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On $\mathbb{Z}_k-$vertex-magic labeling of prime graph $PG(\mathbb{Z}_n)$

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Abstract. Let $G = (V(G), E(G))$ be a graph, $(\mathcal{A}, +)$ be an Abelian group with identity $0_{\mathcal{A}}$, and $(\mathcal{R}, +, \cdot)$ be a ring. The $\mathcal{A}$-vertex-magic labeling of G is a mapping from $V(G)$ to $\mathcal{A} - \{0_{\mathcal{A}}\}$ such that the total labels of every adjacent vertex with u are equal for every u in $V(G)$. The prime graph over $\mathcal{R}$, denoted by $PG(\mathcal{R})$, is a graph with $V(PG(\mathcal{R})) = \mathcal{R}$ such that uv is an edge if and only if $u\mathcal{R}v = \{0_{\mathcal{R}}\}$ or $v\mathcal{R}u = \{0_{\mathcal{R}}\}$, for every vertex $u \neq v$. In this article, we discuss the $\mathbb{Z}_k$-vertex-magic labeling of the prime graph over the ring $\mathbb{Z}_n$. We study some literature to develop the properties of $\mathbb{Z}_k$-vertex-magic labeling of $PG(\mathcal{R})$. We investigate some classes of prime graphs over ring $\mathbb{Z}_n$ for $n = p, n = p^2$, and $n = pq$, with $p \neq q$ primes.

Keywords: Abelian group, Group-vertex-magic labeling, Prime graph, Ring.
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1 Introduction

A graph G is a system consisting of a finite non-empty vertex set and a finite edge set such that every element is identified with a pair of vertices [6]. Many fields have applied graph theory, such as networks, cryptography, transportation, coding theory, chemistry, crystallography, and information systems (see [7], [15], [19], [23], [26], [27], [28], [30], and [33]). The construction of graphs can be related to algebraic structures. Unit graphs [3], zero divisor graphs [1], total graphs [2], and prime graphs [5] are a few examples of the various types of graphs made from rings. The prime graph over a ring $\mathcal{R}$, which is denoted by $PG(\mathcal{R})$, was introduced by Bhavanari *et al.* in 2010 [5].

Graph labeling is a mapping from its vertex set, edge set, or both into the set of positive integers such that it satisfies a certain condition [9]. Rosa first introduced graph labeling in 1966 [29]. Several types of graph labeling are illustrated in, [10], [13], [14], [32], and [34]. There are connections between graph labeling and groups, as Lee *et al.* first introduced the concept

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of graph labeling using Abelian groups in 2001, which is known as group-magic labeling [17]. Many results about group-magic graphs can be seen in [16], [20], [21], [22], and [31]. Recently, in 2020, Kamatchi *et al.* introduced a new concept of graph labeling using Abelian groups called group-vertex-magic labeling [11]. Many results about group-vertex-magic in some classes of graphs can be seen in [4], [12], [18], [24], and [25].

In this article, we specifically discuss the group-vertex-magic labeling of $PG(\mathbb{Z}_n)$ using the Abelian group $\mathbb{Z}_k$. In this research, we investigate the $PG(\mathbb{Z}_n)$ for cases $n = p$, $n = p^2$, and $n = pq$, with primes $p \neq q$. There are four sections in this article. In the second section, we give a literature review used in this article. In the third section, we obtain some results about group vertex-magic labeling of prime graphs over rings. The last section concludes the contents of this article.

2 Preliminaries

This study of graph theory is restricted to simple graphs, that is, undirected graphs that have no loops and multiple edges. More content on graph theory can be studied in [6], while topics on algebraic structure can be studied in [8].

A group $(\mathcal{A}, \star)$ is a system which consists of the non-empty set $\mathcal{A}$ and the binary operation "$\star$", where $(\mathcal{A}, \star)$ is associative, has an identity element, and every element of $\mathcal{A}$ has an inverse element. A group $(\mathcal{A}, \star)$ is said to be Abelian if the operation is commutative. The order of $\mathcal{A}$, denoted by $|\mathcal{A}|$, is the cardinality of $\mathcal{A}$, while the order of $a \in \mathcal{A}$, denoted by $|a|$, is the least positive integer n such that $a^n = e$, with e being the identity element of $\mathcal{A}$. A ring $(\mathcal{R}, +, \cdot)$ is a system that consists of the non-empty set $\mathcal{R}$ with two binary operations "$+$" and "$\cdot$", where $(\mathcal{R}, +)$ is an Abelian group, $(\mathcal{R}, \cdot)$ is associative, and $(\mathcal{R}, +, \cdot)$ is distributive. In this research, we concentrate on the ring $(\mathbb{Z}_n, +, \cdot)$, which is also an Abelian group under addition. We give the special properties of the Abelian group as follows:

Lemma 1. *Let $(\mathcal{A}, +)$ be an Abelian group with $|\mathcal{A}| > 2$ and identity $0_{\mathcal{A}}$. Then, for every $k \in \mathbb{N}$,$0_{\mathcal{A}} \neq a \in \mathcal{A}$ can be expressed as*

$$a_1 + a_2 + \cdots + a_k = a, \tag{1}$$

with non-identity elements $a_1, a_2, \ldots, a_k \in \mathcal{A}$.

Proof. Let $a \in \mathcal{A}$ be a non-identity element.

- Case 1: k is odd.
  We can choose $a_j = a$ if j is odd and $a_j = -a$ if j is even, where $j \in \{1, 2, \ldots, k\}$, so than
  $$a_1 + a_2 + \cdots + a_k = a + (-a) + \cdots + a + (-a) + a = a.$$
- Case 2: k is even.
  Since the order of $\mathcal{A}$ is greater than 2, for every non-identity element $a, b \in \mathcal{A}$ where $a \neq b$, there exists $0_{\mathcal{A}} \neq c \in \mathcal{A}$ such that $b + c = a$. Now, we can choose $a_1 = b, a_2 = c, a_3 = a = 5 \cdots = a_{k-1} = a$ and $a_4 = a_6 = \cdots = a_k = -a$, so than
  $$a_1 + a_2 + \cdots + a_k = b + c + a + (-a) \cdots + a + (-a) = a.$$

This completes the proof. □

A graph $G = (V(G), E(G))$ is a system containing a finite non-empty vertex set $V(G)$ and a finite edge set $E(G)$, which is a subset of $V(G) \times V(G)$. Two vertices u and v are called adjacent if uv is an edge. The open neighborhood set of $u \in V(G)$, denoted by $N(u)$, is the set of vertices that are adjacent to u, and the cardinality of $N(u)$ is called the degree of u, denoted by $\deg(u)$. An isomorphism of graphs G and G' is a bijection $f : V(G) \to V(G')$, such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(G')$. If there is an isomorphism between G and G', then we say that G and G' are isomorphic, denoted by $G \cong G'$.

In this article, we discuss prime graphs over the ring $\mathbb{Z}_n$. The definition of prime graphs over rings is given as follows:

Definition 1 ([5]). *Let $(\mathcal{R}, +, \cdot)$ be a ring. The prime graph over $\mathcal{R}$, denoted by $PG(\mathcal{R})$, is a graph with $V(PG(\mathcal{R})) = \mathcal{R}$ and*

$$E(PG(\mathcal{R})) = \{uv | u\mathcal{R}v = \{0_\mathcal{R}\} \text{ or } v\mathcal{R}u = \{0_\mathcal{R}\} \text{ and } u \neq v\}.$$

In what follows, we present several properties of $PG(\mathcal{R})$. Further properties can be found in [5].

Theorem 1. *Let $(\mathcal{R}, +, \cdot)$ be a ring and $PG(\mathcal{R})$ be its prime graph.*

1. *Every non-zero vertex $v \in \mathcal{R}$ is adjacent to $0_\mathcal{R}$.*
2. *$d(u, v) = 2$ if and only if the vertices u and v are not adjacent.*
3. *If $\mathcal{R}$ is commutative ring unity $1_\mathcal{R}$, then the vertices u and v are adjacent if and only if $uv = 0_\mathcal{R}$.*
4. *If $1_\mathcal{R} \neq u \in \mathcal{R}$ is a unit, then u is only adjacent to $0_\mathcal{R}$.*
5. *If $\mathcal{R} = \mathbb{Z}_p$ for p primes or $p = 4$, then $PG(\mathbb{Z}_p) \cong K_{1,p-1}$.*

Kamatchi *et al.* [11] developed a new concept of graph labeling using Abelian groups, called group-vertex-magic labeling using Abelian groups, as follows:

Definition 2 ([11]). *Let $G = (V(G), E(G))$ be a graph and $(\mathcal{A}, +)$ be an Abelian group with identity $0_\mathcal{A}$. The $\mathcal{A}-$vertex-magic labeling of G is a mapping $\ell : V(G) \to \mathcal{A} - \{0_\mathcal{A}\}$ such that $w : V(G) \to \mathcal{A}$ given by*

$$w(v) = \sum_{u \in N(v)} \ell(u),$$

is a constant map, that is, there exists $a \in \mathcal{A}$ such that $w(v) = a$ for every $v \in V(G)$. We say G that has its labeling as $\mathcal{A}$-vertex-magic graph.

Next, we introduce the definition of a vertex-magic integer set. This definition is a special case for graphs labeled with the group $\mathbb{Z}_k$.

Definition 3 ([11]). *Let G be a graph. The vertex integer-magic set of G, denoted by $VIM(G)$, is the set of all $k \in \mathbb{N}$ such that G is $\mathbb{Z}_k-$vertex-magic graph, or can be written as*

$$VIM(G) = \{k \in \mathbb{N} | G \text{ is a } \mathbb{Z}_k - \text{vertex-magic graph}\}.$$

Here are some previous results that we will use in the next section.

Proposition 1 ([12]). *For every integer $k > 1$, the path graph P_2 is a $\mathbb{Z}_k-$vertex-magic graph.*

Proposition 2 ([12]). *Let $K_{1,n}$ be a star graph with $n > 2$.*

1. *If n is even, then $K_{1,n}$ is not a $\mathbb{Z}_2-$vertex-magic graph.*
2. *If n is odd, then $K_{1,n}$ is a $\mathbb{Z}_2-$vertex-magic graph.*

Theorem 2 ([12]). *For every integer $k > 1$ and $n > 2$, the star graph $K_{1,n}$ is a $\mathbb{Z}_k-$vertex-magic graph.*

Theorem 3 ([12]). *Let $k > 1$ and $n > 2$ be integers. The complete graph K_n is $\mathbb{Z}_k-$vertex-magic graph if and only if $\ell : V(K_n) \to \mathbb{Z}_k - \{\overline{0}\}$ is a constant map.*

3 Main Result

In this section, we obtain results on group-vertex-magic labeling of $PG(\mathbb{Z}_n)$ using an Abelian group $\mathbb{Z}_k$ for certain values of n. In addition, we give a result on the set of vertex integer magic of group-vertex-magic labeling on $PG(\mathbb{Z}_n)$ using the Abelian group $\mathbb{Z}_k$.

Firstly, we establish some properties of the $\mathbb{Z}_k-$vertex-magic labeling of $PG(\mathbb{Z}_p)$, where p is a prime number.

Proposition 3. *For every natural number $k > 1$, the prime graph $PG(\mathbb{Z}_2)$ is $\mathbb{Z}_k-$vertex-magic graph.*

Proof. Since $PG(\mathbb{Z}_2) \cong P_2$, it follows from Proposition 1 that $PG(\mathbb{Z}_2)$ is $\mathbb{Z}_k-$vertex-magic graph. □

Proposition 4. *For every prime number $p > 2$, the prime graph $PG(\mathbb{Z}_p)$ is not a $\mathbb{Z}_2-$vertex-magic graph.*

Proof. Since $PG(\mathbb{Z}_p) \cong K_{1,p-1}$ with $p - 1$ even, it follows from Proposition 2 that $PG(\mathbb{Z}_p)$ is not a $\mathbb{Z}_2-$vertex-magic graph. □

Theorem 4. *If $p > 2$ is prime and $k > 2$ is a natural number, then the prime graph $PG(\mathbb{Z}_p)$ is $\mathbb{Z}_k-$vertex-magic graph.*

Proof. Since $PG(\mathbb{Z}_p) \cong K_{1,p-1}$ with $p - 1$ even, it follows from Theorem 2 that $PG(\mathbb{Z}_p)$ is $\mathbb{Z}_k-$vertex-magic graph. □

Based on Proposition 3, Proposition 4, and Theorem 4, we can compute the vertex integer magic set of $PG(\mathbb{Z}_p)$ as follows:

Corollary 1. *If p is prime and $k > 1$ is a positive integer, then*

$$VIM(PG(\mathbb{Z}_p)) = \begin{cases} \mathbb{N} \setminus \{1\}, & p = 2, \\ \mathbb{N} \setminus \{1, 2\}, & p > 2. \end{cases}$$

In the next results, we have some properties of $\mathbb{Z}_k-$vertex-magic labeling of $PG(\mathbb{Z}_{p^2})$, where p prime numbers.

Proposition 5. *Let $k > 1$ be a natural number.*

1. *The prime graph $PG(\mathbb{Z}_4)$ is $\mathbb{Z}_k-$vertex-magic graph.*
2. *The prime graph $PG(\mathbb{Z}_9)$ is not a $\mathbb{Z}_k-$vertex-magic graph.*

Proof. Let $k > 1$ be a natural number.

1. Since $PG(\mathbb{Z}_4) \cong K_{1,3}$, it follows from Proposition 2 and Theorem 2 that $PG(\mathbb{Z}_4)$ is $\mathbb{Z}_k-$vertex-magic graph.
2. In $PG(\mathbb{Z}_9)$, the vertex $\overline{3}$ is adjacent to $\overline{0}$ and $\overline{6}$, while $\overline{1}$ is adjacent to $\overline{0}$. Suppose that $PG(\mathbb{Z}_9)$ is a $\mathbb{Z}_k-$vertex-magic graph. This means that

$$\ell(\overline{0}) + \ell(\overline{6}) = w(\overline{3}) = w(\overline{1}) = \ell(\overline{0}).$$

This result implies $\ell(\overline{6}) = \overline{0}$ which is a contradiction. Thus, $PG(\mathbb{Z}_9)$ is not a $\mathbb{Z}_k-$vertex-magic graph.

□

Proposition 6. *For every prime number $p > 3$, the prime graph $PG(\mathbb{Z}_{p^2})$ is not a $\mathbb{Z}_2-$vertex-magic graph.*

Proof. Suppose that $PG(\mathbb{Z}_{p^2})$ is a $\mathbb{Z}_2-$vertex-magic graph, and $\ell : V(PG(\mathbb{Z}_{p^2})) \to \mathbb{Z}_2 - \{\overline{0}\}$. Then, the only possible label for every vertex in $PG(\mathbb{Z}_{p^2})$ is $\ell(x) = \overline{1}$ for all $x \in \mathbb{Z}_{p^2}$. Observe that

$$w(\overline{0}) = (p^2 - 1) \cdot \overline{1} = \overline{0}.$$

On the other hand,

$$w(\overline{1}) = \ell(\overline{0}) = \overline{1}.$$

This is a contradiction, so $PG(\mathbb{Z}_{p^2})$ is not a $\mathbb{Z}_2-$vertex-magic graph. □

Theorem 5. *For every natural number $k > 2$ and prime number $p > 3$, the prime graph $PG(\mathbb{Z}_{p^2})$ is $\mathbb{Z}_k-$vertex-magic graph if and only if* $\gcd(k, p - 2) \neq 1$.

Proof. For any prime number $p > 3$, the ring $\mathbb{Z}_{p^2}$ has $p^2 - p$ units and $p - 1$ nontrivial zero divisors. The nontrivial zero divisors in $\mathbb{Z}_{p^2}$ is $Z = \{v_i | v_i = \overline{ip}, i = 1, 2, \ldots p - 1\}$. Let $U = \{t_1 = \overline{1}, t_2, \ldots, t_{p^2-p}\}$ be the set of vertices, which is the unit of the ring $\mathbb{Z}_{p^2}$. Note that every non-zero vertex in $PG(\mathbb{Z}_{p^2})$ is adjacent to $\overline{0}$, and every two distinct vertices in Z are adjacent.

Suppose that $PG(\mathbb{Z}_{p^2})$ is a $\mathbb{Z}_k-$vertex-magic graph, and the map $\ell : V(PG(\mathbb{Z}_{p^2})) \to \mathbb{Z}_k-\{\overline{0}\}$. It follows that

$$\ell(\overline{0}) + \sum_{v_j \neq u \in Z} \ell(u) = w(v_j) = w(v_i) = \ell(\overline{0}) + \sum_{v_i \neq v \in Z} \ell(v),$$

for every $v_i, v_j \in Z$, with $i \neq j$. This implies $\ell(v_i) = \ell(v_j)$, so every vertex in Z must have the same label. Suppose $\ell(v_i) = \overline{a} \neq \overline{0}$. Furthermore, for every $v_i \in Z$, the following also holds

$$\ell(\overline{0}) + (p-2) \cdot \overline{a} = w(v_i) = w(\overline{1}) = \ell(\overline{0}),$$

which implies $(p-2) \cdot \overline{a} = \overline{0}$. This means $|\overline{a}| > 1$ must divide $p-2$. Since $|a|$ also divides k, it must be $\gcd(k, p-2) \geq |a| > 1$ so that $\gcd(k, p-2) \neq 1$.

Conversely, suppose that $\gcd(k, p-2) = d > 1$. Since $\mathbb{Z}_k$ is a cyclic group, there exists $\overline{a} \in \mathbb{Z}_k$ with $\langle \overline{a} \rangle = \mathbb{Z}_k$ and $|\overline{a}| = k$. Since $d|k$ and $d|p-2$, there exist natural numbers m, r such that $k = md, p-2 = rd$, and $m \cdot \overline{a} \in \mathbb{Z}_k$ is an element of order d. Define the mapping $\ell : V(PG(\mathbb{Z}_{p^2})) \to \mathbb{Z}_k - \{\overline{0}\}$ as

$$\ell(x) = \begin{cases} m \cdot \overline{a}, & x = v_i \in Z, \\ a_j, & x = t_j \in U, \\ m \cdot \overline{a} + \sum a_j, & x = \overline{0}, \end{cases}$$

with $\overline{a}, a_j \in \mathbb{Z}_k - \{\overline{0}\}$ and $m \cdot \overline{a} + \sum a_j \neq \overline{0}$. Since

$$\begin{aligned} w(\overline{0}) &= \sum_{t_j \in U} \ell(t_j) + \sum_{v_i \in Z} \ell(v_i) = m \cdot \overline{a} + \sum a_j, \\ w(t_j) &= \ell(\overline{0}) = m \cdot \overline{a} + \sum a_j, && \text{for all } t_j \in U, \\ w(v_i) &= \ell(\overline{0}) + \sum_{v_i \neq v_j \in Z} \ell(v_j) = \ell(\overline{0}) + (p-2) \cdot (m \cdot \overline{a}), \\ &= \ell(\overline{0}) + r \cdot (md \cdot \overline{a}) = \ell(\overline{0}) + \overline{0} = m \cdot \overline{a} + \sum a_j, && \text{for all } v_i \in Z, \end{aligned}$$

it follows that $w(x) = m \cdot \overline{a} + \sum a_j$, for all $x \in V(PG(\mathbb{Z}_{p^2})$. Thus, $PG(\mathbb{Z}_{p^2})$ is $\mathbb{Z}_k-$vertex-magic graph. □

Based on Proposition 5, Proposition 6, and Theorem 5, we can compute the vertex integer magic set of $PG(\mathbb{Z}_{p^2})$ as follows:

Corollary 2. *If p is prime and $k > 1$ is a positive integer, then*

$$VIM(PG(\mathbb{Z}_{p^2})) = \begin{cases} \mathbb{N} \setminus \{1\}, & p = 2, \\ \emptyset, & p = 3, \\ \mathbb{N} \setminus \{k \in \mathbb{N} | \gcd(k, p-2) = 1\}, & p > 3. \end{cases}$$

Lastly, we have some properties of $\mathbb{Z}_k-$vertex-magic labeling of $PG(\mathbb{Z}_{pq})$, where p and q are distinct primes.

Theorem 6. *For every natural number $k > 1$ and prime number $p > 2$, the prime graph $PG(\mathbb{Z}_{2p})$ is not a $\mathbb{Z}_k-$vertex-magic graph.*

Proof. For any prime number $p > 2$, the ring $\mathbb{Z}_{2p}$ has $p-1$ units and p nontrivial zero divisors. The nontrivial zero divisors in $\mathbb{Z}_{2p}$ are $Z_1 = \{\overline{2}, \overline{4}, \dots, \overline{2(p-1)}\}$ and $Z_2 = \{\overline{p}\}$. Note that every nonzero vertex in $PG(\mathbb{Z}_{2p})$ is adjacent to $\overline{0}$, also every vertex in Z_1 is adjacent to every vertex in Z_2, and vice versa. Suppose $PG(\mathbb{Z}_{2p})$ is $\mathbb{Z}_k-$vertex-magic graph. This means that

$$\ell(\overline{p}) + \ell(\overline{0}) = w(\overline{2}) = w(\overline{1}) = \ell(\overline{0})$$

holds. This implies $\ell(\overline{p}) = \overline{0}$, which is a contradiction. Thus, $PG(\mathbb{Z}_{2p})$ is not a $\mathbb{Z}_k-$vertex-magic graph. □

Proposition 7. *For every distinct primes $p, q > 2$, the prime graph $PG(\mathbb{Z}_{pq})$ is not a $\mathbb{Z}_2-$vertex-magic graph.*

Proof. Suppose that $PG(\mathbb{Z}_{pq})$ is a $\mathbb{Z}_2-$vertex-magic graph, and $\ell : V(PG(\mathbb{Z}_{pq})) \to \mathbb{Z}_2 - \{\overline{0}\}$. Then, the only possible label for every vertex in $PG(\mathbb{Z}_{pq})$ is $\ell(x) = \overline{1}$ for all $x \in \mathbb{Z}_{pq}$. Observe that

$$w(\overline{0}) = (pq-1) \cdot \overline{1} = \overline{0}.$$

On the other hand,

$$w(\overline{1}) = \ell(\overline{0}) = \overline{1}.$$

This is a contradiction, so $PG(\mathbb{Z}_{pq})$ is not a $\mathbb{Z}_2-$vertex-magic graph. □

Theorem 7. *For every natural number $k > 1$ and distinct primes $p, q > 2$, the prime graph $PG(\mathbb{Z}_{pq})$ is $\mathbb{Z}_k-$vertex-magic graph.*

Proof. For any primes $p, q > 2$, the ring $\mathbb{Z}_{pq}$ has $(p-1)(q-1) = pq-p-q+1$ units and $p+q-2$ nontrivial zero divisors. The nontrivial zero divisors in $\mathbb{Z}_{pq}$ are

$$Z_1 = \{v_j | v_j = \overline{jp}, j = 1, 2, \dots, q-1\},$$

and

$$Z_2 = \{u_l | u_l = \overline{lq}, l = 1, 2, \dots, p-1\}.$$

Note that every $v_i \in Z_1$ is adjacent to every $u_j \in Z_2$, and vice versa. Suppose the mapping $\ell : V(PG(\mathbb{Z}_{pq})) \to \mathbb{Z}_k - \{\overline{0}\}$. For $PG(\mathbb{Z}_{pq})$ to be a $\mathbb{Z}_k-$vertex-magic graph, it must hold that

$$\ell(\overline{0}) + \sum_{v_j \in Z_1} \ell(v_j) = w(\overline{q}) = w(\overline{1}) = \ell(\overline{0}),$$

which implies

$$\sum_{v_j \in Z_1} \ell(v_j) = \overline{0}. \tag{2}$$

In a similar way, the following must hold:

$$\ell(\overline{0}) + \sum_{u_l \in Z_2} \ell(u_l) = w(\overline{p}) = w(\overline{1}) = \ell(\overline{0}),$$

which implies

$$\sum_{u_l \in Z_2} \ell(u_l) = \overline{0}. \tag{3}$$

Then it must also hold

$$\ell(\overline{0}) = w(\overline{1}) = w(\overline{0}) = \sum_{t_i \in U} \ell(t_i) + \sum_{v_j \in Z_1} \ell(v_j) + \sum_{u_l \in Z_2} \ell(u_l) = \sum_{t_i \in U} \ell(t_i),$$

where $U = \{t_1 = \overline{1}, t_2, \ldots, t_{pq-p-q+1}\}$ is the set of vertices, which is the unit of the ring $\mathbb{Z}_{pq}$. Therefore, define the mapping ℓ as

$$\ell(x) = \begin{cases} a_i, & x = t_i \in U, \\ \sum a_i & x = \overline{0}, \\ \overline{b}, & x = v_j \in Z_1, j \text{ even}, \\ -\overline{b}, & x = v_j \in Z_1, j \text{ odd}, \\ \overline{c}, & x = u_l \in Z_2, l \text{ even}, \\ -\overline{c}, & x = u_l \in Z_2, l \text{ odd}, \end{cases} \tag{4}$$

with $a_i, \overline{b}, \overline{c} \in \mathbb{Z}_k - \{\overline{0}\}$ such that $\sum a_i \neq \overline{0}$. Since

$$\begin{aligned} w(\overline{0}) &= \sum_{t_i \in U} \ell(t_i) = \sum a_i, \\ w(v) &= \ell(\overline{0}) + \sum_{u_l \in Z_2} \ell(u_l) = \ell(\overline{0}) = \sum a_i, && \text{for all } v \in Z_1, \\ w(u) &= \ell(\overline{0}) + \sum_{v_j \in Z_1} \ell(v_j) = \ell(\overline{0}) = \sum a_i, && \text{for all } u \in Z_2, \\ w(t) &= \ell(\overline{0}) = \sum a_i, && \text{for all } t \in U, \end{aligned}$$

it follows that $w(x) = \sum a_i$, for every $x \in V(PG(\mathbb{Z}_{pq})$. Thus, $PG(\mathbb{Z}_{pq})$ is $\mathbb{Z}_k-$vertex-magic graph. □

Based on Theorem 6, Proposition 7, and Theorem 7, we can compute the vertex integer magic set of $PG(\mathbb{Z}_{pq})$ as follows:

Corollary 3. *If p and q are distinct primes and $k > 1$ is a positive integer, then*

$$VIM(PG(\mathbb{Z}_{pq})) = \begin{cases} \emptyset, & p = 2, q > 2, \\ \mathbb{N} \setminus \{1, 2\}, & p, q > 2. \end{cases}$$

4 Conclusion

In this article, we first obtain that $PG(\mathbb{Z}_p)$ is a $\mathbb{Z}_k$-vertex-magic graph, except for pairs $k = 2$ and $p > 2$. Second, we have shown that the $PG(\mathbb{Z}_{p^2})$ is a $\mathbb{Z}_k$-vertex-magic graph for $p = 2$ or pairs

$p > 3$ and $k > 2$ with $\gcd(k, p-2) \neq 1$. Last, the $PG(\mathbb{Z}_{pq})$ is a $\mathbb{Z}_k$-vertex-magic graph for pairs $k > 2$ and $p, q > 2$. For further research, we can investigate the group-vertex-magic labeling of $PG(\mathbb{Z}_n)$ for other cases n and the group-vertex-magic labeling for other graphs constructed from algebraic structures.

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