, $R\Gamma\otimes_{R\Gamma'}F$ is a finitely presented $R\Gamma$-module, $\text{Ext}^{n+1}_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} F, M) = 0$. On the other hand, by [29, Theorem 10.74]

$$\text{Ext}^{n+1}_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} F, M) \cong \text{Ext}^{n+1}_{R\Gamma'}(F, M),$$

so, $\text{Ext}^{n+1}_{R\Gamma'}(F, M) = 0$ and hence $\text{FP-id}_{R\Gamma'}M \leq n$. □

Lemma 7. *Let Γ' be a finite-index subgroup of the group Γ. Then, for every $R\Gamma'$-module M, $\text{FP-id}_{R\Gamma'}M = \text{FP-id}_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} M)$.*

Proof. At the beginning, we assume that $\text{FP-id}_{R\Gamma'}M = n$ is finite. Let us show that $\text{FP-id}_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} M) \leq n$. For this, consider the finitely presented $R\Gamma$-module F. By Lemma 2, F is also finitely presented as an $R\Gamma'$-module, so $\text{Ext}^{n+1}_{R\Gamma'}(F, M) = 0$. But by [29, Theorem 10.75],

$$\begin{aligned}\text{Ext}^{n+1}_{R\Gamma}(F, R\Gamma \otimes_{R\Gamma'} M) &\cong \text{Ext}^{n+1}_{R\Gamma}(F, \text{Hom}_{R\Gamma'}(R\Gamma, M)) \\ &\cong \text{Ext}^{n+1}_{R\Gamma'}(F, M).\end{aligned}$$

Therefore, $\text{FP-id}_{R\Gamma}(R\Gamma\otimes_{R\Gamma'}M) \leq n$ and this means that the inequality $\text{FP-id}_{R\Gamma}(R\Gamma\otimes_{R\Gamma'}M) \leq \text{FP-id}_{R\Gamma'}M$ holds. To prove the converse of the inequality, we assume $\text{FP-id}_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} M) = n < \infty$. By the previous lemma, we have $\text{FP-id}_{R\Gamma'}(R\Gamma \otimes_{R\Gamma'} M) \leq n$. Now, since M is a direct summand of $R\Gamma \otimes_{R\Gamma'} M$ over $R\Gamma'$, therefore $\text{FP-id}_{R\Gamma'}M \leq n$. □

Recall that the finitistic flat dimension of the group ring $R\Gamma$, denoted $\text{fin.f.dim}(R\Gamma)$, is the supremum of the flat dimensions of all $R\Gamma$-modules that have finite flat dimension, i.e.

$$\text{fin.f.dim}(R\Gamma) = \sup\{\text{fd}_{R\Gamma}M \ \mid \ \text{fd}_{R\Gamma}M < \infty\}.$$

By [15, Proposition 2.4], $\text{fin.f.dim}(R\Gamma) \leq \text{sfli}(R\Gamma)$, where $\text{sfli}(R\Gamma)$ denotes the supremum of the flat lengths of injective $R\Gamma$-modules. It should be noted that $\text{sfli}(R\Gamma)$ was first defined in [12] for every ring (not necessarily group rings) and then it was used in [14] and [2] to study some (co)homological properties of groups and especially properties related to $\text{silp}(R\Gamma)$ and $\text{spli}(R\Gamma)$. The last two invariants, which were introduced by Gedrich and Gruenberg in [17], represent the supremum of the injective lengths of projective $R\Gamma$-modules and the supremum of the projective lengths of injective $R\Gamma$-modules, respectively. Now we put

$$\text{fin.fp.dim}(R\Gamma) = \sup\{\text{FP-id}_{R\Gamma}M \ \mid \ \text{FP-id}_{R\Gamma}M < \infty\}.$$

If $R\Gamma$ is a coherent ring, thanks to [30, Proposition 3.4], the equality $\text{fin.fp.dim}(R\Gamma) = \text{fin.f.dim}(R\Gamma)$ holds. Therefore, the following corollary can be recorded immediately.

Corollary 3. *If $R\Gamma$ is a coherent ring, then* $\text{fin.fp.dim}(R\Gamma) \leq \text{sfli}(R\Gamma)$.

Before stating the next proposition, we remind that the complete injective resolution of an $R\Gamma$-module M is a diagram $M \longrightarrow \mathbf{I} \longrightarrow \mathbf{E}$ in which $\mathbf{I}$ is an injective resolution of M and $\mathbf{E}$ is an acyclic complex of injective $R\Gamma$-modules with the property that $\mathbf{I}$ and $\mathbf{E}$ coincides for all $n \gg 0$. In addition, it is said that $M \longrightarrow \mathbf{I} \longrightarrow \mathbf{E}$ is a complete injective resolution in the strong sense if the complex $\text{Hom}_{R\Gamma}(J, \mathbf{E})$ is also acyclic for every injective $R\Gamma$-module J.

Proposition 4. *Let Γ be a group such that $R\Gamma$ is a coherent ring. Every FP-injective $R\Gamma$-module has finite injective dimension provided that* $\text{silp}(R\Gamma) < \infty$.

Proof. Suppose M is an FP-injective $R\Gamma$-module and $\text{silp}(R\Gamma) = n$ is finite. According to 4.1 of [17], a complete injective resolution

$$\begin{array}{ccccccccccccc}
 & & 0 & \longrightarrow & M & \longrightarrow & I^0 & \xrightarrow{\delta^0} \cdots \xrightarrow{\delta^{n-1}} & I^n & \xrightarrow{\delta^n} & I^{n+1} & \longrightarrow & \cdots \\
 & & & & & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & E^{-1} & \xrightarrow{d^{-1}} & E^0 & \xrightarrow{d^0} & \cdots & \xrightarrow{d^{n-1}} & E^n & \xrightarrow{d^n} & E^{n+1} & \longrightarrow & \cdots
\end{array}$$

for M exists, whose proof process shows that $I^i = E^i$ for every $i \geq n$. Now for each $i \in \mathbb{Z}$ set: $\Omega^i = \text{Ker} d^i$. The goal is to show that $\Omega^n = \text{Ker}\delta^n = \text{Ker} d^n$ is injective. For this, it is sufficient to prove that all Ω^i's are FP-injective. Because this shows that the second row of the above diagram, $\mathbf{E}$, is contractible, see [4, Theorem 5.1]. But the fact that M is FP-injective and $R\Gamma$ is coherent in the light of Lemma 3.1 of [30] implies that Ω^n and therefore every Ω^i is FP-injective, for $i \geq n$. On the other hand, applying the inequality $\text{sfli}(R\Gamma) \leq \text{spli}(R\Gamma)$ with the inequality obtained in Corollary 3 gives the finiteness of $\text{fin.fp.dim}(R\Gamma)$ because by [3, Corollary 3.4], $\text{spli}(R\Gamma) = \text{silp}(R\Gamma) = n$. So assume that $\text{fin.fp.dim}(R\Gamma) = d$ and $i < n$. Therefore, $\text{FP-id}_{R\Gamma}\Omega^{i-d-1} \leq d$, because the FP-injective dimension of Ω^{i-d-1} is finite. Now, using [30, Lemma 3.1], it follows that Ω^i is FP-injective in this case as well. $\square$

Corollary 4. *Let $R\Gamma$ be a coherent ring and* $\text{silp}(R\Gamma) \leq r$, *for an integer $r \geq 1$. The following conditions are equivalent for an $R\Gamma$-module M.*

(i) M has finite FP-injective dimension.

(ii) M is a direct limit of finitely presented $R\Gamma$-modules of finite FP-injective dimension.

(iii) M has finite injective dimension.

Proof. (i)$\Longrightarrow$ (ii). Assume that $\text{FP-id}_{R\Gamma}M = n$ is finite. By Lazard's theorem $M = \varinjlim_j M_j$, where every M_j is a finitely presented $R\Gamma$-module. Now, taking a finitely presented $R\Gamma$-module N and applying [30, Theorem 3.2], we have:

$$\varinjlim_j \text{Ext}^{n+1}_{R\Gamma}(N, M_j) \cong \text{Ext}^{n+1}_{R\Gamma}(N, \varinjlim_j M_j) \cong \text{Ext}^{n+1}_{R\Gamma}(N, M) = 0.$$

So, for each j, $\mathrm{Ext}^{n+1}_{R\Gamma}(N, M_j) = 0$, which means that FP-id$_{R\Gamma}M_j$ is finite.

(ii)$\Longrightarrow$ (iii). As we saw in the proof of Proposition 4, since $R\Gamma$ is a coherent ring, we have:

$$\mathrm{fin.fp.dim}(R\Gamma) \leq \mathrm{sfli}(R\Gamma) \leq \mathrm{spli}(R\Gamma) = \mathrm{silp}(R\Gamma) \leq r.$$

If $M = \varinjlim_j M_j$, where each M_j is a finitely presented $R\Gamma$-module of finite FP-injective dimension, then for each j, FP-id$_{R\Gamma}M_j \leq r$. So, in the light of [30, Theorem 3.2], FP-id$_{R\Gamma}M \leq r$. Now consider the exact sequence

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^{r-1} \longrightarrow I^r \longrightarrow 0$$

with injective $R\Gamma$-modules I^i, for $i = 1, 2, \cdots, r-1$. By [30, Lemma 3.1], I^r must be an FP-injective $R\Gamma$-module. So, by using Proposition 4, we can conclude that id$_{R\Gamma}I^r < \infty$, which in turn gives the finiteness of id$_{R\Gamma}M$.

(iii)$\Longrightarrow$ (i). This is clear because the inequality FP-id$_{R\Gamma}M \leq$ id$_{R\Gamma}M$ always holds. □

A two-sided coherent ring R is said to be Ding-Chen, provided that it has left and right self-FP-injective dimensions at most equal to n, for some non-negative integer n. These rings were first introduced and studied in [12] and [13] by Ding and Chen under the name of n-FC rings and then renamed by Gillespie in [18]. It is worth noting that Ding-Chen rings are a generalization of Iwanaga-Gorenstein rings (i.e., two-sided Noetherian rings with finite left and right self-injective dimensions). There are various examples of these types of rings, which are mentioned in [18]. Among these examples, what is interesting to us is that when R is a commutative Ding-Chen ring and Γ is a finite group, then by [18, Proposition 5.1.], $R\Gamma$ is a Ding-Chen ring. Here we intend to show that the converse is also true, that is, we can say that the ring $R\Gamma$ is Ding-Chen if and only if R is so. What we are ready to record now is the general result below.

Corollary 5. *Let Γ' be a subgroup of Γ of finite index. Then the necessary and sufficient condition for the ring $R\Gamma$ to be Ding-Chen is that the ring $R\Gamma'$ is Ding-Chen.*

Proof. Because by Lemma 7, FP-id$_{R\Gamma'}R\Gamma'$ = FP-id$_{R\Gamma}R\Gamma$, the result is obtained directly from the definition along with an application of Proposition 1. □

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