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A GENERALIZATION OF r-SUBMODULES

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Abstract. Let R be a commutative ring with identity $1 \neq 0$ and M a non-zero unital R-module. In this paper, we present the concept of fully I-submodules of M such that I is an ideal of R which is a generalization of r-submodules. Consider that I is an ideal of R, a proper submodule N of M is a fully I-submodule if $JK \subseteq N$ with $\operatorname{ann}_M(J) = 0_M$ results that $K \subseteq N$ for each submodule K of M and each ideal J of R. In addition, we present the concept of fully special I-submodules which is a generalization of special r-submodules. A proper submodule N of M is a fully special I-submodule if the inclusion $IL \subseteq N$ with $\operatorname{ann}_R(L) = 0_R$, implies that $I \subseteq (N : M)$ for each submodule L of M and each ideal J of R. We explore certain outcomes related to these categories of submodules.

1. INTRODUCTION

In this paper, R is a commutative ring with a non-zero identity, and M is a unitary R-module. For each subset S of R, we denote by $\operatorname{ann}_M(S)$ the set of elements $m \in M$ such that $ma = 0$ for each $a \in S$. In particular, for $a \in R$, $\operatorname{ann}_M(a) = \{m \in M : am = 0\}$ is named an *annihilator submodule* of M. An element $a \in R$ is named a *zero-divisor* on M provided that there exists $0 \neq m \in M$ such that $am = 0$, that is $\operatorname{ann}_M(a) \neq 0$. We denote by $\mathrm{Z}_R(M)$ (for short $\mathrm{Z}(M)$) the collection of all zero-divisors of R on M, i.e., $\mathrm{Z}(M) = \{a \in R : \operatorname{ann}_M(a) \neq 0_M\}$. Note that the set $S = R - \mathrm{Z}_R(M)$ is a multiplicatively closed subset of R (i.e., $0 \notin S$, $1 \in S$ and for $a, b \in S$, $ab \in S$). If we consider R as an

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R-module, then we write $\mathrm{Z}(R)$ instead of $\mathrm{Z}_R(R)$. An R-module M is named a *McCoy module* provided that for every finitely generated ideal I of R with $I \subseteq \mathrm{Z}(M)$, $\mathrm{ann}_M(I) \neq 0$. This is a natural extension of the concept of a McCoy ring, see [6]. If N is an irreducible submodule of M, then M/N is a McCoy R-module. For an R-module M, $\mathrm{Z}(M)$ need not be an ideal of R in general. For instance, take $M = \mathbb{Z}_2 \times \mathbb{Z}_3$ as a $\mathbb{Z}$-module. Then one can check that $\mathrm{Z}(M) = 2\mathbb{Z} \cup 3\mathbb{Z}$. Of course, in this case, since $\mathbb{Z}$ is a PID, M is McCoy.

The concept of r-ideals was introduced and studied by Mohamadian in [9]. A proper ideal I of R is named an *r-ideal* if $ab \in I$ and $\mathrm{ann}_R(a) = \{r \in R : ra = 0\} = 0$ results that $b \in I$ for each $a, b \in R$. In [8], the authors introduced and studied two different generalizations of r-ideals to modules by r -submodules and sr -submodules. A proper submodule N of M is named an *r-submodule* if $am \in N$ with $\mathrm{ann}_M(a) = 0_M$ implies that $m \in N$ for each $a \in R$, $m \in M$. Some preliminary properties of r- submodules are given in [10, Remark 2.2, Proposition 2.3]. Also, a proper submodule N of M is named a *special r-submodule* (briefly, *sr-submodule*) if for every $m \in M$ and $a \in R$, $am \in N$ with $\mathrm{ann}_R(m) = 0_R$ results that $a \in (N : M)$. Let R be any ring and let us consider R as a module over itself. Since that the submodules of R are ideals in R, one can easily show that I is an r-ideal if and only if I as a submodule is an r-submodule. The reader is referred to [8] and [9] for a more detailed discussions. The main purpose of this article is to generalize the concepts of r-submodules and sr-submodules to the fully I-submodules and fully special I-submodules, respectively, see Definition 2.1 and Definition 3.1.

An element m of an R-module M is named a *torsion element* if there exists a regular element r of R (r is not a zero-divisor of R) such that $rm = 0$. The collection of all torsion elements of M is named the *torsion submodule* of M, denoted by $\mathrm{T}_R(M)$. In fact, $\mathrm{T}_R(M) = \{m \in M : \mathrm{ann}_R(m) \neq 0\}$. An R-module M satisfies Property $\mathcal{T}$ if for every finitely generated submodule N of M with $N \subseteq T_R(M)$ there exists a non-zero $r \in R$ with $rN = 0$, or equivalently $\mathrm{ann}_R(N) \neq 0$.

Also, $\mathcal{Z}(M) = \{m \in M : \mathrm{ann}_R(m)$ is an essential ideal in $R\}$ is a submodule of M, which is named *singular submodule*. If $\mathcal{Z}(M) = M$, (resp., $\mathcal{Z}(M) = (0)$), then M is named *singular* (resp., *nonsingular*) module. A module M over a ring R is called a *torsion module* if all its elements are torsion elements, i.e., $\mathrm{T}(M) = M$ and *torsion-free* if $\mathrm{T}(M) = 0$. The torsion submodule of M is an r-submodule and also, if R is a domain, then $\mathcal{Z}(M)$ is an r-submodule, see [10, Proposition 2.12].

Note that, if $X \subseteq M$ and N is the submodule of M generated by X, then in general $\mathrm{ann}_R(N)$ is a subset of $\mathrm{ann}_R(X)$, but they are not necessarily equal. If R is commutative, then the equality holds. A module M on an integral domain (a commutative ring without zero-divisors), R with $\mathrm{ann}_R(M) \neq 0$ is a torsion module. In addition, a finitely generated torsion module has a non-zero annihilator. A module M is a *multiplication module* whenever for each submodule N of M there exists an ideal I of R where $N = IM$. It is shown that, in this case, $N = (N :_R M)M$, we direct the reader to [5, 9, 10, 11, 12].

2. A generalization of r-submodules

In this section, we expand upon the definition and findings presented in [8] to encompass a broader fully I-submodule scenario for an ideal I of R. Remember that a proper submodule N of a module M over a commutative ring R is named an *n-submodule*, if for $a \in R$, $x \in M$, $ax \in N$ with $a \notin \sqrt{\mathrm{ann}(M)}$, then $x \in N$. By [1, Theorem 2.2], the following statements are equivalent: (i) N is an n-submodule of M; (ii) $N = (N :_M a)$ for every $a \notin \sqrt{\mathrm{ann}(M)}$; (iii) for every ideal I of R and submodule K of M, $IK \subseteq N$ with $I \nsubseteq \sqrt{\mathrm{ann}(M)}$, implies that $K \subseteq N$.

We initiate with the definition outlined.

Definition 2.1. Let M be a non-zero R-module, N a proper submodule of M and I a non-zero ideal of R. Subsequently, we declare that

(i) N is an *I-submodule* of M if $IK \subseteq N$ with $\mathrm{ann}_M(I) = 0_M$ results that $K \subseteq N$ for each submodule K of M. Equivalently, if for each submodule K of M, $I \subseteq (N :_R K)$ with $\mathrm{ann}_M(I) = 0_M$ results that $(N :_R K) = R$.

(ii) N is a *fully I-submodule* of M if the inclusion $JK \subseteq N$ with $\mathrm{ann}_M(J) = 0_M$ results that $K \subseteq N$ for each submodule K of M and each ideal J of R.

(iii) M is a *fully I-module* if each submodule N of M is a fully I-submodule.

Remark 2.2. Note that $I \subseteq (N : K) \subseteq R$ with $\mathrm{ann}_M(I) = 0_M$ results that $(N : K) = R$ shows that I is a maximal ideal in the following collection

$$\Lambda := \{I \in \mathbb{I}(R) : K \leq M \, and \, I \subseteq (N : K), \, \mathrm{ann}_M(I) = 0_M\}.$$

In particular, when N is an I-submodule, if $IK = N$ and $\mathrm{ann}_M(I) = 0_M$, then $N \subseteq K$ and $K \subseteq N$ result that $K = N$.

In the following example (iii), we show that the concepts of r-submodule and I-submodule are different in general case. In fact, if N is a fully I-submodule of M, then N is an r-submodule of M.

Example 2.3. (i) Let M be a non-zero module on a field R. Then M is a fully I-module.
(ii) Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module, $N = \mathbb{Z}$, $K = \frac{1}{s}\mathbb{Z}$ and $I = s\mathbb{Z}$. Then $\mathrm{ann}_{\mathbb{Q}}(s\mathbb{Z}) = 0$ and $(s\mathbb{Z})(\frac{1}{s}\mathbb{Z}) = \mathbb{Z} \subseteq N$, but $\frac{1}{s}\mathbb{Z} \nsubseteq \mathbb{Z}$. Hence $\mathbb{Z}$ is not a $s\mathbb{Z}$-submodule of $\mathbb{Q}$ as a $\mathbb{Z}$-module for every ideal $s\mathbb{Z}$ of $\mathbb{Z}$.
(iii) Consider $M = \mathbb{Z}$ as a $\mathbb{Z}$-module and the submodule $N = 2\mathbb{Z}$ of M. Clearly, N is not an r-submodule of M since $2.3 \in N$ with $\mathrm{ann}_{\mathbb{Z}}(2) = 0$ but $3 \notin N$. We have two following cases:

case 1. Take, $I = (2t+1)\mathbb{Z}$, then N is an I-submodule of M since for every submodule $K = s\mathbb{Z}$ of $\mathbb{Z}$, $IK \subseteq 2\mathbb{Z}$ with $\mathrm{ann}_{\mathbb{Z}}(I) = 0$ implies that $2 \mid (2t+1)s$ and so $2 \mid s$ hence $K \subseteq N$. Therefore, $N = 2\mathbb{Z}$ is a $(2t+1)\mathbb{Z}$-submodule of $\mathbb{Z}$ for each $t \in \mathbb{N}$.

case 2. Now take, $J = (2t)\mathbb{Z}$, then $N = 2\mathbb{Z}$ is not a J-submodule of M because for $K = 3\mathbb{Z}$, $J(3\mathbb{Z}) \subseteq 2\mathbb{Z}$ and $\mathrm{ann}_{\mathbb{Z}}(J) = 0$ but $3\mathbb{Z} \nsubseteq 2\mathbb{Z}$. Hence N is not a J-submodule of M.

From these two situations, we conclude that $2\mathbb{Z}$ is not a fully I-submodule of $\mathbb{Z}$ as a $\mathbb{Z}$-module.

Theorem 2.4. *Let M be a faithful R-module with prime zero submodule. Then the following statements are equivalent.*

(i) *N is an r-submodule of M;*
(ii) *For every $0 \neq a \in R$ and $m \in M$, $a \in (N :_R m)$ if and only if $(N :_R m) = R$;*
(iii) *For every finitely generated ideal I of R and submodule K of M, $I \subseteq (N :_R K)$ if and only if $(N :_R K) = R$.*
(iv) *For every finitely generated ideal I of R, N is an I-submodule.*

Proof. (i $\Rightarrow$ ii) Clearly, the zero submodule of a faithful module M is prime if and only if $\mathrm{ann}_M(a) = (0)$, for any $0 \neq a \in R$. Now, assume that N is an r-submodule of M and $a \in (N :_R m)$. By assumption, $am \in N$ with $\mathrm{ann}_M(a) = (0)$ results that $m \in N$ so $(N :_R m) = R$.
(ii $\Rightarrow$ i) It is clear.
(i $\Rightarrow$ iii) Assume that $I = (a_1, \ldots, a_s)$. Then

$$\mathrm{ann}_M(I) = \mathrm{ann}_M(\sum_{i=1}^{s} a_i R) = \bigcap_{i=1}^{s} \mathrm{ann}_M(a_i R) = 0.$$

If $I \subseteq (N :_R K)$, then $IK \subseteq N$ with $\mathrm{ann}_M(I) = 0$. Take, $m \in K$. By assumption, for every $0 \neq a_i \in I$ $(1 \leq i \leq s)$, $a_i m \in IK \subseteq N$ with $\mathrm{ann}_M(a_i) = 0$ implies that $m \in N$ so $K \subseteq N$ hence $(N :_R K) = R$, as

needed. The converse is clear.
(iii $\Rightarrow$ ii) Take, $I = (a)$ and $K = Rm$.
(iii $\Leftrightarrow$ iv) It is clear. $\square$

Corollary 2.5. *Let M be a faithful module with $0 \in \mathrm{Spec}(M)$ over a Noetherian ring R and N a proper submodule of M. Then N is an r-submodule if and only if N is a fully I-submodule.*

The following corollary shows that $p\mathbb{Z}$ is not a fully I-submodule of $M = \mathbb{Z}$ as a $\mathbb{Z}$-module.

Corollary 2.6. *Consider $M = \mathbb{Z}$ as a $\mathbb{Z}$-module and $N = p\mathbb{Z}$ such that p is a prime number. Then $p\mathbb{Z}$ is a $k\mathbb{Z}$-submodule of $\mathbb{Z}$ if and only if $(p, k) = 1$.*

Proof. Let $I = k\mathbb{Z}$ be a non-trivial ideal of $\mathbb{Z}$. Clearly, $\mathrm{ann}_{\mathbb{Z}}(k\mathbb{Z}) = 0$. Take, $K = s\mathbb{Z}$ such that $(k\mathbb{Z})(s\mathbb{Z}) = (ks)\mathbb{Z} \subseteq p\mathbb{Z}$. Then $p|ks$, so either $p|k$ or $p|s$. If $p|k$, then $p\mathbb{Z}$ is not a $k\mathbb{Z}$-submodule of $\mathbb{Z}$. Otherwise, $p|s$ and so $s\mathbb{Z} \subseteq p\mathbb{Z}$, as needed. $\square$

Example 2.7. The submodule $N = 3\mathbb{Z}$ is a $2\mathbb{Z}$-submodule of $\mathbb{Z}$, but it is not a $3\mathbb{Z}$-submodule of $\mathbb{Z}$. In fact, $N = 3\mathbb{Z}$ is a $k\mathbb{Z}$-submodule if and only if $(3, k) = 1$.

Proposition 2.8. *Let M be an R-module. Then the following statements are true:*

(i) *The zero submodule of M is a fully I-submodule.*
(ii) *Assume that $\{N_i\}_{i\in\Lambda}$ is a nonempty set of I-submodules of M for an ideal I of R. Then $\bigcap_{i\in\Lambda} N_i$ is an I-submodule of M.*
(iii) *Let $\{N_i\}_{i\in\Lambda}$ be a chain of I-submodules of a finitely generated R-module M. Then $\bigcup_{i\in\Lambda} N_i$ is an I-submodule of M.*
(iv) *If $f \in \mathrm{End}_R(M)$, then $\ker(f) = \{m \in M : f(m) = 0\}$ is a fully I-submodule of M.*
(v) *Every minimal submodule of M is a fully I-submodule.*
(vi) *If $\mathrm{ann}_M(I) = 0$ for some ideal I of R, then $\mathrm{ann}_M(J)$ is an I-submodule for every ideal J of R.*

Proof. (i) Assume that I is an arbitrary ideal of R and $IK = 0$ for some submodule K of M with $\mathrm{ann}_M(I) = 0$. Then $K \subseteq \mathrm{ann}_M(I) = 0$, and so $K = 0$, as needed.
(ii) Assume that $\{N_i\}_{i\in\Lambda}$ is a family of I-submodules of M for some ideal I of R. If $IK \subseteq \bigcap_{i\in\Lambda} N_i$ with $\mathrm{ann}_M(I) = 0$, then $IK \subseteq N_i$ for each $i \in \Lambda$ such that $\mathrm{ann}_M(I) = 0$. By assumption, $K \subseteq N_i$ for each $i \in \Lambda$ so $K \subseteq \bigcap_{i\in\Lambda} N_i$ and the proof is complete.
(iii) The proof is straightforward.

(iv) Suppose that $IK \subseteq \ker(f)$ for some submodule K of M and an arbitrary ideal I of R. If $\mathrm{ann}_M(I) \neq 0$, then we are done. Let $\mathrm{ann}_M(I) = 0$. Then $f(IK) = If(K) = 0$ results that $f(K) \subseteq \mathrm{ann}_M(I) = 0$ and so $K \subseteq \ker(f)$, as needed.
(v) Let N be a minimal submodule of M and $IK \subseteq N$ for some submodule K of M and an arbitrary ideal I of R. If $\mathrm{ann}_M(I) \neq 0$, then we are done. Otherwise, assume that $\mathrm{ann}_M(I) = 0$. By assumption, $IK = 0$ or $IK = N$. If $IK = 0$, then $K \subseteq \mathrm{ann}_M(I) = 0 \subseteq N$. Suppose that $IK = N = Rx$ for some $0 \neq x \in M$. Since $0 \neq IN \subseteq N$, hence $IN = N = IK$. It follows that $(K-N)I = 0$, so $K-N \subseteq \mathrm{ann}_M(I) = 0$. Thus, $K \subseteq N$.
(vi) Assume that $IL \subseteq \mathrm{ann}_M(J)$ with $\mathrm{ann}_M(I) = 0$. Then $J(IL) = I(JL) = 0$, so $JL \subseteq \mathrm{ann}_M(I) = 0$ hence $L \subseteq \mathrm{ann}_M(J)$, as needed. □

Recall that a proper ideal I of R is named an *n-ideal* if the condition $ab \in I$ with $a \notin \mathrm{rad}(0) = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}$ results that $b \in I$, for each $a, b \in R$.

Theorem 2.9. *Let M be a finitely generated multiplication R-module and N a submodule of M such that $(N :_R M)$ is an n-ideal of R. Then for each ideal I of R such that $I \cap (R - \mathrm{rad}(0)) \neq \emptyset$, N is an I-submodule of M.*

Proof. Assume that $IL \subseteq N$ with $\mathrm{ann}_M(I) = 0$, where I is an ideal of R with $I \cap (R - \mathrm{rad}(0)) \neq \emptyset$ and K is a submodule of M. Since $I(L : M)M \subseteq (N : M)M$ hence $I(L : M) \subseteq (N : M)$ because M is a cancellation module. By [13, Theorem 2.7], since $I \cap (R - \mathrm{rad}(0)) \neq \emptyset$ and $(N : M)$ is an n-ideal of R, hence $(L : M) \subseteq (N : M)$ so $L \subseteq N$, as needed. □

Theorem 2.10. *I-submodules are invariant under isomorphisms.*

Proof. Assume that $f : M \to N$ is an R-isomorphism, then for every $I \in \mathbb{I}(R)$, $\mathrm{ann}_M(I) = 0$ if and only if $\mathrm{ann}_N(I) = 0$. Let L be an I-submodule of M, we show that $f(L)$ is an I-submodule of N. Suppose that $IK' \subseteq f(L)$ for some submodule K' of N with $\mathrm{ann}_N(I) = 0$. By assumption, there exists a submodule K of M such that $f(K) = K'$. Hence $IK' = If(K) = f(IK) \subseteq f(L)$, so $IK \subseteq L$ with $\mathrm{ann}_M(I) = 0$. It infer that $K \subseteq L$ since L is an I-submodule of M, so $K' = f(K) \subseteq f(L)$, as needed. □

Remark 2.11. A proper ideal I of R is an A-ideal (resp., fully A-ideal) of R for some ideal A of R, whenever I is an A-submodule (resp., fully A-submodule) of R as an R-module. In the case, $A = R$, $\mathrm{ann}_M(A) = 0_M$ and clearly $AK = K \subseteq N$. Hence as usual every proper submodule N

of M is an R-submodule of M. If $A = 0$, then $\mathrm{ann}_M(0) = M \neq 0_M$, so by Definition 2.1 each proper submodule N of M is a 0-submodule of M. Therefore a proper submodule N of M is a fully A-submodule of M if it is an A-submodule of M for each non-trivial ideal A of R.

By [8, Example 1], every proper submodule of $\mathbb{Z}$-module $\mathbb{Z}_n$ for $n > 2$ is an r-submodule of $\mathbb{Z}_n$. In the following example we show that $\mathbb{Z}_n$ for $n > 2$ is a fully I-module.

Example 2.12. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_n$ for $n > 2$ and the ideal $I = k\mathbb{Z}$ of $\mathbb{Z}$. Let $N = \langle \bar{s} \rangle$ be an arbitrary proper submodule of $\mathbb{Z}_n$ such that $0 \leq s \leq n-1$. Clearly, $(n, s) > 1$, since N is proper. Consider two following cases:
case 1. If $(k, n) = 1$, then $\mathrm{ann}_{\mathbb{Z}_n}(k\mathbb{Z}) = \{\bar{0}\}$. Hence, for submodule $K = \langle \bar{t} \rangle$ of $\mathbb{Z}_n$ the inclusion $IK \subseteq \langle \bar{s} \rangle$ is true only when $s|t$ if and only if $K \subseteq N$, as needed.
case 2. If $(k, n) > 1$, then $\mathrm{ann}_{\mathbb{Z}_n}(I) \neq 0_M$, so we are done. Note that, $\mathrm{ann}_{\mathbb{Z}_n}(k\mathbb{Z}) = \langle \overline{(\frac{n}{(k,n)})} \rangle = \langle \bar{0} \rangle$ if and only if $(k, n) = 1$. Hence, $\mathbb{Z}_n$ as a $\mathbb{Z}$-module is a fully I-module.

Particularly, consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{12}$ and the proper submodule $N = \langle \bar{3} \rangle$. Then $\mathrm{Z}(\mathbb{Z}_{12}) = \{a \in \mathbb{Z} : \mathrm{ann}_{\mathbb{Z}_{12}}(a) \neq \bar{0}\} = \mathbb{Z} - 12\mathbb{Z}$. Take, $I = 5\mathbb{Z}$, then $\mathrm{ann}_{\mathbb{Z}_{12}}(I) = 0_M$. Note that, we have the following cases, $I\langle \bar{0} \rangle \subseteq \langle \bar{3} \rangle$, $I\langle \bar{6} \rangle \subseteq \langle \bar{3} \rangle$ and $I\langle \bar{3} \rangle \subseteq \langle \bar{3} \rangle$, hence N is a $5\mathbb{Z}$-submodule of M. Take, $I = 3\mathbb{Z}$, then $\mathrm{ann}_{\mathbb{Z}_{12}}(3\mathbb{Z}) = \{\bar{0}, \bar{4}, \bar{8}\} = \langle \bar{4} \rangle \neq 0_M$ and so N is a $3\mathbb{Z}$-submodule of M.

Example 2.13. (i) Consider $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. Then

$$E(p) = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = \frac{r}{p^t} + \mathbb{Z}\, for\, t \in \mathbb{N} \cup \{0\}\}$$

is a submodule of $\mathbb{Q}/\mathbb{Z}$, where p is a prime number. Then any proper submodule of $E(p)$ is of the form

$$G_{t_0} = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = \frac{r}{p^{t_0}} + \mathbb{Z} \; for\ some\ r \in \mathbb{Z}\}$$

for some $t_0 \in \mathbb{N} \cup \{0\}$. By [8, Example 2], G_{t_0} is an r-submodule of $E(p)$. Take, $I = m\mathbb{Z}$ such that

$$\mathrm{ann}_{G_{t_0}}(I) = \{\frac{r}{p^{t_0}} + \mathbb{Z} \in G_{t_0} \, : \, (\frac{r}{p^{t_0}} + \mathbb{Z})(m\mathbb{Z}) = \frac{rm}{p^{t_0}} + \mathbb{Z} = \mathbb{Z}\} = 0_{\mathbb{Q}/\mathbb{Z}}.$$

Hence $\frac{rm}{p^{t_0}} \in \mathbb{Z}$, so $p^{t_0} \,|\, rm$. Now since $(p^{t_0}, r) = 1$, hence $p^{t_0} \,|\, m$ and so $m = p^{t_0}s$ for some $s \in \mathbb{Z}$. Then $\mathbb{Z} - \mathrm{Z}(\mathbb{Z}) = \{m \in \mathbb{Z} : m = p^{t_0}s\}$. By Proposition 2.16, for each ideal $I = m\mathbb{Z}$ with $m = p^{t_0}s$ for some $s \in \mathbb{Z}$, G_{t_0} is an I-submodule.

Recall that an R-module M is considered a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = \text{ann}_M(I)$, see [4]. An R-module M satisfies the *double annihilator condition* (briefly, DAC), if for each ideal I of R, $I = \text{ann}_R(\text{ann}_M(I))$. Also, M is named a *strong comultiplication module*, if M is a comultiplication module which satisfies DAC, see [3, Definition 2.1]. For instance, the $\mathbb{Z}$ -module $\mathbb{Z}_{2^\infty}$ is a comultiplication module since all of its proper submodules are of the form $(0 :_M 2^k\mathbb{Z})$ for $k = 0, 1, \ldots$. It is clear that M is comultiplication if and only if for each submodule N of M, $\text{ann}_M(\text{ann}_R(N)) = N$. Note that, if M is a strong comultiplication R-module, then there exists exactly one ideal I of R with $N = \text{ann}_M(I)$.

Theorem 2.14. *Every comultiplication R-module M is a fully I-module.*

Proof. Assume that N is an arbitrary proper submodule of M and I is an arbitrary ideal of R such that $IK \subseteq N$ for some submodule K of M. If $\text{ann}_M(I) \neq 0$, then we are done. Suppose that, $\text{ann}_M(I) = 0$. Then, $\text{ann}_K(I) = K \cap \text{ann}_M(I) = 0$. By [2, Theorem 3.17], every submodule of a comultiplication R-module is a comultiplication R-module, hence K is a comultiplication R-module and $\text{ann}_K(I) = 0$, by [4, Proposition 3.1], $IK = K \subseteq N$, as needed. □

Corollary 2.15. *Every non-zero strong comultiplication R-module M is a fully I-module.*

Proof. Clearly, $I = R$ is the only ideal of R with condition $\text{ann}_M(I) = 0_M$ and so $IK = K \subseteq N$ for every submodule K of M. □

The converse of Corollary 2.15 is not true in general, since by Example 2.12, $\mathbb{Z}_n$ as a $\mathbb{Z}$-module is a fully I-module for every $n > 2$, but $\mathbb{Z}_{12}$ is not a strong comultiplication $\mathbb{Z}$-module, since $\text{ann}_{\mathbb{Z}_{12}}(3\mathbb{Z}) = \text{ann}_{\mathbb{Z}_{12}}(9\mathbb{Z}) = \langle \bar{4} \rangle$ whereas $3\mathbb{Z} \neq 9\mathbb{Z}$.

Remember that an element $r \in R$ is named *prime to* N if $rm \in N$ $(m \in M)$ implies that $m \in N$, that is $(N :_M r) = \{m \in M : rm \in N\} = N$. Suppose that $\mathrm{S}(N)$ is the set of all elements of R that are not prime to N. Then N is named *primal* if $\mathrm{S}(N)$ forms an ideal; this ideal is always a prime ideal, named the *adjoint ideal P of N*. In this case, we also say that N is a *P-primal submodule* of M. If the zero submodule of M is primal, then M will be called a *coprimal module*.

Proposition 2.16. *Let N be a proper submodule of M. Then the following are equivalent.*

(i) *N is an r-submodule of M;*

(ii) $aM \cap N = aN$ *for every* $a \in R - \mathrm{Z}(M)$*;*
(iii) $(N :_M a) = N$ *for every* $a \in R - \mathrm{Z}(M)$*;*
(iv) $aN = N$ *for every* $a \in R - \mathrm{Z}(M)$*;*
(v) $N = \pi^{-1}(L)$*, where* $S = R - \mathrm{Z}(M)$ *and* L *is an* $S^{-1}R$*-submodule of* $S^{-1}M$*;*
(vi) *For an ideal* I *of* R *such that* $I \cap (R - \mathrm{Z}(M)) \neq \emptyset$ *and* L *is a submodule of* M *with* $IL \subseteq N$*, then* $L \subseteq N$*;*
(vii) *Every* $a \in R - \mathrm{Z}(M)$ *is prime to* N*.*

Proof. By [8, Proposition 4], (i)⇔(ii)⇔(iii)⇔(iv) and by [8, Theorem 1], (i)⇔(v). Clearly, (iii)⇔(vii). □

Corollary 2.17. *Let* N *be a proper submodule of* M *and* I *an ideal of* R *with* $I \cap (R - \mathrm{Z}(M)) \neq \emptyset$*. Then* N *is an* r*-submodule if and only if* N *is an* I*-submodule.*

Proof. Assume that $a \in I \cap (R - \mathrm{Z}(M))$. Then $\mathrm{ann}_M(I) \subseteq \mathrm{ann}_M(a) = 0$ and so $\mathrm{ann}_M(I) = 0$. The proof follows from Proposition 2.16. □

Corollary 2.18. *Let* M *be an* R*-module and* N *a proper submodule of* M*. If* $\mathrm{S}(N) \subseteq \mathrm{Z}(M)$*, then* N *is an* r*-submodule.*

Proof. By assumption, $R - \mathrm{Z}(M) \subseteq R - \mathrm{S}(N)$. Hence, each $a \in R - \mathrm{Z}(M)$ is prime to N, i.e., $(N :_M a) = N$ for every $a \in R - \mathrm{Z}(M)$. Then by Proposition 2.16, N is a r-submodule, as needed. □

Theorem 2.19. *Let* M *be a McCoy* R*-module and* I *a finitely generated ideal of* R*. Then the proper submodule* N *is an* r*-submodule of* M *if and only if* N *is an* I*-submodule of* M*.*

Proof. Assume that $IL \subseteq N$ for some submodule L of M. If $\mathrm{ann}_M(I) \neq 0$, then we are done. Let I be a non-zero finitely generated ideal of R with $\mathrm{ann}_M(I) = 0$, then $I \nsubseteq \mathrm{Z}(M)$, since M is MacCoy and so $I \cap (R - \mathrm{Z}(M)) \neq \emptyset$. By Corollary 2.17, in this case, a submodule N of M is an r-submodule if and only if N is an I-submodule. □

Corollary 2.20. *Let* M *be a finitely generated module over a Noetherian ring* R*. Then* N *is an* r*-submodule if and only if* M *is a fully* I*-module.*

Proof. The proof is straightforward, since by [7, Theorem 82], every finitely generated module M over a Noetherian ring R is a MacCoy module. Hence the proof follows from Theorem 2.19. □

Recall that a non-zero submodule N of M is named *second submodule* if for each $a \in R$ the homothety $N \xrightarrow{a} N$ is either surjective or zero. This implies that $\mathrm{ann}_R(N) = \mathfrak{p}$ is a prime ideal of R, and N is named

$\mathfrak{p}$-second submodule of M. Also, the R-module M is named *divisible* if for each $x \in M$ and for each non-zero divisor $r \in R$ there exists $y \in M$ such that $x = ry$. The dual notion of $\mathrm{Z}(M)$, the set of zero divisors of M, is denoted by $\mathrm{W}(M)$, defined as follows:

$$\mathrm{W}(M) = \{r \in R \,|\, M \xrightarrow{r} M \text{ is not surjective}\} = \{r \in R \,|\, rM \neq M\}.$$

Theorem 2.21. *Let M be a R-module and $N \in \mathrm{Spec}(M)$. If $\mathrm{ann}(N) = \mathrm{Z}(M)$, then the following are equivalent.*

(i) *N is a $\mathrm{ann}(N)$-second submodule of M;*
(ii) *N is a divisible $R/\mathrm{ann}(N)$-module;*
(iii) *$rN = N$ for all $r \in R - \mathrm{ann}(N)$;*
(iv) *$IN = N$ for all ideal $I \nsubseteq \mathrm{ann}(N)$;*
(v) *$\mathrm{W}(N) = \mathrm{ann}(N)$;*
(vi) *N is an r-submodule.*

Proof. (i) $\Leftrightarrow$ (ii) The proof is straightforward.
(i) $\Leftrightarrow$ (iii) Clearly, $\mathfrak{p} = \mathrm{ann}(N)$ is a proper prime ideal of R. By assumption, for every $r \in R - \mathfrak{p}$, since $rN \neq 0$, then $rN = N$, because N is a $\mathfrak{p}$-second submodule of M . Conversely, if $r \in \mathfrak{p}$, then $rN = 0$. Otherwise, if $r \in R - \mathfrak{p}$, then by assumption, $rN = N$. Hence, N is a $\mathfrak{p}$-second submodule of M.
(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are clear.
(v) $\Rightarrow$ (i) Assume that, $\mathfrak{p} = \mathrm{ann}(N) = \mathrm{W}(N)$. Then for every $r \in R$ either $r \in \mathfrak{p}$ or $r \in R - \mathfrak{p}$. Hence, either $rN = 0$ or $rN = N$, as needed.
(iii) $\Leftrightarrow$ (vi) The proof is clear by Proposition 2.16 (iv). $\square$

Proposition 2.22. *Let M be an R-module and N a proper submodule of M. If N is an I-submodule for some ideal I of R, then N is an J-submodule for every ideal $J \supset I$ of R with $\mathrm{ann}_M(I) = \mathrm{ann}_M(J)$.*

Proof. Assume that $J \supset I$ is an ideal of R such that $JK \subseteq N$ for some submodule K of M. If $\mathrm{ann}_M(J) \neq 0$, then we are done. Suppose that $\mathrm{ann}_M(J) = 0$. By assumption, since N is an I-submodule and $IK \subseteq JK \subseteq N$ with $\mathrm{ann}_M(I) = 0$ results that $K \subseteq N$, as needed. $\square$

Proposition 2.23. *Let M be a module on a* PID *R and N a proper submodule of M. Then the following assertions hold.*

(i) *N is an r-submodule if and only if N is an I-submodule for ideal $I = Ra$ of R.*
(ii) *M is a fully pure module if and only if M is a fully I-module.*

Proof. (i) Assume that N is an r-submodule, $I = Ra$ is an arbitrary ideal of R for some $a \in R$ and $IK \subseteq N$ for some submodule K of M. If $a \in \mathrm{Z}(M)$, then $\mathrm{ann}_M(a) = \mathrm{ann}_M(I) \neq 0$ and we are done.

Now, if $a \notin \mathrm{Z}(M)$, then $\mathrm{ann}_M(a) = \mathrm{ann}_M(I) = 0$. By Proposition 2.16 (vi), N is an I-submodule, since $I \cap (R - \mathrm{Z}(M)) \neq \emptyset$. Conversely, assume that $am \in N$ with $\mathrm{ann}_M(a) = 0$ for $a \in R$ and $m \in M$. Take, $I = Ra$ and $K = Rm$. Again, since $\mathrm{ann}_M(a) = \mathrm{ann}_M(I) = 0$ and $IK = (Ra)(Rm) \subseteq N$, hence $K \subseteq N$ which results that $m \in N$, as needed.
(ii) Suppose N is an arbitrary submodule of M and $I = Ra$ is an arbitrary ideal of R. By assumption, $IM \cap N = IN$ for each ideal I of R. In particular, $aM \cap N = aN$ for each $a \in R - \mathrm{Z}(M)$. By Proposition 2.16 (ii), N is an r-submodule of M and by part (i), N is an I-submodule of M, as needed. The converse is clear by part (i) and Proposition 2.16 (ii). □

Proposition 2.24. *Suppose that N is an I-submodule of M for some ideal I of R, and S is a nonempty subset of R. Then $(N :_M S)$ is an I-submodule of M.*

Proof. Assume that $IK \subseteq (N :_M S)$ for some submodule K of M and ideal I of R with $\mathrm{ann}_M(I) = 0_M$. Then $s(IK) = I(sK) \subseteq N$ for each $s \in S$. By assumption, since N is an I-submodule of M, hence $sK \subseteq N$ for each $s \in S$ and so $K \subseteq (N :_M S)$, as needed. □

Corollary 2.25. *For every ideal I of R, $\mathrm{ann}_M(I)$ is an I-submodule.*

Proof. If $\mathrm{ann}_M(I) \neq 0$, then we are down. Otherwise, take, $S = I$ and $N = 0_M$ in Propostion 2.24. □

Recall that a proper submodule P of M is named *prime* if $rm \in P$ for $r \in R$ and $m \in M$ conclude that $m \in P$ or $r \in (P :_R M)$. Let $\mathrm{Spec}_R(M)$ denote the collection of prime submodules of M. It is shown that a proper submodule P of M is prime if and only if for each ideal I of R and submodule K of M with $IK \subseteq P$, then either $I \subseteq (P :_R M)$ or $K \subseteq P$.

Proposition 2.26. *Let M be an R-module, P a proper submodule of M and I a non-zero ideal of R. If P is prime with $I \nsubseteq \mathrm{Z}(M/P)$, then P is an I-submodule of M.*

Proof. Assume that $P \in \mathrm{Spec}_R(M)$. Clearly, if for some ideal I of R, $I \nsubseteq (P : M)$, then P is an I-submodule of M. Let $IK \subseteq P$ for some ideal I of R and submodule K of M with $\mathrm{ann}_M(I) = 0$. Since P is a prime submodule, hence $\mathrm{Z}(M/P) = (P :_R M)$. It results that $K \subseteq P$ since $I \nsubseteq \mathrm{Z}(M/P)$, as needed. □

In the following definition, we generalize the definition of nonregular submodule to the nonregular submodule with respect to an ideal I of

R. Recall that a proper submodule N of an R-module M is called *nonregular*, if $aM \subseteq N$ implies that $\operatorname{ann}_M(a) \neq 0_M$, for each $a \in R$. If we consider R as an R-module, then this definition agrees with the concept of nonregular ideal, see ([10, Definition 2.17]).

Definition 2.27. A proper submodule N of an R-module M is *nonregular with respect to an ideal* I of R, if $I \subseteq (N : M)$ results that $\operatorname{ann}_M(I) \neq 0_M$.

Theorem 2.28. *Let M be an R-module and I an ideal of R. Then the following statements are true.*

(i) *Every I-submodule of M is nonregular with respect to I.*
(ii) *Every prime nonregular submodule N of M with respect to I is an I-submodule.*

Proof. (i) Assume that N is an I-submodule and $I \subseteq (N : M)$. Then $IM \subseteq N$. If $\operatorname{ann}_M(I) = 0_M$, then by assumption, $N = M$ which is a contradiction. Hence $\operatorname{ann}_M(I) \neq 0_M$, as needed.
(ii) Suppose that $IK \subseteq N$ for some submodule K of M with $\operatorname{ann}_M(I) = 0_M$. It results that $I \nsubseteq (N : M)$, because otherwise by assumption $\operatorname{ann}_M(I) \neq 0_M$ which is a contradiction. Since N is prime, hence $K \subseteq N$, as needed. □

3. Special I-submodules

In this section, we present the concept of special I-submodules of an R-module M. Also, we extend the concept of special r-submodules to the fully special I-submodules. Remember that a submodule N of M is named a *special r-submodule* (briefly, sr-submodule) if $N \neq M$, for each $a \in R$, $m \in M$ with $am \in N$ and $\operatorname{ann}_R(m) = 0$, then $a \in (N :_R M)$, see [8, Definition 4].

Definition 3.1. Let M be an R-module and N a proper submodule of M.

(i) N is a *special I-submodule* for an ideal I of R if for each submodule L of M, $IL \subseteq N$ with $\operatorname{ann}_R(L) = 0$ results that $I \subseteq (N :_R M)$.
(ii) N is a *fully special I-submodule*, if N is a special submodule with respect to every proper ideal J of R. Equivalently, N is a fully special I-submodule of M, if $JL \subseteq N$ with $\operatorname{ann}_R(L) = 0$ results that $J \subseteq (N :_R M)$ for each submodule L of M and each ideal J of R.
(iii) If each proper submodule N of M is a fully special I-submodule of M, then we say that M is a *fully special I-module*.

In particular, an ideal I of R is a special A-ideal of R, if I is a special A-submodule of R as an R-module, i.e., $AB \subseteq I$ for each ideal B of R with $\operatorname{ann}_R(B) = 0$ results that $A \subseteq I$.

Remark 3.2. Cleary, every fully special I-submodule is a sr-submodule (take, $I = Ra$ and $L = Rm$). The converse is not true in general case.

Recall that an R-module M is named a *cancellation module*, if $IM \subseteq JM$ results that $I \subseteq J$ for all ideals I and J of R .

Theorem 3.3. *Let M be a faithful multiplication R-module. If N is a special A-submodule of M, then $(N : M)$ is a special A-ideal of R. The converse is true if M is finitely generated.*

Proof. Assume that N is a special A-submodule of M such that $AB \subseteq (N : M)$ for some ideals A, B of R with $\operatorname{ann}_R(A) = 0$. Then, $ABM = B(AM) \subseteq N = (N : M)M$. By assumption, $\operatorname{ann}_R(A) = \operatorname{ann}_R(AM) = 0$ implies that $BM \subseteq N$. Hence $B \subseteq (N : M)$, as needed. Conversely, suppose that $(N : M)$ is a special A-ideal of R and $AL \subseteq N$ such that A is an ideal of R and L a submodule of M with $\operatorname{ann}_R(L) = 0$. Then, $A(L : M)M \subseteq (N : M)M$, so $A(L : M) \subseteq (N : M)$ because every finitely generated faithful R-module M is a cancellation module and also, $\operatorname{ann}_R(L) = \operatorname{ann}_R((L : M)) = 0$. By assumption, $A \subseteq (N : M)$. □

Proposition 3.4. *Let M be an R-module. If a proper submodule N of M is a fully special I-submodule, then N is an sr-submodule. The converse is true if M is torsion-free.*

Proof. Assume that $a \in R$, $m \in M$ with $am \in N$ and $\operatorname{ann}_R(m) = 0$. Take, $I = Ra$, $L = Rm$. Hence $IL \subseteq N$ and $\operatorname{ann}_R(L) = 0$. By assumption, $I = Ra \subseteq (N :_R M)$ and so $a \in (N :_R M)$, as needed.
Conversely, assume that L is a submodule of M with $IL \subseteq N$ and $\operatorname{ann}_R(L) = 0$ for each arbitrary ideal I of R. Since $T(M) = 0$, hence $L \cap (M - T(M)) = L \neq \emptyset$. By [8, Theorem 10 (i)], $I \subseteq (N :_R M)$, as needed. □

Example 3.5. Consider $M = \mathbb{Z}_n$ as a $\mathbb{Z}$-module. Take, $N = \langle \bar{s} \rangle$ $(0 \leq s \leq n-1)$ with $(n, s) = d > 1$ and $I = k\mathbb{Z}$ an ideal of $\mathbb{Z}$. Let $L = \langle \bar{t} \rangle$ such that $IL = (k\mathbb{Z})(\langle \bar{t} \rangle) \subseteq \langle \bar{s} \rangle$. Then we have the following cases,

(i) If $t = 0$, then $I\langle \bar{0} \rangle \subseteq \langle \bar{s} \rangle$, with $\operatorname{ann}_{\mathbb{Z}}(\langle \bar{0} \rangle) = \mathbb{Z} \neq 0$.
(ii) If $t \neq 0$, then $s \mid kt$ with $\operatorname{ann}_{\mathbb{Z}}(\langle \bar{t} \rangle) = \frac{n}{(n,t)}\mathbb{Z} \neq 0$.

Hence N is a fully special I-submodule, so M is fully special I-module. In this case, all proper submodules of $\mathbb{Z}_n$ are sr-submodules. If $(t, n) =$

1, then $L = M$, so $\mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}_n) = n\mathbb{Z} \neq 0$ and we are done. Also, if $(t, n) > 1$, then $\mathrm{ann}_{\mathbb{Z}}(L) = \mathrm{ann}_{\mathbb{Z}}(\langle \bar{t} \rangle) = \frac{n}{(t,n)}\mathbb{Z} \neq 0$.

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