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A class of number fields without odd rational prime index divisors

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Abstract. In this work, for every number field K generated by a root of a monic irreducible trinomial $F(x) = x^7 + a.x^6 + b \in \mathbb{Z}[x]$, we show that no odd rational prime p divides the index $i(K)$, and we give the necessary and sufficient conditions on a, b such that 2 divides $i(K)$. Specifically, we provide adequate requirements for K to be non-monogenic. Finally, several computational examples are used to illustrate our conclusions.

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1 Introduction

Let K be a number field of degree n and $\mathfrak{o}_K$ its ring of integers; it is a free $\mathbb{Z}$-module of rank $n = [K : \mathbb{Q}]$. For any primitive element $\theta \in \mathfrak{o}_K$, let $F(x)$ be the minimal polynomial of θ, and $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$ be the index of $\mathbb{Z}[\theta]$ in $\mathfrak{o}_K$. It is well known that $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$ is a finite abelian group, and the index formula is given by:

$$disc_{\mathbb{Z}}(F) = \pm[\mathfrak{o}_K : \mathbb{Z}[\theta]]^2 \cdot d_K,$$

where d_K is the absolute discriminant of K and $disc_{\mathbb{Z}}(F)$ is the discriminant of $F(x)$. The number field K is said to be monogenic if there is some non-trivial $\beta \in \mathfrak{o}_K$, such that $(1, \beta, \ldots, \beta^{n-1})$ is an integral basis of K. If K has no such β we say that K is non-monogenic.
The number $i(K) = gcd\{[\mathfrak{o}_K : \mathbb{Z}[\beta]], \beta \in \mathfrak{o}_K \textit{ generates } K\}$ is called the field index of K. A rational prime number p that divides $i(K)$ is called the common index prime divisor of K. If $\mathfrak{o}_K$

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has a power integral basis, then the index $i(K) = 1$. Thus, the field K is not monogenic if it has a common index prime divisor. The construction of power integral bases (PIBs) or relative power integral bases (RPIBs) and testing the monogenity of number fields is a traditional problem - [4]- that has been the focus of in-depth research during the last century and this century, mainly by; Jakhar et al. [20–23], Gaál et al. [14–18], Sahmoudi et al. [2,3,28,29,31,32], El Fadil et al. [7–9], Nakahara et al. [11–13,24], and others.
There are efficients algorithms to solve index form equations in low degree fields, and for some special higher degree fields, see Gaál [17]. In [24], Khan, Nakahara and Sekiguchi studied the monogenity of a family of cyclic sextic composite fields $K.L$ over the field $\mathbb{Q}$, where K is a cyclic cubic field of prime conductor p and L is a quadratic field with the field discriminant d_k such that $(p, d_L) = 1$. In addition to his studies of the pure case, El. Fadil was also interested in studying trinomial cases of degree 6, 7, ..., for example, he characterized when a prime p is a common index divisor of the number field defined by $x^6 + a.x^3 + b \in \mathbb{Z}[x]$ (see [7]). In [29], [30] and [1] Sahmoudi et al. looked at the relative monogenity of $L = K(\theta)$, where θ is a root of the monic irreducible polynomial, respectively: $F(x) = x^{p^n} - \beta \in \mathfrak{o}_K[x]$ and $F(x) = x^{p^n} + a.x^{p^m} + b \in \mathfrak{o}_K[x]$ with $p = 3$ and, generally, for $(p > 3)$. In [32] Sahmoudi and Charkani examined the relative monogenity of number fields L defined by $x^p - \beta$ over an arbitrary number field K using a straightforward and useful application of Dedekind's criterion that characterized the existence of power integral bases over any Dedekind ring by employing the Gauss valuation. In the same way, Jakhar et al. established certain conditions on a and b that ensure the non-monogenity of any sextic number field defined by a trinomial $x^6 + a.x + b \in \mathbb{Z}[x]$ (see [22]). In [5], for K defined by a trinomial $x^4 + a.x + b$, Davis and Spearman characterized when $p = 2, 3$ divided $i(K)$.
Let $F(x) = x^7 + a.x^6 + b$ be an irreducible monic trinomial and $K = \mathbb{Q}(\theta)$ be a number field generated by a complex root θ. The purpose of this work is to present necessary and sufficient conditions on a and b such that p serves as a common index divisor of K for any rational prime p. Particularly under these circumstances, K is non-monogenic. The strategy used is primarily based on Dedekind's criterion, Newton's polygons and the factorization onto prime ideals.

2 Main results

Assume that a and b are two rational integers, and $F(x) = x^7 + ax^6 + b$ is an irreducible polynomial over $\mathbb{Q}$, and let $K = \mathbb{Q}(\theta)$ be a number field generated by a root θ of $F(x)$. We assume, without losing generality, that for any rational prime p, $v_p(a) \leq 0$ or $v_p(b) \leq 6$.
The following theorem characterizes the integral closedness of $\mathbb{Z}[\theta]$.

Theorem 1. *Let K be a septic number field generated by a root θ of an irreducible trinomial $F(x) = x^7 + ax^6 + b \in \mathbb{Z}[x]$, with $v_p(a) \leq 0$ or $v_p(b) \leq 6$ for any prime number p. The ring $\mathbb{Z}[\theta]$ is integrally closed if and only if the conditions (1), (2) and (3) hold:*

1. *the integer b is square-free,*
2. *If $a \equiv 0 \pmod 7$ and 7 does not divide b, then, $v_7(1 - b^6 + ab^5) = 1$.*
3. *For every rational prime $p \notin \{2, 3, 7\}$, if $v_p(ab) = 0$, then $v_p(7^7 \times b + 2^6 \times 3^6 \times a^7) \leq 1$.*

Specifically, in view of all these conditions, we have $i(K) = 1$.

Remark 1. *Unfortunately, the monogenity of K is not equivalent to the fact that $\mathbb{Z}[\theta]$ is integrally closed, and we can construct examples of which K is monogenic without the integral closedness of $\mathbb{Z}[\theta]$.*

The following theorem is our second important main result, in which we offer conditions for the prime integer 2 to divide $i(K)$.

Theorem 2. *We put $b = 2^m.c$ for some integer m, and $c \equiv 1 \ (mod 2)$. The index $i(K)$ is divisible by the prime number 2 if and only if one of the following conditions hold:*

1. *If $m \geq 6v_2(ca+1)$ and $v_2(ca+1) > 2$.*
2. *If $m \leq 6v_2(ca+1)$ and $m > 12$.*

Specifically, K is not monogenic.

Theorem 3. *Let $K = \mathbb{Q}(\theta)$, where θ is a root of a monic irreducible polynomial $F(x) = x^7 + ax^6 + b \in \mathbb{Z}[x]$. Then, the index $i(K)$ is not divisible by p for every prime $p \geq 3$ and every (a, b) in $\mathbb{Z}^2$.*

3 A prime ideal factorization description based on Newton polygons

In order to prove our major theorems, we describe some important approaches for prime ideal factorization and index calculation: $Ind_{\mathbb{Z}}(\theta) = [\mathfrak{o}_K : \mathbb{Z}[\theta]]$. Let $K = \mathbb{Q}(\theta)$ be a number field where $\theta \in \mathfrak{o}_K$ be an algebraic integer over $\mathbb{Z}$.

3.1 Dedekind criterion

Dedekind's criterion allows testing whether a prime integer p divides or not the index $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$:

Theorem 4 ([32], Dedekind Criterion)**.** *Let $F = Irrd(\theta, \mathbb{Z}) \in \mathbb{Z}[x]$ be the monic irreducible polynomial of θ. Let p be a non-zero prime integer, and $\bar{F} = \prod_{i=1}^{r} \bar{\phi_i}^{l_i}$ is the primary decomposition of $\bar{F}$ in $\mathbb{F}_p[x]$ for some monic polynimial $\phi_i \in \mathbb{Z}[x]$. Let $R_i \in \mathbb{Z}[x]$ be the Remainder of the Euclidean division of F by ϕ_i. Let υ_{G^p} be the Gauss valuation on $\mathbb{Q}[x]$. Then, p doesn't divide the index integer $Ind_{\mathbb{Z}}(\theta)$ if and only if either $l_i = 1$ or $\upsilon_{G^p}(R_i) = 1$ for all $i = 1, \ldots, r$ such that $l_i \geq 2$.*

Remark 2. *Dedekind's criterion fails sometimes, that is to say, the prime p divides the index* $\mathrm{Ind}_{\mathbb{Z}}(\theta)$ *for every primitive element $\theta \in \mathfrak{o}_K$, then for such primes and number fields, we are not able to find the factorization of the prime ideal of $p\mathfrak{o}_K$. So, to resolve this problem, we start by recalling some fundamental notions about Newton's polygon. For additional information, we refer to [19, 27].*

3.2 Ore's Theorem

Let p be a rational prime integer and $\phi \in \mathbb{Z}[x]$ a monic polynomial, which is irreducible modulo p. Upon the Euclidean division by successive powers of ϕ, we can expand $F(x)$ as follows: $F(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_n(x)\phi(x)^n$, with $deg\ (a_i(x)) < deg\ (\phi(x))$.
Any such expansion is unique, which is called the ϕ-adic expansion of $F(x)$. Let $u_i = \nu_p(a_i(x))$, for every $i = 0, \dots, n$. The lower boundary convex envelope of the collection of points $\{(i, u_i), 0 \leq i \leq n, a_i(x) \neq 0\}$ in the Euclidean plane, denoted by $N_\phi(F)$, is The ϕ-Newton polygon of $F(x)$. The polygon$N_\phi(F)$ is formed by the union of many neighboring sides $S_1, S_2, \dots, S_t$ with increasing slopes $\lambda_1 < \lambda_2 < \cdots < \lambda_t$, we write $N_\phi(F) = S_1 + S_2 + \cdots + S_g$. The polygon formed by the sides of negative slopes of $N_\phi(F)$ is known as the ϕ-principal Newton polygon of $F(x)$ and is denoted by $N_\phi^+(F)$.
The length of $N_\phi^+(F)$ is $\nu_{\overline{\phi}}(\overline{F(x)})$; the greatest power of $\overline{\phi}$ dividing $\overline{F(x)}$ in $\mathbb{F}_p[x]$.
Let $\mathbb{F}_\phi$ be the finite field $\mathbb{Z}[x]/(p, \phi(x)) \simeq \mathbb{F}_p[x]/(\overline{\phi(x)})$ (remember that if $deg(\phi) = 1$, then $\mathbb{F}_\phi \simeq \mathbb{F}_p$). For every $i = 0, \dots, n$, we assign the following residue coefficient: $c_i \in \mathbb{F}_\phi$ as

$$c_i = \begin{cases} 0, & \text{if } (i, u_i) \text{ lies strictly above } N_\phi^+(F), \\ \left(\dfrac{a_i(x)}{p^{u_i}}\right) \pmod{(p, \phi(x))}, & \text{if } (i, u_i) \text{ lies on } N_\phi^+(F). \end{cases}$$

Let S be a side of $N_\phi^+(F)$ and $\lambda = -\frac{h}{e}$ its slope, with e and h being positive coprime integers. When S is projected to the horizontal axis, its length is marked by the symbol $l(S)$, and when it is projected to the vertical axis, it is marked by the symbol $h(S)$. S's degree is $d = d(S) = gcd(l(S), h(S))$. It is worth noting that if if (s, u_s) is the starting point of S, then the points with integer coordinates lying on S are exactly (s, u_s), $(s+e, u_s - h), \cdots, (s+de, u_s - dh)$. We attach to S the residual polynomial defined by

$$R_\lambda(F)(y) = c_s + c_{s+e}y + \cdots + c_{s+(d-1)e}y^{d-1} + c_{s+de}y^d \in \mathbb{F}_\phi[y].$$

The $ind_\phi(F)$ symbol stands for the ϕ-index of $F(x)$, which is defined in [8, Def. 1.3] as deg(ϕ) times the number of points with natural integer coordinates that strictly lie above the horizontal axis, strictly above the vertical axis, below or on the polygon $N_\phi^+(F)$ (see FIGURE 1).
The polynomial $F(x)$ is said to be ϕ-regular with regard to p if the associated residual polynomial $R_\lambda(F)(y)$ is separable in $\mathbb{F}_\phi[y]$ for each side S of$N_\phi^+(F)$.

Now, let $\overline{F(x)} = \prod_{i=1}^{t} \overline{\phi_i}^{l_i}$ be the factorization of $\overline{F(x)}$ into monic irreducible polynomials $\phi_1, \dots, \phi_t$ in $\mathbb{F}_p[x]$. The polynomial $F(x)$ is said to be p-regular if $F(x)$ is ϕ_i-regular for $i = 1, \dots, t$.
Let $N_{\phi_i}^+(F) = S_{i1} + \cdots + S_{it_i}$; be the principal ϕ_i- Newton polygon of F with regard to p for each $i = 1, \dots, t$. For every $j = 1, \dots, t_i$ let $-\lambda_{ij}$ the slope of the side S_{ij}, and let $R_{\lambda_{ij}}(F)(y) = \prod_{s=1}^{s_{ij}} \psi_{ijs}^{n_{ijs}}(y)$ be the factorization of $R_{\lambda_{ij}}(F)(y)$ in $\mathbb{F}_{\phi_i}[y]$. Then, we have the Ore index theorem, which plays a significant role in the proof of our theorems (see [8, Theorem 1.7 and Theorem 1.9], and [27]).

Theorem 5 (Ore's Theorem)**.** *With the notation above, we have*

1. $\nu_p([\mathfrak{o}_K : \mathbb{Z}[\theta]]) \geq \sum_{i=1}^{t} ind_{\phi_i}(F)$, *with equality if* $F(x)$ *is* p*-regular.*

2. *If* $F(x)$ *is* p*-regular, then;*

$$p\mathfrak{o}_K = \prod_{i=1}^{t}\prod_{j=1}^{r_i}\prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}}, \tag{1}$$

where e_{ij} *is the ramification index of the side* S_{ij}*, i.e, the smallest positive integer satisfying* $e_{ij}\lambda_{ij} \in \mathbb{Z}$ *and* $f_{ijs} = deg(\phi_i) \times deg(\psi_{ijs})$ *is the residue degree of* $\mathfrak{p}_{ijs}$ *over* p*.*

Corollary 1 ([6], Corollary 1.3)**.** *Under the assumptions of theorem 5 ,* p *does not divide the index* $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$ *if and only if* $l_i = 1$ *or* $N_{\phi_i}^{+}(F) = S_i$ *has a single side of height* 1 *for every* $i = 1, \dots, r$*, then,* $F(x)$ *is* $p-$*regular and* $\nu_p([\mathfrak{o}_K : \mathbb{Z}[\theta]]) = 0$

Example 1. Consider the monic irreducible polynomial $F(x) = x^8 + 3x^2 + 30$ which is 3-Eisenstein and factors in $\mathbb{F}_3[x]$ into $F(x) = \overline{\phi_1}^2.\overline{\phi_2}^2.\overline{\phi_3}^2 \pmod 2$ where $\phi_1 = x$, $\phi_2 = x + 1$ and $\phi_3 = x^2 + x + 1$, then the ϕ_i-adic development of $F(x)$ for $i = 1, 2, 3$, respectively are

$$\begin{aligned}
F(x) &= 30 + 3\phi_1(x)^2 + \phi_1(x)^8,\\
F(x) &= 34 - 14\phi_2(x) + 31\phi_2(x)^2 - 56\phi_2(x)^3 + 70\phi_2(x)^4 - 56\phi_2(x)^5\\
&\quad + 28\phi_2(x)^6 - 8\phi_2(x)^7 + \phi_2(x)^8,\\
F(x) &= (26 - 4x) + (8 + 2x)\phi_3(x) + (-7 + 2x)\phi_3(x)^2 + (2 - 4x)\phi_3(x)^3 + \phi_3(x)^4.
\end{aligned}$$

Thus, $N_{\phi_i}^{+}(F) = S_i$, $i = 1, 2, 3$ with respect to ν_2 has one side joining points $(0, 1)$ and $(2, 0)$ in the Euclidean plane (see FIGURE 1) of degree 1 with slopes $\lambda_i = \frac{-1}{2}$ $i = 1, 2, 3$.

The residual polynomials attached to the side of $N_{\phi_i}^{+}(F)$, $i = 1, 2, 3$ are

$$R_{\lambda_1}(F)(y) = R_{\lambda_2}(F)(y) = R_{\lambda_3}(F)(y) = 1 + y,$$

Which are irreducible polynomials in $\mathbb{F}_{\phi_i}[y] \simeq \mathbb{F}_2[y]$. Therefore, $F(x)$ is ϕ_i-regular, where $i = 1, 2, 3$. Hence, it is 2-regular. By Theorem 5, $\nu_2(ind(F)) = \nu_2((\mathfrak{o}_K : \mathbb{Z}[\theta])) = ind_{\phi_1}(F) + ind_{\phi_2}(F) + ind_{\phi_3}(F) = \deg(\phi_1) \times 0 + \deg(\phi_2) \times 0 + \deg(\phi_3) \times 0 = 0 + 0 + 0 = 0$ and $2\mathfrak{o}_K = \mathfrak{p}_1^2\mathfrak{p}_2^2\mathfrak{p}_3^2$ with $f(\mathfrak{p}_i/2) = 1$, $i = 1, 2$ and $f(\mathfrak{p}_3/2) = 2$.

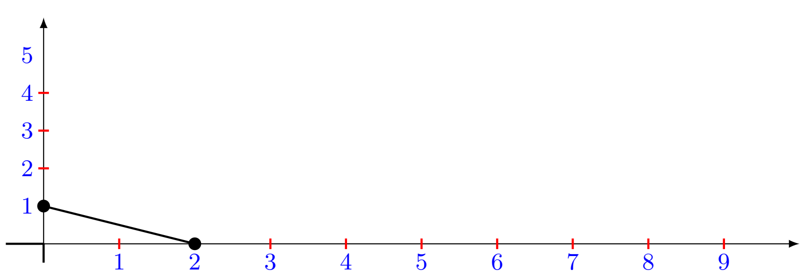

Figure 1: The ϕ_i-principal Newton polygon $N_{\phi_i}^{+}(F)$, $i = 1, 2, 3$ with respect to ν_2.

4 Proofs of main results

When the polynomial $F(x)$ is not p-regular; certain factors of residual polynomials $R_{\lambda_{ij}}(f)(y)$ are not irreducible in $\mathbb{F}_{\phi_i}[y]$. Montes et al. recently introduced an efficient algorithm to factorize the principal ideal $p\mathfrak{o}_K$ (see [19]). They defined the Newton polygon of order r and proved an extension of the theorem of the product, polygon, residual polynomial, and index in order r. As we use this algorithm in the second order, $r = 2$, we briefly recall the concepts that will be used throughout this work. Let ϕ be a monic irreducible factor of $F(x)$ modulo p. Let S be a side of $N_1 = N_\phi^+(F)$, with slope $\lambda = -\frac{h}{e}$, with h and e are two coprime positive integers such that the associated residual polynomial $R_\lambda(f)(y) = \psi_1(y)^s$, with $deg(\psi_1) = f \geq 2$.

A type of order 2 is a chain: $(\phi(x); \lambda, \phi_2(x); \lambda_2, \psi_2(y))$, where the key polynomial $\phi_2(x)$ is a monic irreducible polynomial in $\mathfrak{o}_p[x]$ of degree $m_2 = e \cdot f \cdot \deg(\phi)$, λ_2 is a negative rational number and $\psi_2(y) \in \mathbb{F}^2 = \mathbb{F}_\phi[y]/(\psi_1(y))$ such that

1. $\phi_2(x)$ is congruent to a power of x modulo p.
2. $N_{\phi_2}^+(F)$ is one-sided with slope λ.
3. The residual polynomial in order 1 of ϕ_2; $R_\lambda(\phi_2)(y) \simeq \psi_1(y)$ in $\mathbb{F}_\phi[y]$.
4. λ_2 is a slope of a certain side of ϕ_2-Newton polygon of second order (To be specified below) and $\psi_2(y) = R^2_{\lambda_2}(f)(y)$ is the associated residual polynomial of second order.

The key polynomial ϕ_2 induces a valuation ν_p^2 on $\mathbb{Q}_p(x)$, called the augmented valuation of ν_p of second order with respect to ϕ and λ. By [19, Proposition 2.7], If $P(x) \in \mathfrak{o}_p[x]$ such that $P(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_l(x)\phi(x)^l$. Then $\nu_p^2(P(x)) = e \times \min_{0\leq i\leq l}\{\nu_p(a_i(x)) + i(\nu_p(\phi(x)) + |\lambda|)\}$, in particular $\nu_p^2(\phi_2(x)) = e \cdot f \cdot \nu_p(\phi(x))$. Let $F(x) = a_0(x) + a_1(x)\phi_2(x) + \cdots + a_t(x)\phi_2(x)^t$ be the ϕ_2-adic development of $F(x)$ and let $\mu_i = \nu_p^2(a_i(x)\phi_2(x)^i)$ for every $0 \leq i \leq t$. The ϕ_2-Newton polygon of $F(x)$ of second order with respect ν_p^2 is the lower boundary of the convex envelope of the set of points $\{(i, \mu_i), 0 \leq i \leq t\}$ in the Euclidean plane, which we denote by $N^2(f)$. We use the theorem of the polygon and the theorem of residual polynomial in the second order (see [19, Theorem 3.1 and 3.4] for more general treatment).

The following lemma provides a sufficient condition for a rational prime integer p to be a prime common index divisor of the field K. For the proof, see [26, Theorems 4.33 and 4.34].

Lemma 1. *Let p be a rational prime integer and K be a number field. For every positive integer m, let $P_p(m)$ be the number of distinct prime ideals of $\mathfrak{o}_K$ lying above p with residue degree m and $N_p(m)$ be the number of monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree m. Then, p divides $ind(K)$ if and only if $P_p(m) > N_p(m)$ for some positive integer m.*

To apply the last lemma, one needs to know the number $N_p(m)$ of the monic irreducible polynomial over $\mathbb{F}_p$ of degree m. This number was found by Gauss, which is given by the following proposition (see [26, Proposition 4.35.]).

Proposition 1. *The number of monic irreducible polynomials of degree m over the field $\mathbb{F}_p[X]$ is*

$$N_p(m) = \frac{1}{m}\sum_{d|m}\mu(d)p^{\frac{m}{d}},$$

where μ is the Möbius function.

Proof of Theorem 1. Let p be a rational integer dividing $(\mathfrak{o}_K : \mathbb{Z}[\theta])$. Since $\triangle(F) = -b^5(7^7.b + 2^6.3^6.a^7)$ is the discriminant of $F(x)$ and thanks to the index formula $\triangle(F) = \pm[\mathfrak{o}_K : \mathbb{Z}[\theta]]^2 \times d_K$, we have that p^2 divides $\triangle(F)$; that is, p divides b or p^2 divides $7^7.b + 2^6.3^6.a^7$. Hence if p $= 2, 3$, then p divides b. If p $= 7$, then p divides ab. If $p \notin \{2, 3, 7\}$, then p divides b or p does not divide ab and p^2 divides $7^7.b + 2^6.3^6.a^7$.

1. For $p = 2$, then 2 divides b.
   - If 2 divides a, then 2 does not divide $(\mathfrak{o}_K : \mathbb{Z}[\theta])$ if and only if $v_2(b) = 1$.
   - If 2 does not divide a, then $F(x) \equiv x^6(x-1) (\text{mod } 2)$. Let $\phi_1 = x$ and $\phi_2 = x - 1$, then $F(x) = \phi_2^7 \cdots + (a + b + 1)$ and $v_{\phi_2}(F(x) = 1$ (The Gauss valuation under ϕ_2) . We conclude that 2 does not divide $(\mathfrak{o}_K : \mathbb{Z}[\theta])$ if and only if $v_2(b) = 1$
2. For $p = 3$, then 3 divides b.
   - If 3 divides a, then 3 does not divide $(\mathfrak{o}_K : \mathfrak{o}[\theta])$ if and only if $v_3(b) = 1$.
   - If $a \equiv 1 (\text{mod } 3)$, then $F(x) \equiv x^6(x + 1) (\text{mod } 3)$.
     Let $\phi_1 = x$ and $\phi_2 = x + 1$, $F(x) = \phi_1^7 + a\phi_1^6 + b$ and we have $v_{\phi_2}(F(x) = 1$. Also in this case, 3 does not divide $(\mathfrak{o}_K : \mathbb{Z}[\theta])$ if and only if $v_3(b) = 1$
   - If $a \equiv -1 (\text{mod } 3)$, then $F(x) \equiv x^6(x - 1) (\text{mod } 3)$.
     Let $\phi_1 = x$ and $\phi_2 = x - 1$, then $F(x) = \phi_1^7 + a\phi_1^6 + b$ and we have $v_{\phi_2}(F(x) = 1$. Also in this case, 3 does not divide $(\mathfrak{o}_K : \mathbb{Z}[\theta])$ if and only if $v_3(b) = 1$
3. For $p = 7$, then 7 divides ab.
   - If 7 divides a and b, then
     $$F(x) \equiv \phi(x)^7 (\text{mod } 7),$$
     Where $\phi(x) = x$. We have $F(x) = \phi^7(x) + a\phi^6(x) + b$. Also in this case, 7 does not divide $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$ if and only if $v_7(b) = 1$
   - If 7 does not divide a and 7 divides b, then
     $$F(x) \equiv \phi(x)^6(x + a) (\text{mod } 7).$$
     Let $\phi_1 = x$ and $\phi_2 = x + a$, $F(x) = \phi_2^7 + \cdots + b$ and $F(x) = \phi_1^7 + a\phi_1^6 + b$. Also in this case, 7 does not devide $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$ if and only if $v_7(b) = 1$.

- If 7 divides a and 7 does not divide b, then

$$F(x) \equiv \phi(x)^7 (\text{mod } 7).$$

Let $\phi(x) = x + b$, $F(x) = \phi(x)^7 + \cdots + b(1 + ab^5 - b^6)$. Also in this case, 7 does not devide $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$ if and only if $v_7(1 + ab^5 - b^6) = 1$.

4. For $p \notin \{2, 3, 7\}$, and p divides b.

- If p divides a, then $F(x) \equiv x^7 (\text{mod } 7)$.
Let $\phi(x) = x$, $F(x) = \phi(x)^7 + a\phi(x)^6 + b$. In addition, p does not divide $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$ if and only if $v_p(b) = 1$.
- If p does not divides a, then $F(x) \equiv x^6(x + a)(\text{mod } 7)$.
Let $\phi_1 = x$ and $\phi_2 = x + a$, $F(x) = \phi_2^7 + \cdots + b$ and $F(x) = \phi_1^7 + a\phi_1^6 + b$. Also in this case, p does not devide $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$ if and only if $v_p(b) = 1$.

5. For $p \notin \{2, 3, 7\}$, p does not divide ab, and p^2 divides $(7^7 \times b + 2^6 \times 3^6 a^7)$, then by [[25], Theorem 1]

$$v_p(d_K) = \begin{cases} 0 \text{ if } & v_p(7^7 \times b + 2^6 \times 3^6 a^7) \textit{ is even}, \\ 1 \text{ if } & v_p(7^7 \times b + 2^6 \times 3^6 a^7) \textit{ is odd}. \end{cases}$$

Thus, by the index formula $\triangle(F) = \pm[\mathfrak{o}_K : \mathbb{Z}[\theta]]^2 \times d_K$, p divides the index $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$.

□

For the proof of Theorems 2 and 3, we use the lemma 1.

Proof of Theorem 2. As $\triangle(F) = -b^5(7^7 \times b + 2^6 \times 3^6 \times a^7)$ is the discriminant of $F(x)$, by the index formula $\triangle(F) = \pm[\mathfrak{o}_K : \mathbb{Z}[\theta]]^2 \times d_K$, if 2 does not divide b, then 2 does not divide $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$, and so 2 does not divide $i(K)$.
Now assume that 2 divides b. Then we have the following cases:

1. If $v_2(a) = 0$, then $F(x) \equiv x^6(x + 1)(\text{mod } 2)$. Let $\phi_1 = x$ and $\phi_2 = x + 1$.

- For ϕ_2, we have $N^+_{\phi_2}(F) = S_2$ has one side of degree 1, then ϕ_2 provides a unique prime ideal $\mathfrak{p}_1$ of $\mathfrak{o}_K$ lying above 2 with residue degree 1.
- For ϕ_1, we have $F(x) = \phi_1^7 + a\phi_1^6 + b$, and so $N^+_{\phi_1}(F) = S_1$ has a single side joining $(0,\ v_2(b))$ and $(6, 0)$ of degree $d(S_1) \in \{1, 2, 3\}$.
  - If $v_2(b) \not\equiv 0(\text{mod } 2)$ and $v_2(b) \not\equiv 0(\text{mod } 3)$, so $N^+_{\phi_1}(F)$ has a single side joining $(0,\ v_2(b))$ and $(6, 0)$ of degree $d(S_1) = 1$. then ϕ_1 provides a unique prime ideal $\mathfrak{p}_2$ of $\mathfrak{o}_K$ lying above 2 with residue degree 1. Thus $2\mathfrak{o}_K = \mathfrak{p}_1^6.\mathfrak{p}_2$ with residue degree 1 for each prime ideal $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Hence 2 does not divide $i(K)$.

– If $v_2(b) \equiv 0(\bmod 2)$ and $v_2(b) \not\equiv 0(\bmod 3)$, then $N^+_{\phi_1}(F) = S_1$ has a single side of degree 2, $R_{\lambda_1}(F)(y) = y^2+1 = (y+1)^2$ in $\mathbb{F}_{\phi_1}[y]$. Hence, $R_{\lambda_1}(F)(y)$ is inseparable in $\mathbb{F}_{\phi_1}[y]$.
We use the Newton polygon of $F(x)$ of second order, we have the ramification index of S_1 is $e_1 = 3$ and the slope of $F(x)$ is $\lambda_1 = -\frac{v_2(b)}{6}$. Put $F(x) = x^7 + ax^6 + 2^m.c$, where $m = v_2(b)$ and 2 is not divide c. In addition, the key polynomial $\psi(x) = x^3 - 2^{\frac{m}{2}}.c$, where 2 does not divide c. Let the ψ-adic development of $F(x)$ is: $F(x) = \ (x+a)\psi(x)^2 + (2.2^{\frac{m}{2}}c(x+a))\psi(x) + (2^m c^2(x+a)+b)$ Let, $u_0 = v_2^2(a_0)$, $u_1 = v_2^2(a_1)$ and $u_2 = v_2^2(a_2)$, then

$$\begin{aligned} u_0 &= \ v_2^2((2.2^{2^{\frac{m}{2}}}c^2(x+a)+b)) = \ 3\ min(m+\frac{m}{6}, m+v_2(ca+1)), \\ u_1 &= \ v_2^2((2.2^{\frac{m}{2}}c(x+a)\psi(x)) = \ 3\ m+3, \\ u_2 &= \ v_2^2((x+a)\psi(x)^2) = 3m. \end{aligned}$$

So, we have two cases:
- If $\frac{m}{6} \geq v_2(ca+1)$ and $v_2(ca+1) > 2$, then $N^2_{\psi}(F) = S_{11} + S_{12}$ has two sides S_{11}, S_{12} of degree 1. Then we will have $2\mathfrak{o}_K = \mathfrak{p}_2.\mathfrak{p}_{11}\mathfrak{p}_{12}$ with $f_2 = f_{11} = f_{12} = 1$. Then 2 divide $i(K)$.
- If $12 < m$ and $m \ < \ 6.v_2(ca+1)$, then $N^2_{\psi}(F) = S_{11} + S_{12}$ has two sides S_{11}, S_{12} of degree 1. Then we will have $2\mathfrak{o}_K = \mathfrak{p}_2.\mathfrak{p}_{11}\mathfrak{p}_{12}$ with $f_2 = f_{11} = f_{12} = 1$. Then 2 divide $i(K)$.

– If $v_2(b) \equiv 0(\bmod 3)$ and $v_2(b) \not\equiv 0(\bmod 2)$, then $N^+_{\phi_1}(F) = S_1$ has a single side joining $(0, v_2(b))$ and $(6, 0)$ of degree $d(S_1) = 3$. with $R_{\lambda_1}(F)(y) = y^3 + 1 = (y+1)(y^2+y+1)$ in
F_{ϕ_1}. Hence $2\mathfrak{o}_K = \mathfrak{p}_{11}^2.\mathfrak{p}_{12}^2.\mathfrak{p}_2$ with residue degree 1 for prime ideals $\mathfrak{p}_{11}$ and $\mathfrak{p}_2$, and residue degree 2 for prime ideal $\mathfrak{p}_{12}$. Hence 2 does not divide $i(K)$.

2. If $v_2(a) \geq 1$, then $F(x) \equiv \phi(x)^6((\bmod 2)$. where $\phi(x) = x$. Hence we have $f(x) = \phi(x)^7 + a\phi(x)^6 + b$. As $v_2(a) \geq 1$, this implies $v_2(b) \leq 6$. Therefore, $N^+_{\phi}(F) = S$ has a single side of degree 1, then ϕ provides a unique prime ideal of $\mathfrak{o}_K$ lying above 2 with residue degree 1. Namely $2\mathfrak{o}_K = \mathfrak{p}^7$, $f(\mathfrak{p}/2) = 1$. Finally, 2 dose not divide $i(K)$.

□

Proof of Theorem 3. Since the dgree of K is 7, by the result of *Żyliński* [33], if p divides $i(K)$, then $p < 7$, see also [10]. Therefore, the candidate rational primes to be a common index divisor of K are 2; 3, and 5. So, to prove this theorem, it is sufficient to show that 3 does not divide $i(K)$ and 5 does not divide $i(K)$.

For $p = 3$. Since $\triangle(F) = -b^5(7^7 \times b + 2^6 \times 3^6 \times a^7)$, by the index formula

$$\triangle(F) = \pm[\mathfrak{o}_K : \mathbb{Z}[\theta]]^2 \times d_K$$

if 3 does not divide b, then 3 does not divide $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$, and so 3 does not divide $i(K)$. Now assume that 3 divides b. Then we have the following cases:

1. If $a \equiv 1 (\mathrm{mod}\ 3)$, then $F(x) \equiv x^6(x+1)(\mathrm{mod}\ 3)$, let $\phi_1 = x$ and $\phi_2 = x+1$. Recall that 3 divides $i(K)$ if and only if there are at least four prime ideals of $\mathfrak{o}_K$ lying above 3 with residue degree 1 each. In this case, we have $F(x) = \phi_1^7(x) + a\phi_1^6(x) + b$ and $F(x) = \phi_2^7(x) + \cdots + (1-a)\phi_2(x) + (a+b+1)$. The factor $\phi_2(x)$ gives us a single side of height one. And we have

   (a) If $v_3(b) \not\equiv 0 (\mathrm{mod}\ 3)$ and $v_3(a) \not\equiv 0 (\mathrm{mod}\ 3)$, then $N^+_{\phi_1}(F) = S_1$ has a single side of degree 1, then we will have $3\ \mathfrak{o}_K = \mathfrak{p}_1^6\mathfrak{p}_2$ with $f_1 = f_2 = 1$. Hence 3 does not divide $i(K)$.

   (b) If $v_3(b) \equiv 0 (\mathrm{mod}\ 2)$ and $v_3(b) \not\equiv 0 (\mathrm{mod}\ 3)$, then $N^+_{\phi_1}(F) = S_1$ has a single side of degree 2, thus we will have either two prime ideals of degree residual 1 or one prime ideal of degree residual 2. Then we will have $3\mathfrak{o}_K = \mathfrak{p}_{11}^3\mathfrak{p}_{12}^3\mathfrak{p}_2$ with $f_{11} = f_{12} = f_2 = 1$ or $3\mathfrak{o}_K = \mathfrak{p}_1^3\mathfrak{p}_2$ with $f_1 = 2$ et $f_2 = 1$. Finally, 3 does not divide $i(K)$.

   (c) If $v_3(b) \equiv 0 (\mathrm{mod}\ 3)$ and $v_3(b) \not\equiv 0 (\mathrm{mod}\ 2)$, then $N^+_{\phi_1}(F) = S_1$ has a single side of degree 3, $R_{\lambda_1}(F)(y) = y^3 + b_3 = (y \pm 1)^3$ in $F_{\phi_1}[y]$. Hence, $R_{\lambda_1}(F)(y)$ is not separable in $F_{\phi_1}[y]$.
   We use the Newton polygon of $F(x)$ of the second order, the ramification index of S_1 is $e_1 = 2$ and the slope of $F(x)$ is $\lambda_1 = \frac{v_3(b)}{-6}$. Put $F(x) = x^7 + ax^6 + 3^m.c$ where $m = v_3(b)$ and 3 is not divide c. In addition, the key polynomial $\psi(x) = x^2 - 3^{\frac{m}{3}}.s$ where 3 is not divide s and $s = c+2$ if $c \equiv 2(\mathrm{mod}\ 3)$ and $s = c+1$ if $c \equiv 1(\mathrm{mod}\ 3)$. Also, Let the ψ-adic development of $F(x)$ is: $F(x) = (x+a)\psi(x)^3 + (3.3^{\frac{m}{3}}sx + 3a.3^{\frac{m}{3}}.s)\psi(x)^2 + (3.3^{2\frac{m}{3}}s^2x + 3a.3^{2\frac{m}{3}}.s^2)\psi(x) + (3^m s^3 x + 3^m c + a.3^m.s^3)$ Let, $u_0 = v_3^2(a_0)$, $u_1 = v_3^2(a_1)$, $u_2 = v_3^2(a_2)$ and $u_3 = v_3^2(a_2)$, then

$$
\begin{aligned}
u_0 &= v_3^2((3^m s^3 x + 3^m c + a.3^m.s^3)) = 2m + 2min(\frac{m}{6}, v_3(c + as^3)),\\
u_1 &= v_3^2((3.3^{2\frac{m}{3}}s^2x + 3a.3^{2\frac{m}{3}}.s^2)\psi(x)) = 2m + 2,\\
u_2 &= v_3^2((3.3^{\frac{m}{3}}sx + 3a.3^{\frac{m}{3}}.s)\psi(x)^2) = 2m + 2,\\
u_3 &= v_3^2((x+a)\psi(x)^3) = 2m.
\end{aligned}
$$

   If $min(\frac{m}{6}, v_3(c + as^3)) = 1$, then $N^2_\psi(F) = S_1$ has one side of the degree $deg(S_1) = 1$. Then, we have $3\mathfrak{o}_K = \mathfrak{p}_2.\mathfrak{p}_\psi^3$ with $f(\mathfrak{p}_\psi) = f_{\mathfrak{p}_2} = 1$. Hence, 3 does not divide $i(K)$.
   If $min(\frac{m}{6}, v_3(c + as^3)) \geq 2$, then $N^2_\psi(F) = S_{11} + S_{12}$ has two sides of the degree $deg(S_{11}) = 1$ and $deg(S_{12}) = 2$. Then, we have $3\mathfrak{o}_K = \mathfrak{p}_{11}\mathfrak{p}_{12}^2\mathfrak{p}_2$. with $f_2 = f_{11} = 1$ and $f_{12} = 2$. Hence, 3 does not divide $i(K)$.

2. If $a \equiv 2 (\mathrm{mod}\ 3)$, then $F(x) \equiv x^6(x-1)(\mathrm{mod}\ 3)$, let $\phi_1 = x$, $\phi_2 = x-1$. Recall that 3 divides $i(K)$ if and only there are at least four prime ideals of $\mathfrak{o}_K$ lying above 3 with a residue degree of 1 each. In this case, we have: $F(x) = \phi_1^7(x) + a\phi_1^6(x) + b$ and $F(x) = \phi_2^7(x) + \cdots + (1-a)\phi_2(x) + (a+b+1)$. The factor $\phi_2(x)$ gives us a single side of height 1. And we have

   (a) If $v_3(b) \not\equiv 0 (\mathrm{mod}\ 2)$ and $v_3(b) \not\equiv 0 (\mathrm{mod}\ 3)$, then $N^+_{\phi_1}(F) = S_1$ has a single side of degree 1, then we will have $3\mathfrak{o}_K = \mathfrak{p}_1^6\mathfrak{p}_2$ with $f_1 = f_2 = 1$. Hence, 3 does not divide $i(K)$.

(b) If $v_3(b) \equiv 0 (\text{mod } 2)$ and $v_3(b) \not\equiv 0 (\text{mod } 3)$, then $N^+_{\phi_1}(F) = S_1$ has a single side of degree 2, thus we will have either two prime ideals of degree residual 1 or one prime ideal of degree residual 2. Then we will have $3\mathfrak{o}_K = \mathfrak{p}^3_{11}\mathfrak{p}^3_{12}\mathfrak{p}_2$ with $f_{11} = f_{12} = f_2 = 1$ or $3\mathfrak{o}_K = \mathfrak{p}^3_1\mathfrak{p}_2$ with $f_1 = 2$ et $f_2 = 1$. Finally, 3 does not divide $i(K)$.

(c) If $v_3(b) \equiv 0 (\text{mod } 3)$ and $v_3(b) \not\equiv 0 (\text{mod } 2)$, then $N^+_{\phi_1}(F) = S_1$ has a single side of degree 3, $R_{\lambda_1}(F)(y) = -y^3 + b_3 = (-y \pm 1)^3$ in $F_{\phi_1}[y]$. Hence, $R_{\lambda_1}(F)(y)$ is not separable in $F_{\phi_1}[y]$.
We use the Newton polygon of $F(x)$ of second order, we have the ramification index of S_1 is $e_1 = 2$ and the slope of $F(x)$ is $\lambda_1 = \frac{v_3(b)}{-6}$. Put $F(x) = x^7 + ax^6 + 3^m.c$, where $m = v_3(b)$ and 3 is not divide c. Also set a key polynomial $\psi(x) = x^2 - 3^{\frac{m}{3}}.c$ where 3 is not divide c. Let the ψ-adic development of $F(x)$ is $F(x) = (x + a)\psi(x)^3 + (3.3^{\frac{m}{3}}cx + 3a.3^{\frac{m}{3}}.c)\psi(x)^2 + (3.3^{2\frac{m}{3}}c^2x + 3a.3^{2\frac{m}{3}}.c^2)\psi(x) + (3^mc^3x + 3^mc + a.3^m.c)$ Let, $u_0 = v^2_3(a_0)$, $u_1 = v^2_3(a_1)$, $u_2 = v^2_3(a_2)$ and $u_3 = v^2_3(a_2)$, then

$$\begin{aligned}
u_0 &= v^2_3(3^mc^3x + 3^mc + a.3^m.c) = 2m + 2min(\frac{m}{6}, v_3(c + ac)),\\
u_1 &= v^2_3((3.3^{2\frac{m}{3}}c^2x + 3a.3^{2\frac{m}{3}}.c^2)\psi(x)) = 2m + 2,\\
u_2 &= v^2_3((3.3^{\frac{m}{3}}cx + 3a.3^{\frac{m}{3}}.c)\psi(x)^2) = 2m + 2,\\
u_3 &= v^2_3((x + a)\psi(x)^3) = 2m.
\end{aligned}$$

Then we have two cases:
If $min(\frac{m}{6}, v_3(c + ac)) = 1$, $N^2_\psi(F) = S$ has one side of degree $deg(S_1) = 1$. Then we will have $3\mathfrak{o}_K = \mathfrak{p}_2.\mathfrak{p}^3_\psi$ with $f(\mathfrak{p}_\psi) = f_{\mathfrak{p}_2} = 1$.
If $min(\frac{m}{6}, v_3(c+ac)) \geq 2$, then $N^2_\psi(F) = S_{11}+S_{12}$ has two sides of degree $deg(S_1) = 1$ and $deg(S_2) = 2$. Then, we have $3\mathfrak{o}_K = \mathfrak{p}_2.\mathfrak{p}_{11}\mathfrak{p}^2_{12}$ with $f_2 = f_{11} = 1$ and $f_{12} = 2$. Hence, 3 does not divide $i(K)$.

3. If $a \equiv 0 (\text{mod } 3)$, then $F(x) \equiv x^7 (\text{mod } 3)$, let $\phi = x$. In this case, $F(x) = \phi^7(x)+a\phi^6(x)+b$. Therefore, $v_3(b) \leq 6$, as $a \equiv 0 (\text{mod } 3)$, therefore $v_3(b) \leq 6$. Then, $N^+_\phi(F) = S$ has a single side of degree 1, and we will have $3\mathfrak{o}_K = \mathfrak{p}^7$ with $f_1 = 1$. Hence 3 does not divide $i(K)$.

For $p = 5$, since $\triangle(F) = -b^5(7^7.b + 2^6 \times 3^6 \times a^7)$ and thanks to the index formula $\triangle(F) = \pm[\mathfrak{o}_K : \mathbb{Z}[\theta]]^2 \times d_K$; if $(a,\ b) \notin \{(k, 0);\ 0 \leq k \leq 4\ , (3, 1),\ (4,\ 2),\ (1,\ 3),\ (2,\ 4)\} (\text{mod } 5)$, then 5 does not divide $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$, and so 5 does not divide $i(K)$.
Assume that :

$$(a,\ b) \in \{(k, 0);\ 0 \leq k \leq 4\ , (3, 1),\ (4,\ 2),\ (1,\ 3),\ (2,\ 4)\} (\text{mod } 5)$$

We obtain the following cases:

1. If $(a,\ b) \in \{(3, 1),\ (4,\ 2),\ (1,\ 3),\ (2,\ 4)\} (\text{mod } 5)$, then $F(x) \equiv \phi^2_1 \cdot \phi_2 U(x) (\text{mod } 5)$ with ϕ_1, ϕ_2, and $U(x)$ are monic irreducible coprime polynomials over $\mathbb{F}_5$, $\deg(\phi_1) = \deg(\phi_2) = 1$,

and $\deg(U(x)) = 4$. Recall that 5 divides $i(K)$ if and only if there are at least six prime ideals of $\mathfrak{o}_K$ lying above 5 with a residue degree of 1 each. In this case, there are at most three prime ideals of $\mathfrak{o}_K$ lying above 5, with residue degree 1 for each. Hence, 5 does not divide $i(K)$.

2. If $(a,\ b) \equiv (0,\ 0) (\bmod\ 5)$, then $F(x) \equiv x^7 (\bmod\ 5)$. Let $\phi = x$, then $F(x) = \phi^7 + a\phi^6 + b$. As $v_5(a) \geq 1$, $N_\phi^+(F) = S$ has a single side of degree 1, because $v_5(b) \leq 6$ by assumption. Thus, $5\mathfrak{o}_K = \mathfrak{p}^7$ with a residual degree of 1. Hence, 5 does not divide $i(K)$.

3. If $v_5(a) = 0$, then we have the following cases:

   (a) If $v_5(b) \not\equiv 0 (\bmod\ 2)$ and $v_5(b) \not\equiv 0 (\bmod\ 3)$, then $N_\phi^+(F) = S$ has one side joining $(0,\ v_5(b)), (6, 0)$. In this case, there is one prime ideal of $\mathfrak{o}_K$ lying above 5, with a residue degree of 1. Hence, 5 does not divide $i(K)$.

   (b) If $v_5(b) \equiv 0 (\bmod\ 2)$ and $v_5(b) \not\equiv 0 (\bmod\ 3)$, then $N_\phi^+(F) = S$ has one side joining $(0,\ v_5(b)), (6, 0)$. In this case, there is at most two prime ideals of $\mathfrak{o}_K$ lying above 5 with residue degree 1 each. Hence, 5 does not divide $i(K)$.

   (c) If $v_5(b) \equiv 0 (\bmod\ 3)$ and $v_5(b) \not\equiv 0 (\bmod\ 2)$, then $N_\phi^+(F) = S$ has one side joining $(0,\ v_5(b)), (6, 0)$. In this case, there is at most three prime ideals of $\mathfrak{o}_K$ lying above 5 with residue degree 1 each. Hence, 5 does not divide $i(K)$. We conclude that in every case, 5 does not divide $i(K)$.

This completes this proof.

□

Remark 3. *Now, we ask why 7 does not divide $i(K)$?. Suppose there exist distinct non-zero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of $\mathfrak{o}_K$ such that $7\mathfrak{o}_K = \mathfrak{p}_1^{e_1} \times \cdots \times \mathfrak{p}_r^{e_r}$ where $e_i \geq 1,\ i = 1, \ldots, r$, then by the Fundamental Equality, $e_1 \times f_1 + \cdots + e_r \times f_r = 7$ where $f_i,\ i = 1, \ldots, r$, is the residual degree of p_i for $i = 1, 2, \ldots, r$. Since $e_i \geq 1$ for all $i = 1, 2, \ldots, r$, therefore, there can be at most 5 prime ideals lying above 5, but for every positive integer h the number of monic irreducible polynomials of degree h in $\mathbb{F}_7[Y]$ is greater than or equal to 7. So by Lemma 1 , 7 does not divide $i(K)$.*

5 Examples

Let $F(x)$ be a monic irreducible polynomial and K the number field defined by a complex root of $F(x)$.

1. If $F(x) = x^7 + 5x^6 + 20$. As $F(x)$ is 5-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. So, by Theorem 1 (2), K is monogenic.

2. If $F(x) = x^7 + 7x^6 + 14$. As $F(x)$ is 7-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. So, by Theorem 1 (2), K is monogenic.

3. If $F(x) = x^7 + 20x^6 + 30$. As $F(x)$ is 5-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. So, by Theorem 1 (3), K is monogenic.

4. If $F(x) = x^7 + 10x^6 + 66$. As $F(x)$ is 2-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. So, by Theorem 1 (3), K is monogenic.

5. If $F(x) = x^7 + 3x^6 + 12$. As $F(x)$ is 3-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. So, by Theorem 1 (3), K is monogenic.

6. If $F(x) = x^7 + 15x^6 + b$ where $b = 2^k \times 5$ and $k \geq 12$. As $F(x)$ is 5-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. Thus, according to Theorem 2 (1), K is not monogenic.

7. If $F(x) = x^7 + 77x^6 + 90112$. As $F(x)$ is 11-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. Thus, according to Theorem 2 (2), K is not monogenic.

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