Journal of Algebra and Related Topics
Vol. 13, No. 01, 2025, pp. 109-116.

On primal topological groups

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Abstract. In this paper, we introduce the notion of primal topological group, which is a generalization of topological group by a primal. We discuss the characterizations of primal topological groups with illustrative examples.

Keywords: Primal, Topology, Primal topological group.
AMS Subject Classification 2010: 22A05, 22A10, 22A30.

1 Introduction

The theory of topological algebra clenches a part of the literature on topology. Generalization of algebraic structures, especially groups and vector spaces through numerous generalized topological spaces emanates in the field of research over recent decades. The topological structures enriches with a finer topology by endowing filter, ideal, partially ordered set and grill. Recently primal was introduced as dual of grill. In this paper, we define primal topological group.

2 Terminology

In this paper, M connotes a group and Υ a topology on M with no separation axioms assumed. In a group M, we write mn instead of $m \cdot n$ for $m, n \in \mathsf{M}$ and m^{-1} indicates the inverse of m for $m \in \mathsf{M}$. Let $\mathsf{ST} = \{st : s \in \mathsf{S}, t \in \mathsf{T}\}$ and $\mathsf{S}^{-1} = \{s^{-1} : s \in \mathsf{S}\}$ where $\mathsf{S}, \mathsf{T} \subseteq \mathsf{M}$. We denote Left translation, right translation and inversion by $\lambda_m(a) = ma, \rho_m(a) = am$, $i(m) = m^{-1}$ and S symmetric if $\mathsf{S} = \mathsf{S}^{-1}$. The power set of R is denoted by $\mathsf{P}(\mathsf{R})$ for a set R. A subset $\mathcal{P}$ of $\mathsf{P}(\mathsf{M})$ is a primal on M if (i) $\mathsf{M} \notin \mathcal{P}$, (ii) If $\mathsf{S} \in \mathcal{P}$ and $\mathsf{T} \subseteq \mathsf{S}$, then $\mathsf{T} \in \mathcal{P}$, (iii) If $\mathsf{S} \cap \mathsf{T} \in \mathcal{P}$, then

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Received: 28 October 2023/ Revised: 28 January 2024/ Accepted: 29 January 2024
DOI: 10.22124/JART.2024.25894.1593

$\mathsf{S} \in \mathcal{P}$ or $\mathsf{T} \in \mathcal{P}$.The notions $int(\mathsf{S})$, $cl(\mathsf{S})$ connotes the interior, closure of S in M and $\mathsf{O}(\mathsf{M})$ denotes the collection of open sets in M respectively. In a primal topological space M, the local function [1] of $\mathsf{T} \subseteq \mathsf{M}$, $\wedge : \mathsf{P}(\mathsf{M}) \mapsto \mathsf{P}(\mathsf{M})$ concerning a primal $\mathcal{P}$ and a topology Υ is given by $\wedge(\mathsf{T}, \mathcal{P}, \Upsilon) = \{g \in \mathsf{M} : \mathsf{E}^c \cup \mathsf{T}^c \in \mathcal{P}$ for $\mathsf{E} \in \mathsf{O}(\mathsf{M})$ of $g\}$. We use $\wedge(\mathsf{T})$ instead of $\wedge(\mathsf{T}, \mathcal{P}, \Upsilon)$ and T is primal-open [1] if $\mathsf{T} \subseteq int(\wedge(\mathsf{T}))$. The primal-open sets are closed under union and intersection of primal-open set with an open set is primal open. On using the Kuratowski closure operator [5] $cl(.)$ defined by $cl(\mathsf{T}) = \mathsf{T} \cup \wedge(\mathsf{T})$, we obtain a topology Υ^* which is finer than Υ. The collection of closed (respectively, primal-open, primal-closed) sets in a primal topological space M is denoted by $\mathsf{C}(\mathsf{M})$ (respectively, $\mathsf{O}_{\mathsf{prim}}(\mathsf{M}), \mathsf{C}_{\mathsf{prim}}(\mathsf{M})$). For primal interior and primal closure we use $\mathsf{Int}_{\mathsf{prim}}$ and $\mathsf{Cl}_{\mathsf{prim}}$. A group M binded with a topology Υ and a primal $\mathcal{P}$ is denoted by the 3 - tuple $(\mathsf{M}, \Upsilon, \mathcal{P})$. In addition, we set forth some definitions and results required for the sequel.

Definition 1. *A map* $f : \mathsf{M} \mapsto \mathsf{N}$ *is*

(i) ***primal-continuous*** *if inverse image of an open set is primal-open.*

(ii) ***primal open*** *if image of an open set is primal-open*

(iii) ***primal-irresolute*** *if inverse image of a primal open set is primal-open.*

Definition 2. *A topological space* M *is*

(i) ***primal-irresolute*** *if primal open set of* M *is open.*

(ii) ***primal-regular*** *if for a closed set* C *of* M *and* $k \in \mathsf{M} \setminus \mathsf{C}$*, there is disjoint* $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ *such that* $\mathsf{C} \subseteq \mathsf{E}$ *and* $k \in \mathsf{F}$*.*

Lemma 1. *In a topological space* M *the following are equivalent:*

(i) M *is primal-regular.*

(ii) For $\mathsf{E} \in \mathsf{O}(\mathsf{M})$ *and* $x \in \mathsf{E}$*, there exists* $\mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ *of* x *such that* $\mathsf{Cl}_{prim}(\mathsf{F}) \subseteq \mathsf{E}$*.*

(iii) For a closed set C *of* M*,* $\cap\{\mathsf{Cl}_{prim}(\mathsf{F}) : \mathsf{C} \subseteq \mathsf{F}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})\} = \mathsf{C}$*.*

(iv) For a subset K *of* M *and* $\mathsf{E} \in \mathsf{O}(\mathsf{M})$ *such that* $\mathsf{K} \cap \mathsf{E} \neq \emptyset$*, there exists* $\mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ *such that* $\mathsf{K} \cap \mathsf{F} \neq \emptyset$ *and* $\mathsf{Cl}_{prim}(\mathsf{F}) \subseteq \mathsf{E}$*.*

(v) For a non-empty set K *of* M *and closed set* C *with* $\mathsf{K} \cap \mathsf{C} = \emptyset$*, there exists* $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ *such that* $\mathsf{K} \cap \mathsf{E} \neq \emptyset$*,* $\mathsf{C} \subseteq \mathsf{F}$ *and* $\mathsf{E} \cap \mathsf{F} = \emptyset$*.*

Proof. $(i) \Rightarrow (ii)$ Let $\mathsf{E} \in \mathsf{O}(\mathsf{M})$ with $x \in \mathsf{E}$, then $\mathsf{D} = \mathsf{M} \setminus \mathsf{E}$ is closed and $x \notin \mathsf{D}$. By (i), there exists $\mathsf{F}, \mathsf{H} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ such that $x \in \mathsf{F}, \mathsf{D} \subseteq \mathsf{H}$ and $\mathsf{F} \cap \mathsf{H} = \emptyset$ and so $\mathsf{Cl}_{prim}(\mathsf{F}) \cap \mathsf{H} = \emptyset$. Thus, $x \in \mathsf{F} \subseteq \mathsf{Cl}_{prim}(\mathsf{F}) \subseteq \mathsf{E}$.

$(ii) \Rightarrow (iii)$ Let C be a closed set in M. Since, closed set is primal closed, $\cap\{\mathsf{Cl}_{prim}(\mathsf{F}) : \mathsf{C} \subseteq \mathsf{F}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})\} \subseteq \mathsf{C}$. Conversely, let $x \notin \mathsf{C}$ then $\mathsf{K} = \mathsf{M} \setminus \mathsf{C} \in \mathsf{O}(\mathsf{M})$ and $x \in \mathsf{K}$. By (ii), there exists $\mathsf{R} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ such that $x \in \mathsf{R} \subseteq \mathsf{Cl}_{prim}(\mathsf{R}) \subseteq \mathsf{K}$. Put $\mathsf{F} = \mathsf{M} \setminus \mathsf{Cl}_{prim}(\mathsf{R})$. Then, $\mathsf{C} \subseteq \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ and $x \notin \mathsf{Cl}_{prim}(\mathsf{F})$ which implies that $\cap\{\mathsf{Cl}_{prim}(\mathsf{F}) : \mathsf{C} \subseteq \mathsf{F}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})\} = \mathsf{C}$.

$(iii) \Rightarrow (iv)$ Let $\mathsf{K} \subseteq \mathsf{M}$ and $\mathsf{E} \in \mathsf{O}(\mathsf{M})$ with $\mathsf{K} \cap \mathsf{E} \neq \emptyset$. Let $x \in \mathsf{K} \cap \mathsf{E}$ then $\mathsf{C} = \mathsf{M} \setminus \mathsf{E}$ is a closed set and $x \notin \mathsf{C}$. By (iii), there exists $\mathsf{H} \in \mathsf{O}_{prim}(\mathsf{M})$ such that $\mathsf{C} \subseteq \mathsf{H}$ and $x \in \mathsf{Cl}_{prim}(\mathsf{H})$. Put $\mathsf{F} = \mathsf{M} \setminus \mathsf{Cl}_{prim}(\mathsf{H})$ then $\mathsf{F} \in \mathsf{O}_{prim}(\mathsf{M})$, $x \in \mathsf{F} \cap \mathsf{K}$ and $\mathsf{Cl}_{prim}(\mathsf{F}) \subseteq \mathsf{Cl}_{prim}(\mathsf{M} \setminus \mathsf{H}) = \mathsf{M} \setminus \mathsf{H} \subseteq \mathsf{E}$.

$(iv) \Rightarrow (v)$ Let K be a non-empty subset and C be a closed set in M with $\mathsf{K} \cap \mathsf{C} = \emptyset$. Since $\mathsf{M} \setminus \mathsf{C}$ is open in M and $\mathsf{K} \neq \emptyset$, by (iv) there exists $\mathsf{E} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ such that $\mathsf{K} \cap \mathsf{E} \neq \emptyset$ and $\mathsf{Cl}_{prim}(\mathsf{E}) \subseteq \mathsf{M} \setminus \mathsf{C}$. Put $\mathsf{F} = \mathsf{M} \setminus \mathsf{Cl}_{prim}(\mathsf{E})$ then $\mathsf{C} \subseteq \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ and $\mathsf{E} \cap \mathsf{F} = \emptyset$.

$(v) \Rightarrow (i)$ By definition of primal-regular. □

Lemma 2. *Let* $(\mathsf{M}, \Upsilon, \mathcal{P})$ *be a primal topological space.*

(i) For $\mathsf{K} \in \mathsf{O}(\mathsf{M})$ *of* xy *there exist* $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ *of* x, y *such that* $\mathsf{EF} \subseteq \mathsf{K}$ *if and only if the map* $f : \mathsf{M} \times \mathsf{M} \mapsto \mathsf{M}$ *by* $f(x, y) = xy$ *is primal-continuous.*

(ii) For $\vartheta \in \mathsf{O}(\mathsf{M})$ *of* x^{-1} *there exists* $\mathsf{Q} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ *of* x *such that* $\mathsf{Q}^{-1} \subseteq \vartheta$ *if and only if the map* $i : \mathsf{M} \mapsto \mathsf{M}$ *by* $i(x) = x^{-1}$ *is primal-continuous.*

Proof. (i) Suppose, for $\mathsf{K} \in \mathsf{O}(\mathsf{M})$ of xy there exist $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of x, y such that $\mathsf{EF} \subseteq \mathsf{K}$. Let $\mathsf{D} = \mathsf{E} \times \mathsf{F} \subseteq \mathsf{M} \times \mathsf{M}$ then

$$\begin{aligned} \mathsf{D} = \mathsf{E} \times \mathsf{F} &\subseteq int(\wedge(\mathsf{E})) \times int(\wedge(\mathsf{F})) \quad \text{since } \mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M}) \\ &\subseteq int(\wedge(\mathsf{E}) \times \wedge(\mathsf{F})) \quad \text{since } int(\mathsf{A} \times \mathsf{B}) = int(\mathsf{A}) \times int(\mathsf{B}) \\ &\subseteq int(\wedge(\mathsf{E} \times \mathsf{F})) \quad \text{since } \wedge(\mathsf{A} \times \mathsf{B}) = \wedge(\mathsf{A}) \times \wedge(\mathsf{B}) \\ &\subseteq int(\wedge(\mathsf{D})) \quad \text{since } \mathsf{D} = \mathsf{E} \times \mathsf{F}. \end{aligned}$$

Thus, (x, y) is an element of $\mathsf{D} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M} \times \mathsf{M})$ and $f(\mathsf{D}) \subseteq \mathsf{K}$. Hence, for $\mathsf{K} \in \mathsf{O}(\mathsf{M})$ of xy, the inverse image $\mathsf{D} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M} \times \mathsf{M})$ and thus f is primal-continuous. The converse implication can be proved by reversing the above argument.

(ii) Suppose, for an open neighbourhood ϑ of x^{-1} there exists $\mathsf{Q} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of x such that $\mathsf{Q}^{-1} \subseteq \vartheta$ then $i(\mathsf{Q}) = \mathsf{Q}^{-1} \subseteq \vartheta$. Thus, inverse image of an open set ϑ of x^{-1} is Q which is primal-open and so i is primal-continuous. The converse implication can be proved by reversing the above argument. □

3 Primal topological group

In this section, we introduce the concept of primal topological group and investigate its basic properties with illustrated examples.

Definition 3. *A 3 - tuple* $(\mathsf{M}, \Upsilon, \mathcal{P})$ *is* ***primal topological group*** *if:*

for $\mathsf{K} \in \mathsf{O}(\mathsf{M})$ *of* xy *there exist* $\mathsf{S}, \mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ *of* x, y *such that* $\mathsf{ST} \subseteq \mathsf{K}$.

for $\mathsf{S} \in \mathsf{O}(\mathsf{M})$ *of* x^{-1} *there exists* $\mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ *of* x *such that* $\mathsf{T}^{-1} \subseteq \mathsf{S}$.

By Lemma 2.4, It is equivalently saying that, in a primal topological group, multiplication and inversion are primal-continuous.

Example 1. Let M be group of order greater than 2, $x \in \mathsf{M}$ binded with a topology $\Upsilon = \{\emptyset, \mathsf{M} \setminus \{x\}, \mathsf{M}\}$ and a primal $\mathcal{P} = \mathsf{P}(\mathsf{M}) \setminus \{\mathsf{M} \setminus \{x\}, \mathsf{M}\}$. Then $\wedge(\{x\}) = \emptyset$. For $\mathsf{T} \subseteq \mathsf{M}$ other than $\{x\}$, we have $\wedge(\mathsf{T}) = \mathsf{M}$. Thus, $\mathsf{O}_{\mathsf{prim}}(\mathsf{M}) = \mathsf{P}(\mathsf{M}) \setminus \{x\}$. By Definition 3.1, $(\mathsf{M}, \Upsilon, \mathcal{P})$ is a primal topological group.

Example 2. Consider the addition modulo group $\mathbb{Z}_2$ with discrete topology and a primal $\mathcal{P} = \mathsf{P}(\mathsf{M}) \setminus \{\mathsf{M} \setminus \{0\}, \mathsf{M}\}$. Then $\wedge(\{0\}) = \emptyset, \wedge(\{1\}) = \wedge(\{0,1\}) = \{0,1\}$ and so $\mathsf{O}_{\mathsf{prim}}(\mathsf{M}) = \mathsf{P}(\mathsf{M}) \setminus \{0\}$. By Definition 3.1, $(\mathbb{Z}_2, \Upsilon, \mathcal{P})$ is not a primal topological group.

Proposition 1. *If* $(\mathsf{M}, \Upsilon, \mathcal{P})$ *is a primal topological group, then*

(i) $\mathsf{K} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ *if and only if* $\mathsf{K}^{-1} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$.

(ii) If $\mathsf{K} \in \mathsf{O}(\mathsf{M})$ *and* $\mathsf{N} \subseteq \mathsf{M}$, *then* KN *and* NK *are both in* $\mathsf{O}_{\mathsf{prim}}(\mathsf{M})$.

Proof. (i) Let $\mathsf{K} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$. Then there exists $\mathsf{S} \in \mathsf{O}(\mathsf{M})$ such that $\mathsf{K} \subseteq \mathsf{S}$. Now, $\mathsf{K}^{-1} \subseteq \mathsf{S}^{-1}$ and $\wedge(\mathsf{S}^{-1}) \subseteq \wedge(\mathsf{K}^{-1})$. Since inversion is primal-continuous, then S^{-1} is primal-open and so K^{-1} is primal-open in M. Hence, for $\mathsf{K} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$, we have $\mathsf{K}^{-1} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$.

(ii) Let $n \in \mathsf{N}$, $a \in n\mathsf{K}$, then $a = nk$ for some $k \in \mathsf{K}$. Now, $k = n^{-1}a$ and by Definition 3, there exist $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of n^{-1} and a such that $\mathsf{EF} \subseteq \mathsf{K}$ which implies $a \in \mathsf{F} \subseteq n\mathsf{K}$. Hence $n\mathsf{K}$ is primal-open. Since primal-open sets are closed under union, $\mathsf{NK} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$. By the same token, we can prove that $\mathsf{KN} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$. □

Proposition 2. *Let* C *be closed subset of a primal topological group* M. *Then* $a\mathsf{C}$ *and* $\mathsf{C}a$ *are primal-closed, for* $a \in \mathsf{M}$.

Proof. Let $x \in \mathsf{Cl}_{prim}(a\mathsf{C})$, $b = a^{-1}x$ and D be an open neighbourhood of b. Then by Definition 3, there exist $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of a^{-1} and x in M such that $\mathsf{EF} \subseteq \mathsf{D}$. Since $x \in \mathsf{Cl}_{prim}(a\mathsf{C})$, $\mathsf{F} \cap \mathsf{aC} \neq \emptyset$. Let $c \in \mathsf{F} \cap a\mathsf{C}$, then $a^{-1}c \in \mathsf{C} \cap \mathsf{EF} \subseteq \mathsf{C} \cap \mathsf{D}$ which implies $\mathsf{C} \cap \mathsf{D} \neq \emptyset$. Thus b is a limit point of C. Since C is closed, $\mathsf{b} \in \mathsf{C}$. Now $x = ab$ and so $x \in a\mathsf{C}$. By the above argument, $\mathsf{Cl}_{prim}(a\mathsf{C}) \subseteq a\mathsf{C}$ and since $a\mathsf{C} \subseteq \mathsf{Cl}_{prim}(a\mathsf{C})$ is trivial then $a\mathsf{C} = \mathsf{Cl}_{prim}(a\mathsf{C})$. Hence $a\mathsf{C}$ is Primal-closed. Proof of $\mathsf{C}a$ is similar. □

Theorem 1. *Let* K *and* N *be subsets of primal topological group* M. *Then* $\mathsf{Cl}_{prim}(\mathsf{K}).\mathsf{Cl}_{prim}(\mathsf{N}) \subseteq \mathsf{Cl}(\mathsf{KN})$.

Proof. Let $a \in \mathsf{Cl}_{prim}(\mathsf{K}).\mathsf{Cl}_{prim}(\mathsf{N})$ and D be an open neighbourhood of a in M where $a = kn$ for some $k \in \mathsf{Cl}_{prim}(\mathsf{K})$ and $n \in \mathsf{Cl}_{prim}(\mathsf{N})$. By definition of Primal topological group, there exists $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ containing k and n, respectively such that $\mathsf{EF} \subseteq \mathsf{D}$. Since $k \in \mathsf{Cl}_{prim}(\mathsf{K})$ and $n \in \mathsf{Cl}_{prim}(\mathsf{N})$ there exist $c \in \mathsf{K} \cap \mathsf{E}$ and $d \in \mathsf{N} \cap \mathsf{F}$. Now $cd \in (\mathsf{MN}) \cap (\mathsf{EF}) \subseteq \mathsf{KN} \cap \mathsf{D}$ which implies that $\mathsf{KN} \cap \mathsf{D} \neq \emptyset$. Hence a is a limit point of KN and therefore $a \in \mathsf{Cl}(\mathsf{KN})$. □

Definition 4. *A mapping* $f : \mathsf{S} \mapsto \mathsf{T}$ *is* ***primal-homeomorphism*** *if* f *is bijective, primal-continuous and primal-open.*

Theorem 2. *Let* $(\mathsf{M}, \Upsilon, \mathcal{P})$ *be a primal topological group. Then left (right) translations and inversion are primal-homeomorphisms.*

Proof. (i) Let $a, b \in \mathsf{M}$ and $\mathsf{D}_1 \in \mathsf{O}(\mathsf{M})$ with $ab \in \mathsf{D}_1$. By Definition 3, for $\mathsf{D}_1 \in \mathsf{O}(\mathsf{M})$ of ab there exist $\mathsf{E}_1, \mathsf{F}_1 \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of a and b such that $\mathsf{E}_1\mathsf{F}_1 \subseteq \mathsf{D}_1$ which implies $\mathsf{a}\mathsf{F}_1 \subseteq \mathsf{D}_1$ and so left translation is primal-continuous. Let $g \in \mathsf{M}$ and $\mathsf{D}_2 \in \mathsf{O}(\mathsf{M})$ of g. The element g can be written as $g = a^{-1}ag$. Since left translation is primal-continuous, there exist $\mathsf{E}_2, \mathsf{F}_2 \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of a^{-1} and ag such that $\mathsf{E}_2\mathsf{F}_2 \subseteq \mathsf{D}_2$. Hence, left translation is primal-homeomorphism. The proof of right translations is similar.

(ii) Let S_1 be an open neighbourhood of a^{-1}. Since M is primal topological group, for $\mathsf{S}_1 \in \mathsf{O}(\mathsf{M})$ of a^{-1} there exists $\mathsf{T}_1 \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of a such that $\mathsf{T}_1^{-1} \subseteq \mathsf{S}_1$. Thus, the inversion mapping is primal-continuous. Let S_2 be an open neighbourhood of a. Since inversion is primal-continuous there exists $\mathsf{T}_2 \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of a^{-1} such that $\mathsf{T}_2^{-1} \subseteq \mathsf{S}_2$. Hence the inversion is primal-homeomorphism. □

Theorem 3. *Let* $(\mathsf{M}, \Upsilon, \mathcal{P})$ *be a primal topological group and let* $\mathfrak{B}_e$ *be the base at identity element* e *of* M. *Then*

(i) for $\mathsf{S} \in \mathfrak{B}_e$*, there exists* $\mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})_e$ *such that* $\mathsf{T}^2 \subseteq \mathsf{S}$.

(ii) for $\mathsf{S} \in \mathfrak{B}_e$*, there exists* $\mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})_e$ *such that* $\mathsf{T}^{-1} \subseteq \mathsf{S}$.

(iii) for $\mathsf{S} \in \mathfrak{B}_e, g \in \mathsf{S}$*, there exists* $\mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})_e$ *such that* $g.\mathsf{T} \subseteq \mathsf{S}$ $(\mathsf{T}.g \subset \mathsf{S})$.

Proof. (i) Let $\mathsf{S} \in \mathfrak{B}_e$. Then S is an open neighbourhood of e. We know that $e = e.e$ and by definition of primal topological group, there exist $\mathsf{O}, \mathsf{P} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of e such that OP is contained in S. Let K be the smallest among O and P and so there exists $\mathsf{K} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})_e$ such that $\mathsf{K}^2 \subseteq \mathsf{S}$.

(ii) Let $\mathsf{S} \in \mathfrak{B}_e$. Then S is an open neighbourhood of e. We know that the inverse of e is itself. Since the inversion mapping $a \mapsto a^{-1}$ is primal-continuous on M, there exists $\mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of e such that $\mathsf{T}^{-1} \subseteq \mathsf{S}$.

(iii) Let $\mathsf{S} \in \mathfrak{B}_e$ and $g \in \mathsf{S}$. We know that $g = g.e$ $(g = e.g)$. Since M is a primal topological group, by Definition 3, there exist $\mathsf{P} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of g and $\mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of e such that $\mathsf{PT}(\mathsf{TP})$ is contained in S. So for $g \in \mathsf{S}$, there is a $\mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})_e$ such that $g\mathsf{T} \subseteq \mathsf{S}$ $(\mathsf{T}g \subseteq \mathsf{S})$. □

Theorem 4. *Let* $(\mathsf{M}, \Upsilon, \mathcal{P})$ *be a primal topological group and* K *a subgroup of* M.

(i) If K *contains a non - empty set* $\mathsf{S} \in \mathsf{O}(\mathsf{M})$ *then* $\mathsf{K} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$.

(ii) An open subgroup of M *is primal-closed.*

(iii) An open subgroup K *is also a primal topological group.*

Proof. (i) Suppose K contains a non - empty set $\mathsf{S} \in \mathsf{O}(\mathsf{M})$. By Theorem 3.9, translation is primal-homeomorphism, so $\mathsf{S}m$ is primal-open in M for $m \in \mathsf{K}$. Since primal-open sets are closed under union, then $\mathsf{K} = \cup_{m\in\mathsf{K}}\mathsf{S}m$ is primal-open in M.

(ii) Let K be an open subgroup of M. Then $\gamma = \{\mathsf{K}a_i : a_i \in \mathsf{M}\}$ is the family of right cosets of K which is disjoint primal-open covering of M. Thus, $\mathsf{M} = \cup_{a_i\in\mathsf{M}}\mathsf{K}a_i$ and so $\mathsf{K}a_i = \left(\cup_{a_j\neq a_i\in\mathsf{M}}\mathsf{K}a_j\right)^c$. Therefore, an element of γ is both primal-open and primal-closed. In particular, $\mathsf{K} = \mathsf{K}e$ is primal-closed in M.

(iii) We have to show that for $a, b \in \mathsf{K}$ and $\mathsf{D} \in \mathsf{O}(\mathsf{K})$ of ab^{-1} in K, there exist $\mathsf{S} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{K})$ of a and $\mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{K})$ of b such that $\mathsf{ST}^{-1} \subseteq \mathsf{D}$. Since M is a primal topological group, there exist $\mathsf{E} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of a and $\mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of b such that $\mathsf{EF}^{-1} \subseteq \mathsf{D}$. Since K is open, the sets $\mathsf{S} = \mathsf{K} \cap \mathsf{E}$ and $\mathsf{T} = \mathsf{K} \cap \mathsf{F}$ are primal-open. Thus, $\mathsf{ST}^{-1} \subseteq \mathsf{EF}^{-1} \subseteq \mathsf{D}$. □

Theorem 5. *Let* M *and* S *be primal topological groups,* S *primal irresolute and* f *be a homomorphism which is primal-irresolute at identity* e_{M}*. Then* f *is primal-irresolute.*

Proof. Let $a \in \mathsf{M}$ and E be a primal-open set in S containing $f(a) = b$. Since primal-open sets is open in S, E is open. By Proposition 3.4, left translation of an open set is primal-open and thus $b^{-1}\mathsf{E} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{S})$ containing e_{S}. Since f is primal-irresolute at identity e_{M}, there exists $\mathsf{F} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ containing e_{M} such that $f(\mathsf{F}) \subseteq b^{-1}\mathsf{E}$. Given that f is homomorphism, it follows that $f(a\mathsf{F}) = f(a)f(\mathsf{F}) \subseteq \mathsf{E}$. This means that f is primal-irresolute. □

Theorem 6. *Let* M *be a primal topological group with base* $\mathfrak{B}_e$ *at the identity element* e *such that for* $\mathsf{S} \in \mathfrak{B}_e$ *there is a symmetric open neighbourhood* ϑ *of* e *such that* $\vartheta^2 \subseteq \mathsf{S}$*. Then* M *satisfies primal-regularity at* e*.*

Proof. Let $\mathsf{S} \in \mathsf{O}(\mathsf{M})$ containing the identity e. By assumption, there is a symmetric $\vartheta \in \mathsf{O}(\mathsf{M})$ of e such that $\vartheta^2 \subseteq \mathsf{S}$. We have to prove that primal-closure of ϑ is contained in S. Let $a \in \mathsf{Cl}_{prim}(\vartheta)$. The set $a\vartheta$ is a primal-open neighbourhood of a, which implies $a\vartheta \cap \vartheta \neq \emptyset$. Therefore, there exists points $b, c \in \vartheta$ such that $c = ab$ and so, $a = cb^{-1} \in \vartheta\vartheta^{-1} = \vartheta\vartheta \subseteq \mathsf{S}$. Thus $\mathsf{Cl}_{prim}(\vartheta) \subseteq \mathsf{S}$. □

Definition 5. *A primal topological group* M *is primal-connected if* M *cannot be written as union of two disjoint non - empty primal-open sets in* M*.*

Theorem 7. *Let* M *be a primal topological group which is primal irresolute and* K *be a subgroup of* M*. If* K*,* M/K *are primal-connected, then* M *is primal-connected.*

Proof. Suppose M is not primal-connected. Assume that $\mathsf{M} = \mathsf{E} \cup \mathsf{F}$ where E and F are disjoint non - empty primal-open sets. Since K is primal-connected, coset of K is either a subset of E or a subset of F. Thus,

$$\begin{aligned} \mathsf{M}/\mathsf{K} &= \{a\mathsf{K} : a\mathsf{K} \subseteq \mathsf{E}\} \cup \{a\mathsf{K} : a\mathsf{K} \subseteq \mathsf{F}\} \\ &= \{a\mathsf{K} : a \in \mathsf{E}\} \cup \{a\mathsf{K} : a \in \mathsf{F}\}. \end{aligned}$$

So, M/K is expressed as union of disjoint non - empty primal-open sets which is a contradiction to primal-connectedness of M/K. Thus, M is primal-connected. □

Theorem 8. *Let* M *be a primal topological group which is primal-connected and primal irresolute with identity* e*. If* S *is primal-open neighbourhood of* e*, then* M *is generated by* S*.*

Proof. Let S be a primal-open neighbourhood of e. For $n \in \mathbb{N}$, we denote S^n by the set of elements of the form $s_1.s_2 \dots s_n$ where $s_i \in \mathsf{S}$. Let $\mathsf{T} = \cup_{n=1}^{\infty} \mathsf{S}^n$. Since M is primal-connected, suppose if we prove $\mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ and $\mathsf{T} \in \mathsf{C}_{\mathsf{prim}}(\mathsf{M})$, then $\mathsf{M} = \mathsf{T}$ and so M is generated by S. Since S^n is primal-open and union of primal-open sets is primal-open, T is primal-open. Let us prove that T is primal-closed. Let $a \in \mathsf{Cl}_{prim}(\mathsf{T})$. Since $a\mathsf{S}^{-1}$ is a primal-open neighbourhood of a, it must intersect T. Thus, let $b \in \mathsf{T} \cap a\mathsf{S}^{-1}$. Since $b \in a\mathsf{S}^{-1}$ then $b = a.s^{-1}$ for some $s \in \mathsf{S}$. Since $b \in \mathsf{T}$ then $b \in \mathsf{S}^n$ for some $n \in \mathbb{N}$ which implies $b = s_1 s_2 \dots s_n$ with $s_i \in \mathsf{S}$. Now, $a = s_1 s_2 \dots s_n.s$ and so $a \in \mathsf{S}^{n+1} \subseteq \mathsf{T}$. Hence T is primal-closed. Since M is primal-connected with $\mathsf{T} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ and $\mathsf{T} \in \mathsf{C}_{\mathsf{prim}}(\mathsf{M})$ then $\mathsf{T} = \mathsf{M}$. Thus, M is generated by S. □

Theorem 9. *If* M *is a primal topological group which is primal-connected and primal irresolute with* H*, a discrete invariant subgroup of* M*, then* $\mathsf{H} \subseteq \mathsf{Z}(\mathsf{M})$*, where* $\mathsf{Z}(\mathsf{M})$ *denotes the center of* M*.*

Proof. Suppose $\mathsf{H} = \{e\}$, then the result is trivial. Suppose H is non - trivial. Let $h \neq e \in \mathsf{H}$. Since H is discrete, we can find $\mathsf{D} \in \mathsf{O}_{\mathsf{prim}}(\mathsf{M})$ of h in M such that $\mathsf{D} \cap \mathsf{H} = \{h\}$. Now, by definition of primal topological group, there exists a primal-open neighbourhood E of e and a primal-open neighbourhood $\mathsf{E}.h$ of h in M such that $(\mathsf{E}.h).\mathsf{E}^{-1} \subseteq \mathsf{D}$. Let $b \in \mathsf{E}$ be arbitrary. Since H is an invariant subgroup of M, $b.\mathsf{H} = \mathsf{H}.b$ which implies that $b.h \in \mathsf{H}.b$ and so $b.h.b^{-1} \in \mathsf{H}$. Also, $b.h.b^{-1} \in \mathsf{E}h\mathsf{E}^{-1} \subseteq \mathsf{D}$. Therefore, $b.h.b^{-1} \in \mathsf{D} \cap \mathsf{H} = \{h\}$ which implies $b.h.b^{-1} = h$. Thus, $b.h = h.b$ for $b \in \mathsf{E}$. Since M is primal-connected, E^n with $n \in \mathbb{N}$ covers M. Thus, $a \in \mathsf{M}$ can be written in the form $a = b_1.b_2 \dots b_n$ where $b_1, b_2, \dots, b_n \in \mathsf{E}$ and $n \in \mathbb{N}$. Since h commutes with every element of E,

$$\begin{aligned}
a.h &= b_1.b_2 \dots b_n.h \\
&= b_1.b_2 \dots h.b_n \\
&\vdots \\
&= b_1.h.b_2 \dots b_n \\
&= h.b_1.b_2 \dots b_n \\
&= h.a
\end{aligned}$$

Hence $h \in \mathsf{H}$ is in the center of M. Since h is arbitrary, we proved that center of M contains H. □

4 Conclusion

In this article, the notion of generalized topological group by endowing a primal and also proved some characterizations. We observe that, endowment of structures together with a topology will bestow generalized topological groups with ambivalent properties. Similar generalized version of topological groups were discussed in [2–4, 6].

Acknowledgments

The authors would like to thank the referee for careful reading.

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