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On the genus and crosscap of the total graph of commutative rings with respect to multiplication

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Abstract. Let $\mathcal{S}$ be a commutative ring and $Z(\mathcal{S})$ be its zero-divisors set. The total graph of $\mathcal{S}$ with respect to multiplication, denoted by $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}))$, is an undirected graph with vertex set as the ring elements $\mathcal{S}$ and two distinct vertices α and β are adjacent if and only if $\alpha\beta \in Z(\mathcal{S})$. The graph $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a subgraph of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}))$ with vertex set $\mathcal{S}^*$ (set of nonzero elements of $\mathcal{S}$). In this paper, we characterize finite rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ belongs to some well-known families of graphs. Further, we classify the finite rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is planar, toroidal or double toroidal. Finally, we analyze the finite rings $\mathcal{S}$ for which the graph $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ has crosscap at most two.

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Throughout this paper, all rings $\mathcal{S}$ are commutative rings with unity which are not fields. The set of zero-divisors and unit elements of $\mathcal{S}$ are denoted by $Z(\mathcal{S})$ and $U(\mathcal{S})$, respectively. We denote $A^* = A \setminus \{0\}$ for any subset A of $\mathcal{S}$. The element $e_i \in \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$ denotes $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ for each $1 \leq i \leq n$. For more details on ring theory, we refer the reader to [5].

A variety of graphs attached to rings or other algebraic structures can be found in the literature. Beck in [6] introduced by the first time a graph associated to a commutative ring $\mathcal{S}$ with the elements of ring $\mathcal{S}$ as its vertices, who was mainly interested in coloring. In [3], Anderson and Livingston introduced the zero-divisor graph of $\mathcal{S}$, denoted by $\Gamma(\mathcal{S})$, with vertex

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set $Z(\mathcal{S})^*$ and two vertices $\alpha \neq \beta \in Z(\mathcal{S})^*$ are adjacent if and only if $\alpha\beta = 0$ (see [1,2,11–15] for more details).

In [4], Ashraf et al. introduced and studied the total graph with respect to multiplication of a commutative ring $\mathcal{S}$, denoted by $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}))$. It is an undirected graph with vertex set $\mathcal{S}$ and two distinct vertices α and β are adjacent if and only if $\alpha\beta \in Z(\mathcal{S})$. The graph $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a subgraph of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}))$ with vertex set $\mathcal{S}^*$. The authors in [4], discussed some basic properties of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}))$ and proved that $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}))$ is a connected graph with diameter at most 2.

In this paper, we characterize the finite rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a tree, a unicycle and a split graph. Then we classify the finite rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a planar, ring, outerplanar, toroidal or double toroidal graph. Finally, we classify the finite rings $\mathcal{S}$ for which the graph $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ has crosscap at most two.

1 Preliminaries

Let $\mathcal{G}$ be a graph with vertex set $V(\mathcal{G})$. A *cycle* is a closed path in $\mathcal{G}$. A cycle with length n is denoted by C_n. A graph is said to be *complete graph* if all its vertices are adjacent to each other. A complete graph with n vertices is denoted by K_n. A *bipartite graph* is a graph $\mathcal{G}$ whose vertex set $V(\mathcal{G})$ can be partitioned into two subsets V_1 and V_2 such that every edge in $\mathcal{G}$ has one end in V_1 and the other end in V_2. Further, if each vertex of V_1 is adjacent to every vertex of V_2, then $\mathcal{G}$ is called a *complete bipartite graph*. The complete bipartite graph with partition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. We write $K_{m,\infty}$ (respectively, $K_{\infty,\infty}$) if one (respectively, both) of the disjoint vertex sets is infinite. The *complement* of $\mathcal{G}$, denoted by $\overline{\mathcal{G}}$, is a graph with vertex set $V(G)$ and $x \sim y$ is an edge of $\overline{\mathcal{G}}$ if and only if $x \sim y$ is not an edge of $\mathcal{G}$. The *join* of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \vee \mathcal{G}_2$, is a graph formed from disjoint copies of $\mathcal{G}_1$ and $\mathcal{G}_2$ by connecting each vertex of $\mathcal{G}_1$ to every vertex of $\mathcal{G}_2$. A connected graph $\mathcal{G}$ is said to be a *tree* if it does not contain cycles. A graph $\mathcal{G}$ is said to be *unicycle* whenever it contains a unique cycle. A graph $\mathcal{G}$ is said to be a *split graph* if the vertex set of $\mathcal{G}$ can be partitioned into a clique and an independent set. A graph is *planar* whenever it can be drawn in the plane in such a way that its edges intersect only at their endpoints. A *subdivision* of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. An undirected graph $\mathcal{G}$ is said to be *outerplanar* if it can be embedded in the plane in such a way that all the vertices lie on the unbounded face of the drawing. For more details on graph theory, we refer the reader to [16,17].

In the following table some finite local rings and their zero-divisor sets are given:

2 Basic properties of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$

In this section, we classify the finite rings for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a unicycle graph, a tree or a split graph.

Lemma 1 (Theorem VI-2 [9])**.** *Let $\mathcal{S}$ be a finite commutative ring. Then $\mathcal{S}$ decomposes (up to order of summands) uniquely as a direct sum of local rings.*

Theorem 1. *Let $\mathcal{S}$ be a finite commutative ring, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a unicycle graph if and only if $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Table 1: Zero-divisors set of some finite local commutative rings

$\lvert\mathcal{S}\rvert$	Local Ring $\mathcal{S}$	$\lvert Z(\mathcal{S})^*\rvert$
4	$\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$	1
9	$\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$	2
8	$\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$	3
16	$\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$	3
25	$\mathbb{Z}_{25}$, $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$	4
49	$\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{\langle x^2\rangle}$	6

Proof. Assume that $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a unicycle graph. Since $\mathcal{S}$ is finite, by Lemma 1, $\mathcal{S}\cong\mathcal{S}_1\times\mathcal{S}_2\times\cdots\times\mathcal{S}_n$, where $\mathcal{S}_i$ is a local ring for each i and $n\geq 1$. If $n\geq 3$, then $e_1\sim e_2\sim e_1+e_2+e_3\sim e_1$ and $e_3\sim e_1+e_2\sim e_1+e_3\sim e_3$ are two different cycles in $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$, a contradiction. Hence $n\leq 2$.
Suppose $n=2$. If $|\mathcal{S}_1|\geq 3$, then $(1,0)\sim(0,1)\sim(1,1)\sim(1,0)$ and $(\alpha,0)\sim(0,1)\sim(\alpha,1)\sim(\alpha,0)$, where $1\neq\alpha\in\mathcal{S}_1^*$, are two different cycles in $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$, a contradiction. Hence $|\mathcal{S}_1|=2$. Similarly, one can show that $|\mathcal{S}_2|=2$. This shows that $\mathcal{S}\cong\mathbb{Z}_2\times\mathbb{Z}_2$.
Suppose $n=1$. Then $\mathcal{S}$ is a local ring. If $|Z(\mathcal{S})^*|\geq 2$, then the subgraph $K_2\vee\overline{K}_m$ of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains at least two different cycles, where $m=|U(\mathcal{S})|\geq 3$, a contradiction. Hence $|Z(\mathcal{S})^*|=1$. Thus, $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))\cong K_{1,2}$ does not contain any cycle, again a contradiction.
If $\mathcal{S}\cong\mathbb{Z}_2\times\mathbb{Z}_2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))\cong K_3$ is a unicycle graph. □

Theorem 2. *Let $\mathcal{S}$ be a finite commutative ring, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a tree if and only if $\mathcal{S}\cong\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$.*

Proof. Assume that $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a tree. Since $\mathcal{S}$ is finite, by Lemma 1, $\mathcal{S}\cong\mathcal{S}_1\times\mathcal{S}_2\times\cdots\times\mathcal{S}_n$, where $\mathcal{S}_i$ is a local ring for each i and $n\geq 1$. If $n\geq 2$, then $e_1\sim e_1+e_2\sim e_2$ is a cycle in $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$, a contradiction. Hence $n=1$, that is $\mathcal{S}$ is a local ring. If $|Z(\mathcal{S})^*|\geq 2$, then the subgraph $K_2\vee\overline{K}_m$ of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains a cycle, where $m=|U(\mathcal{S})|\geq 3$, again a contradiction. Hence $|Z(\mathcal{S})^*|=1$, which implies that $\mathcal{S}\cong\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$.
Conversely, if $\mathcal{S}\cong\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))\cong K_{1,2}$, which is a tree.

□

Theorem 3 ([7])**.** *Let $\mathcal{G}$ be a connected graph, then $\mathcal{G}$ is a split graph if and only if $\mathcal{G}$ contains no induced subgraph isomorphic to $2K_2$, C_4, or C_5.*

Theorem 4. *Let $\mathcal{S}$ be a finite commutative ring, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a split graph if and only if $\mathcal{S}\cong\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$ or $\mathbb{Z}_2\times\mathbb{Z}_2$.*

Proof. Assume that $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a split graph. Since $\mathcal{S}$ is finite, by Lemma 1, $\mathcal{S}\cong\mathcal{S}_1\times\mathcal{S}_2\times\cdots\times\mathcal{S}_n$, where $\mathcal{S}_i$ is a local ring for each i and $n\geq 1$. If $n\geq 3$, then $e_1\sim e_2\sim e_3\sim e_1+e_2\sim e_1$ is C_4 in $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$, a contradiction by Theorem 3. Hence $n\leq 2$.

Suppose $n = 2$. If $|\mathcal{S}_1| \geq 3$, then $(1,0) \sim (\alpha, 0) \sim (0,1) \sim (1,1) \sim (1,0)$, where $1 \neq \alpha \in \mathcal{S}_1^*$, is C_4 in $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$, a contradiction by Theorem 3. Hence $|\mathcal{S}_1| = 2$. Similarly, one can show that $|\mathcal{S}_2| = 2$. This shows that $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Suppose $n = 1$ that is, $\mathcal{S}$ is a local ring. If $|Z(\mathcal{S})^*| \geq 2$, then the subgraph $K_2 \vee \overline{K}_m$ of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains C_4, where $m = |U(\mathcal{S})| \geq 3$, a contradiction by Theorem 3. Hence $|Z(\mathcal{S})^*| = 1$, which implies that $\mathcal{S} \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.
Conversely, if $\mathcal{S} \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_{1,2}$ is a split graph. If $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_3$ is also a split graph by Theorem 3. □

Theorem 5 (Kuratowski's theorem [16]). *A graph $\mathcal{G}$ is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Theorem 6. *Let $\mathcal{S}$ be a finite commutative ring, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a planar graph if and only if $\mathcal{S} \cong \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$.*

Proof. Assume that $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a planar graph. Since $\mathcal{S}$ is finite, by Lemma 1, $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$, where $\mathcal{S}_i$ is a local ring for each i and $n \geq 1$. If $n \geq 3$, then the subgraph induced by the set $\{e_1, e_2, e_3, e_1 + e_2, e_1 + e_3\}$ is K_5 in $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$, a contradiction by Theorem 5. Hence $n \leq 2$.
Suppose $n = 2$. If $|\mathcal{S}_1| \geq 4$, then the subgraph induced by the set $\{(1,0), (\alpha, 0), (0,1), (\beta, 0), (1,1)\}$, where $1 \neq \alpha, \beta \in \mathcal{S}_1^*$, is K_5 in $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$, a contradiction by Theorem 5. Hence $|\mathcal{S}_1| \leq 3$. Similarly, $|\mathcal{S}_2| \leq 3$. If $|\mathcal{S}_1| = |\mathcal{S}_2| = 3$, then the subgraph induced by the set $\{(1,0), (\delta, 0), (0,1), (0, \omega)$ $, (1,1)\}$, where $1 \neq \delta \in \mathcal{S}_1^*$ and $1 \neq \omega \in \mathcal{S}_2^*$, is K_5 in $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$, a contradiction by Theorem 5. Hence $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$.
Suppose $n = 1$ that is $\mathcal{S}$ is a local ring. If $|Z(\mathcal{S})^*| \geq 3$, then the subgraph $K_3 \vee \overline{K}_m$ of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains $K_{3,3}$, where $m = |U(\mathcal{S})| \geq 4$, a contradiction. Hence $|Z(\mathcal{S})^*| \leq 2$. This shows that $\mathcal{S} \cong \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$.
Conversely, if $\mathcal{S} \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_{1,3}$ which is a planar graph. If $\mathcal{S} \cong \mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_2 \vee \overline{K}_6$ which is a planar graph by Theorem 5. If $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_3$ is a planar graph. If $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, then planar embedding of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is shown in Figure 1. □

Let C be a cycle of $\mathcal{G}$. Any edge in $\mathcal{G}$ that connects two nonadjacent vertices in C is called a *chord*. A *primitive cycle* is one that has no chords. Furthermore, the graph $\mathcal{G}$ has the *primitive cycle property* **(PCP)** if any two primitive cycles intersect in at most one edge. The *frank* of $\mathcal{G}$ is denoted by $frank(\mathcal{G})$ and equals the number of primitive cycles of $\mathcal{G}$. Also, $rank(\mathcal{G}) = q - n + \mathcal{A}$, where q, n, and $\mathcal{A}$ denote the number of edges, vertices, and connected components of $\mathcal{G}$, respectively. Section 2 of [8] contains a detailed definition of a ring graph. The authors in [8] also demonstrated that the following conditions are equivalent:

Theorem 7 (Theorem 2.13 [8]). *If $\mathcal{G}$ is a connected graph, then following are equivalent:*

1. *$\mathcal{G}$ is a ring graph,*
2. *$rank(\mathcal{G}) = frank(\mathcal{G})$,*

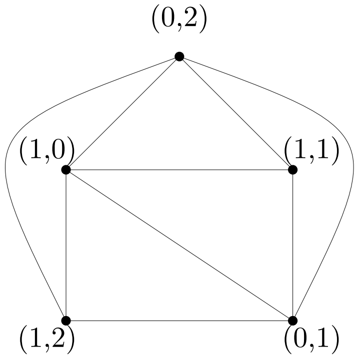

Figure 1: Planar embedding of $T_{Z(\mathcal{S})}(\Gamma((\mathbb{Z}_2 \times \mathbb{Z}_3)^*))$.

3. $\mathcal{G}$ satisfies PCP and $\mathcal{G}$ does not contain a subdivision of K_4 as a subgraph.

As a result, each ring graph is planar. In the following theorem, we characterize finite commutative rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a ring graph.

Theorem 8. *Let $\mathcal{S}$ be a finite commutative ring, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a ring graph if and only if $\mathcal{S} \cong \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Since every ring graph is a planar graph, it is enough to deal with rings for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a planar graph. If $\mathcal{S} \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_{1,2}$ is a ring graph by Theorem 7. If $\mathcal{S} \cong \mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_2 \vee \overline{K}_6$ is a ring graph by Theorem 7. If $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_3$ is again a ring graph by Theorem 7. If $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_4 induced by the set $\{(1,0),(0,1),(0,2),(1,1)\}$. Thus, $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is not a ring graph. □

Theorem 9 ([16]). *A graph $\mathcal{G}$ is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.*

In the next theorem, we determine finite commutative rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is an outerplanar graph.

Theorem 10. *Let $\mathcal{S}$ be a finite commutative ring, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is an outerplanar graph if and only if $\mathcal{S} \cong \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. In view of Theorems 7 and 9, one can say that every outerplanar graph is a ring graph. Thus it is enough to deal with the rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a ring graph. Hence the results follows from Theorem 8. □

3 Genus of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$

In this section, we classify the finite commutative rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ has genus at most two.

The minimal integer k such that the graph can be drawn without crossing itself on a sphere with k handles (i.e., an oriented surface of genus k) is the *genus* of a graph $\mathcal{G}$, denoted by $\gamma(\mathcal{G})$. A planar graph has genus 0 because it can be drawn on a sphere without self-crossing. The following results deal with genus features of complete and complete bipartite graphs.

Lemma 2 (Theorem 6.37 [17]). $\gamma(K_m) = \lceil \frac{(m-3)(m-4)}{12} \rceil$ *if* $m \geq 3$. *In particular,* $\gamma(K_m) = 1$ *if* $m = 5, 6, 7$.

Lemma 3 (Theorem 6.38 [17]). $\gamma(K_{n,m}) = \lceil \frac{(n-2)(m-2)}{4} \rceil$ *if* $n, m \geq 2$. *In particular,* $\gamma(K_{4,4}) = \gamma(K_{3,m}) = 1$ *if* $m = 3, 4, 5, 6$. *Also* $\gamma(K_{5,4}) = \gamma(K_{6,4}) = \gamma(K_{m,3}) = 2$ *if* $m = 7, 8, 9, 10$.

Now, we are in position to characterize the finite commutative rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ has genus one.

Theorem 11. *Let* $\mathcal{S}$ *be a finite commutative ring, then* $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$ *if and only if* $\mathcal{S}$ *is isomorphic to one of the following rings:* $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2, xy, y^2 \rangle}$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Assume that $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$. Since $\mathcal{S}$ is finite, by Lemma 1, $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$, where $\mathcal{S}_i$ is a local ring for each i and $n \geq 1$. If $n \geq 4$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_8 induced by the set $\{e_1, e_2, e_3, e_4, e_1+e_2, e_1+e_3, e_1+e_4, e_2+e_3\}$. Thus by Lemma 2, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 1$, a contradiction. Hence $n \leq 3$. Consider the following cases:

Case(i) Suppose $n = 3$. If $|\mathcal{S}_1| \geq 3$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_8 as a subgraph induced by the set $\{(1,0,0), (\alpha,0,0), (0,1,0), (0,0,1), (1,1,0), (\alpha,1,0), (1,0,1), (\alpha,0,1)\}$, where $1 \neq \alpha \in \mathcal{S}_1^*$. Thus by Lemma 2, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 1$, a contradiction. Hence $|\mathcal{S}_1| = 2$. Similarly, one can show that $|\mathcal{S}_2| = |\mathcal{S}_3| = 2$. This shows that $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case(ii) Suppose $n = 2$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ both are non-reduced ring with nonzero maximal ideals $\Im_1$ and $\Im_2$, respectively. Then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_8 induced by the set $\{(1,0), (\beta_1,0), (\nu_1,0), (\beta_1,1), (0,1), (0,\nu_2), (0,\beta_2), (1,\beta_2)\}$, where $\beta_i \in \Im_i^*$ and $1 \neq \nu_i \in U(\mathcal{S}_i)$ for $i = 1, 2$. Thus, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 1$ by Lemma 2, a contradiction. Hence $\mathcal{S}_1$ or $\mathcal{S}_2$ is a reduced ring. Consider the following subcases:

Subcase(a) Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ both are reduced. If $|\mathcal{S}_i| \geq 4$ for each $i = 1, 2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains $K_{5,4}$ induced by the set $\{(1,0), (\delta_1,0), (\delta_2,0), (0,1), (0,\omega_2)\} \cup \{(1,1), (1,\omega_1), (1,\omega_2), (\delta_1,1)\}$, where $1 \neq \delta_1, \delta_2 \in \mathcal{S}_1^*$ and $1 \neq \omega_1, \omega_2 \in \mathcal{S}_2^*$. Thus, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 1$ by Lemma 3, a contradiction. Hence without loss of generality, we can assume that $|\mathcal{S}_1| \leq 3$ and $|\mathcal{S}_2| \leq 4$. If $|\mathcal{S}_1| = 2$ and $|\mathcal{S}_2| \leq 3$, then by Theorem 6, $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a planar graph. If $|\mathcal{S}_1| = 3$ and $|\mathcal{S}_2| = 4$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains $K_{5,4}$ induced by the set $\{(1,0), (\lambda,0), (0,\xi_1), (0,1), (0,\xi_2)\} \cup \{(1,1), (1,\xi_1), (1,\xi_2), (\lambda,1)\}$, where $1 \neq \lambda \in \mathcal{S}_1^*$ and $1 \neq \xi_1, \xi_2 \in \mathcal{S}_2^*$, a contradiction by Lemma 3. Hence $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{F}_4$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Subcase(b) Suppose $\mathcal{S}_1$ is a non-reduced ring with maximal ideal $\Im \neq (0)$ and $\mathcal{S}_2$ is a reduced ring. If $|\Im^*| \geq 2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_8 induced by the set $\{(1,0), (\pi,0), (\epsilon,0), (\mu_1,0),$

$(\mu_2,0),(\pi,1),(\epsilon,1),(0,1)\}$, where $\pi,\epsilon\in\Im^*$ and $1\neq\mu_1,\mu_2\in U(\mathcal{S}_1)$. Thus $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)))\geq 3$ by Lemma 2, a contradiction. Hence $|\Im_1^*|=1$, which shows that $\mathcal{S}_1\cong\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$.

Finally, if $|\mathcal{S}_2|\geq 3$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_8 induced by the set $\{(1,0),(\zeta,0),(\nu,0),(0,\mu),(1,1),(0,1),(\zeta,1),(\zeta,\mu)\}$, where $\zeta\in\Im^*$, $1\neq\nu\in U(\mathcal{S}_1)$ and $1\neq\mu\in\mathcal{S}_2^*$, a contradiction by Lemma 2. Hence $|\mathcal{S}_2|=2$, which implies that $\mathcal{S}_2\cong\mathbb{Z}_2$.

Case(iii) Suppose $n=1$ that is, $\mathcal{S}$ is a local ring. If $|Z(\mathcal{S})^*|\geq 4$, then the subgraph $K_4\vee\overline{K}_m$ of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains $K_{4,5}$, where $m=|U(\mathcal{S})|\geq 5$, a contradiction by Lemma 3. Hence $|Z(\mathcal{S})^*|\leq 3$. If $|Z(\mathcal{S})^*|\leq 2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a planar graph by Theorem 6. Thus, $|Z(\mathcal{S})^*|=3$, which implies that $\mathcal{S}\cong\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$. Finally, if $\mathcal{S}\cong\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))\cong K_3\vee\overline{K}_{12}$ contains $K_{3,7}$ as a subgraph, a contradiction by Lemma 3.
Conversely, if $\mathcal{S}\cong\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$ or $\mathbb{Z}_2\times\mathbb{F}_4$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))\cong K_3\vee\overline{K}_4$. Thus, by Lemmas 2 and 3,
$\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)))=1$. If $\mathcal{S}\cong\mathbb{Z}_3\times\mathbb{Z}_3$, then $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)))=1$ by Figure 2. If $\mathcal{S}\cong\mathbb{Z}_4\times\mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}\times\mathbb{Z}_2$, then the toroidal embedding of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is shown in Figure 3. If $\mathcal{S}\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))\cong K_7$. Hence $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)))=1$ by Lemma 2. □

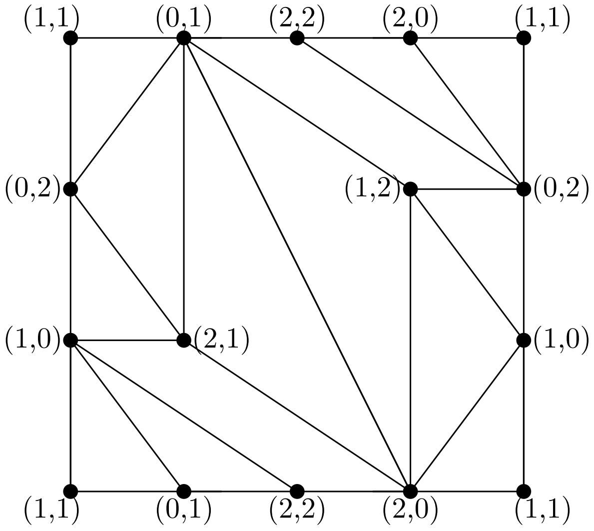

Figure 2: Toroidal embedding of $T_{Z(\mathcal{S})}(\Gamma((\mathbb{Z}_3\times\mathbb{Z}_3)^*))$.

We end this section with the classification of finite commutative rings $\mathcal{S}$ with genus two $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$.

Theorem 12. *Let $\mathcal{S}$ be a finite commutative ring, then*
$\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)))\neq 2$.

Proof. Since $\mathcal{S}$ is finite, by Lemma 1, $\mathcal{S}\cong\mathcal{S}_1\times\mathcal{S}_2\times\cdots\times\mathcal{S}_n$, where $\mathcal{S}_i$ is a local ring for each i and $n\geq 1$. If $n\geq 4$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_9 induced by the set $\{e_1,e_2,e_3,e_4,e_1+$

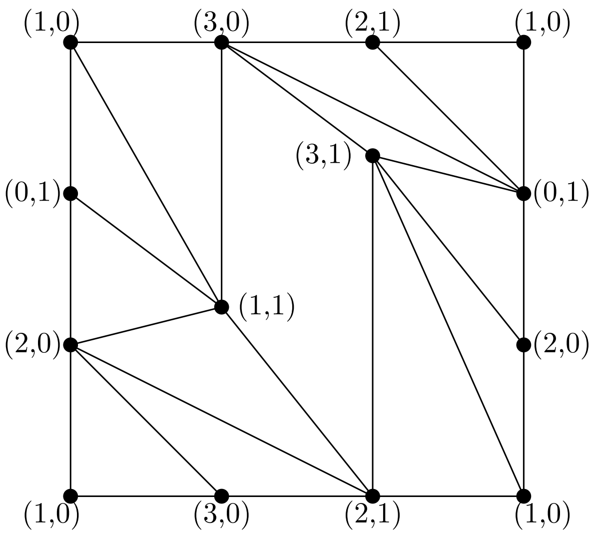

Figure 3: Toroidal embedding of $T_{Z(\mathcal{S})}(\Gamma((\mathbb{Z}_4 \times \mathbb{Z}_2)^*)) \cong T_{Z(\mathcal{S})}(\Gamma((\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2)^*))$.

$e_2, e_1 + e_3, e_1 + e_4, e_2 + e_3, e_2 + e_4\}$. Thus, by Lemma 2, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 2$. Hence $n \leq 3$. Consider the following cases:

Case(i) Suppose $n = 3$. If $|\mathcal{S}_1| \geq 3$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_9 induced by the set $\{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1), (1,1,1), (\alpha,0,0), (\alpha,1,0)\}$, where $0 \neq \alpha \in \mathcal{S}_1^*$. Thus, by Lemma 2, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 2$. Hence $|\mathcal{S}_i| = 2$ for each $i = 1,2,3$. This implies that $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$ by Theorem 11.
Case(ii) Suppose $n = 2$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ both are non-reduced ring with nonzero maximal ideals $\Im_1$ and $\Im_2$ respectively. Then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_9 induced by the set $\{(1,0), (\beta_1,0), (\nu_1,0), (\beta_1,1), (0,1), (0,\nu_2), (0,\beta_2), (1,\beta_2), (\beta_1,\beta_2)\}$, where $\beta_i \in \Im_i^*$ and $1 \neq \nu_i \in U(\mathcal{S}_i)$ for $i = 1,2$. Thus, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 2$ by Lemma 2. Hence $\mathcal{S}_1$ or $\mathcal{S}_2$ is a reduced ring. Consider the following subcases:

Subcase(a) Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ both are reduced. If $|\mathcal{S}_i| \geq 4$ for each $i = 1,2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains $K_{5,6}$ induced by the set $\{(1,0), (\omega_1,0), (\omega_2,0), (0,1), (0,\delta_1), (0,\delta_2)\} \cup \{(1,1), (1,\delta_1), (1,\delta_2), (\omega_1,1), (\omega_2,1)\}$, where $1 \neq \omega_1, \omega_2 \in \mathcal{S}_1^*$ and $1 \neq \delta_1, \delta_2 \in \mathcal{S}_2^*$. Thus, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 2$ by Lemma 3. Hence without loss of generality, we can assume that $|\mathcal{S}_1| \leq 3$ and $|\mathcal{S}_2| \leq 4$. If $|\mathcal{S}_1| = 2$ and $|\mathcal{S}_2| \leq 3$, then by Theorem 6, $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a planar graph. If $|\mathcal{S}_1| = 2$ and $|\mathcal{S}_2| = 4$, then by Theorem 11, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$. If $|\mathcal{S}_1| = 3$ and $|\mathcal{S}_2| = 3$, then again by Theorem 11, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$. If $|\mathcal{S}_1| = 3$ and $|\mathcal{S}_2| = 4$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains $K_{5,6}$ induced by the set $\{(1,0), (\xi,0), (0,\zeta_1), (0,1), (0,\zeta_2)\} \cup \{(1,1), (1,\zeta_1), (1,\zeta_2), (\xi,1), (\xi,\zeta_1), (\xi,\zeta_2)\}$, where $1 \neq \xi \in \mathcal{S}_1^*$ and $1 \neq \zeta_1, \zeta_2 \in \mathcal{S}_2^*$. Thus, by Lemma 3, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 2$.
Subcase(b) Suppose $\mathcal{S}_1$ is a non-reduced ring with maximal ideal $\Im \neq (0)$ and $\mathcal{S}_2$ is a reduced ring. If $|\Im^*| \geq 2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_9 induced by the set $\{(1,0), (\pi,0), (\theta,0), (\kappa_1,0), (\kappa_2,0), (\pi,1), (\theta,1), (0,1), (1,1)\}$, where $\pi, \theta \in \Im^*$ and $1 \neq \kappa_1, \kappa_2 \in U(\mathcal{S}_1)$. Thus,

$\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 2$ by Lemma 2. Hence $|\Im_1^*| = 1$, which shows that $\mathcal{S}_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.

If $|\mathcal{S}_2| \geq 3$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains K_8 induced by the set $\{(1,0), (\epsilon,0), (\upsilon,0), (0,\vartheta), (0,1), (\epsilon,1)\} \cup \{(\epsilon,\vartheta), (1,1), (\upsilon,1), (\upsilon,\vartheta), (1,\vartheta)\}$, where $\epsilon \in \Im^*$, $1 \neq \upsilon \in U(\mathcal{S}_1)$ and $1 \neq \vartheta \in \mathcal{S}_2^*$. Thus $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 2$ by Lemma 2. Hence $|\mathcal{S}_2| = 2$, which implies that $\mathcal{S}_2 \cong \mathbb{Z}_2$. Thus, by Theorem 2, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$.

Case(iii) Suppose $n = 1$ that is, $\mathcal{S}$ is a local ring. If $|Z(\mathcal{S})^*| \geq 4$, then the subgraph $K_4 \vee \overline{K}_m$ of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ contains $K_{4,7}$, where $m = |U(\mathcal{S})| \geq 8$. Thus, by Lemma 3, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 2$. Hence $|Z(\mathcal{S})^*| \leq 3$. If $|Z(\mathcal{S})^*| \leq 2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is a planar graph by Theorem 6. If, $|Z(\mathcal{S})^*| = 3$ and $|U(\mathcal{S})| = 4$, then by Theorem 2, $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$. If $|Z(\mathcal{S})^*| = 3$ and $|U(\mathcal{S})| = 12$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_3 \vee \overline{K}_{12}$ contains $K_{3,12}$. Thus, by Lemma 3 $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 2$.

□

4 Crosscap of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$

In this section, we characterize the finite commutative rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ has crosscap at most two.

Let N_k denote the sphere with k crosscaps, where k is a non-negative integer, that is, N_k is a non-oriented surface with k crosscaps. The *crosscap number* of a graph $\mathcal{G}$, denoted by $\overline{\gamma}(\mathcal{G})$, is the minimal integer k such that the graph $\mathcal{G}$ can be embedded in N_k. Intuitively, $\mathcal{G}$ is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. It is easy to see that $\overline{\gamma}(H) \leq \overline{\gamma}(\mathcal{G})$ for all subgraphs H of $\mathcal{G}$. The crosscap of various particular types of graphs are given in the following results, which are useful for proving the results of this section.

Lemma 4 (Theorems 4.4.5 and 4.4.6 [10]). *If $m \geq 3$, then*

$$\overline{\gamma}(K_m) = \begin{cases} \left\lceil \frac{(m-3)(m-4)}{6} \right\rceil & \text{if } m \geq 3 \text{ and } m \neq 7; \\ 3 & \text{if } m = 7. \end{cases}$$

Lemma 5 (Theorem 4.4.7 [10]). *If $n, m \geq 2$, then*

$$\overline{\gamma}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil.$$

Lemma 6 (Proposition 4.4.4 [10]). *If $\mathcal{G}$ is a connected graph with q edges and $m \geq 3$ vertices, then*

$$\overline{\gamma}(\mathcal{G}) \geq \left\lceil \tfrac{q}{3} - m + 2 \right\rceil.$$

Now, we are in position to characterize the finite commutative rings $\mathcal{S}$ with crosscap at most two $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$.

Theorem 13. *Let $\mathcal{S}$ be a finite commutative ring, then $\overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$ if and only if $\mathcal{S}$ is isomorphic to of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2, xy, y^2 \rangle}$ or $\mathbb{Z}_2 \times \mathbb{F}_4$.*

Proof. Since $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) \leq \overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)))$, it is enough to deal with the rings $\mathcal{S}$ for which $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$. If $\mathcal{S} \cong \mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$ or $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_3 \vee \overline{K}_4$. Thus by Figure 4, $\overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$. If $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{F}_4$, then projective embedding of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ is shown in Figure 5. If $\mathcal{S} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_4 \vee \overline{K}_4$ contains $K_{4,4}$. Thus, by Lemma 5, $\overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 1$. If $\mathcal{S} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_5 \vee \overline{K}_2$ contains 20 edges and 7 vertices. Thus, by Lemma 6, $\overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 1$. Finally, if $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_7$. Thus, by Lemma 4, $\overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) > 1$. □

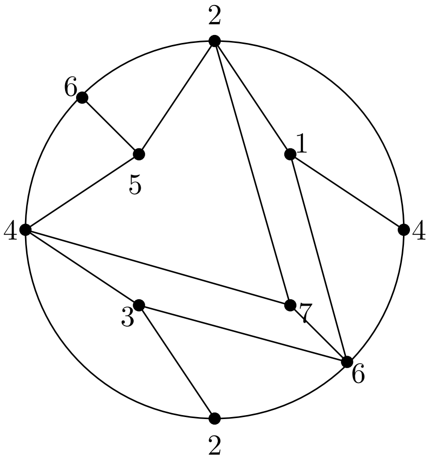

Figure 4: Projective embedding of $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$, where $\mathcal{S} \cong \mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$.

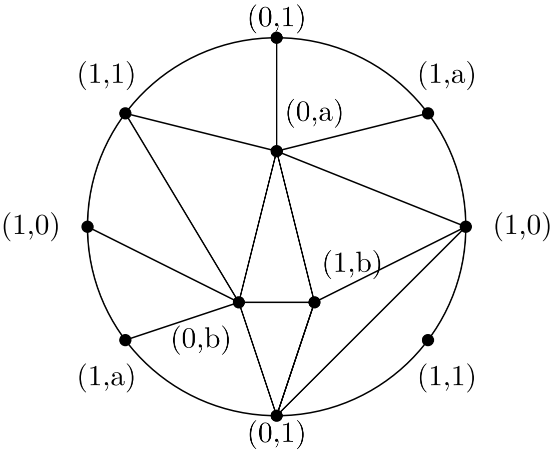

Figure 5: Projective embedding of $T_{Z(R)}(\Gamma((\mathbb{Z}_2 \times \mathbb{F}_4)^*))$.

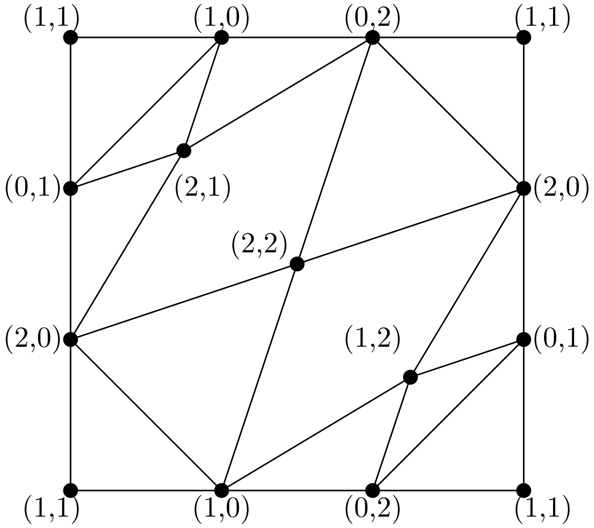

Figure 6: Embedding of $T_{Z(R)}(\Gamma((\mathbb{Z}_3 \times \mathbb{Z}_3)^*))$.

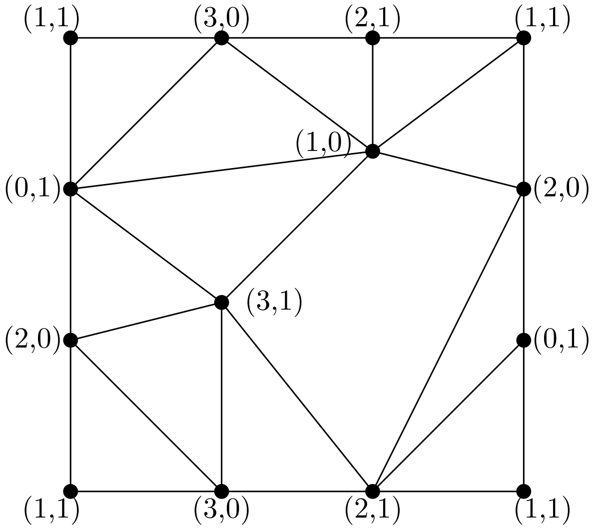

Figure 7: Embedding of $T_{Z(\mathcal{S})}(\Gamma((\mathbb{Z}_4 \times \mathbb{Z}_2)^*)) \cong T_{Z(\mathcal{S})}((\Gamma(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2)^*))$ on N_2.

Theorem 14. *Let $\mathcal{S}$ be a finite commutative ring, then $\overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 2$ if and only if $\mathcal{S} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$.*

Proof. Since $\gamma(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) \leq \overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)))$, it is enough to deal with the rings $\mathcal{S}$ for which $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))$ has genus at most two. It is clear from Theorem 13 that, if $\mathcal{S} \cong \mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2, xy, y^2 \rangle}$ or $\mathbb{Z}_2 \times \mathbb{F}_4$, then $\overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 1$. If $\mathcal{S} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_4 \vee \overline{K}_4$. Thus, by Figure 6, $\overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 2$. If $\mathcal{S} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_5 \vee \overline{K}_2$. Thus, by Figure 7, $\overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 2$. Finally, if $\mathcal{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*)) \cong K_7$. Thus, by Lemma 4, $\overline{\gamma}(T_{Z(\mathcal{S})}(\Gamma(\mathcal{S}^*))) = 3$.

□

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