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New models of four dimensional absolute valued algebras

Noureddine Motya [†*], **Mouanis Hakima**[‡], **Abdelhadi Moutassim**[§]

[†] *Department of Mathematics, Sidi Mohamed Ben Abdellah University, Faculty of Scinces Dhar El Mahraz, Fez-Atlas, Morocco*
[‡] *Department of Mathematics, Sidi Mohamed Ben Abdellah University, Faculty of Scinces Dhar El Mahraz, Fez-Atlas, Morocco*
[§] *Regional Center For Education And Training Professions, Casablanca-Settat, Morocco*
Emails: noureddine.motya@usmba.ac.ma, hmouanis@yahoo.fr, moutassim-1972@hotmail.fr

Abstract. This paper deals with some results concerning the 4-dimensional absolute valued algebras with left omnipresent unit. We also construct, by algebraic methods some new models of 4-dimensional absolute valued algebras with left omnipresent unit. These new algebras contain at least one 2-dimensional sub-algebra, and aren't isomorphic to $\mathbb{H}$ (The quaternion algebra).

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1 Introduction

Let A be a non necessarily associative real algebra which is normed as real vector space. We say that A is a pre-Hilbert algebra, if it's norm $\|.\|$ comes from an inner product $(./.)$ (See [12, 13, 19, 21] and [10]), and is said to be absolute valued algebra (AVA), if it's satisfy the equality $\|ab\| = \|a\|\|b\|$, for all $a, b \in A$. The readers are referred to [1], for basis facts and intrinsic characterizations of these classical AVA. The last decades have known several works in the theme for algebras which either having left-unit ([14, 15, 17] and [7]). Or finite-dimensional ([4] and [9]). We recall that every AVA, A is a normed algebra. Note that, the norm of any AVA with left unit (or finite dimensional) comes from an inner product (see [5, 15]). In 1947 Albert proved that the finite dimensional unital AVA are classified by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ and that every finite dimensional AVA is isotopic to one of the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ and so has dimension 1, 2, 4 or 8 (See [1]). Urbanik and Wright proved in 1960 that all unital AVA are classified by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$

[*]Corresponding author

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(See [20]). It is easily seen that the 1-dimensional AVAs are classified by $\mathbb{R}$, and it is well-known that the 2-dimensional AVAs are isomorphic to $\mathbb{C}$, $^*\mathbb{C}$, $\mathbb{C}^*$ or $\overset{*}{\mathbb{C}}$ (see [16]). The 4-dimensional one have been described by M.I. Ramírez Álvarez in 1997 (See [14]). Moreover, El-Mallah([6]) proved that if an AVA, A satisfies the identities $(x^2, x, x) = 0$ and contains a nonzero central element a which is orthogonal to a^2, then A has a left unit $e = -a^2$ and is isomorphic to $\mathbb{C}$. This result was extended to the following case: if A has a left unit and contains a nonzero central element a, then A is finite dimensional and isomorphic to $\mathbb{R}$, $\mathbb{C}$ or new classes of 4 and 8-dimensional AVAs with left unit (see [2] and [3]). In the other hand we classify algebraically all 4-dimensional AVAs containing a non zero omnipresent idempotent ([11]). The problem of classifying all (4 or 8)-dimensional AVAs seems still to be open.
Motivated by these facts, we became interested in the study of 4-dimensional AVAs with left omnipresent unit, such an algebra contains at least one 2-dimensional subalgebra. On the other hand we note that there exists a 4-dimensional AVAs with left unit containing no 2-dimensional subalgebra (see [14])). We get that a left omnipresent unit is a left unit, but the reciprocal case does not hold in general. So, a natural question may be posed as: what is the classification of 4-dimensional AVA with left omnipresent unit? This paper is devoted to shed some light on this problem.
In Section 2, we introduce the basic tools for the study of 4-dimensional AVAs with left omnipresent unit. Moreover, in Section 3, we construct by an algebraic method, all 4-dimensional AVAs with left omnipresent unit, namely $\mathbb{M}_1(\alpha,\beta)$, $\mathbb{M}_2(\alpha,\beta)$, $^*\mathbb{M}_1(\alpha,\beta)$ and $^*\mathbb{M}_2(\alpha,\beta)$, with $(\alpha,\beta) \in \mathbb{R}^2$ such that $\alpha^2+\beta^2 = 1$. These new algebras contains at least one 2-dimensional subalgebra. In Section 4 we give the algebraic calssification of all four dimensional AVA with left omnipresent unit.
The paper ends, with the following main result.

Theorem 1. *Let A be a 4-dimensional AVA with left omnipresent unit e, and B a 2-dimensional subalgebra of A. Then the following table specifies the isomorphisms classes,*

Table 1

B isomorphic to	A isomorphic to
$\mathbb{C}$	$\mathbb{M}_1(\alpha,\beta)$, $\mathbb{M}_2(\alpha,\beta)$, $\mathbb{M}_1(0,\pm1)$, $\mathbb{M}_2(0,\pm1)$
$^*\mathbb{C}$	$^*\mathbb{M}_1(\alpha,\beta)$, $^*\mathbb{M}_2(\alpha,\beta)$, $^*\mathbb{M}_1(0,\pm1)$, $^*\mathbb{M}_2(0,\pm1)$

with $(\alpha,\beta) \in \mathbb{R}^2$ such that $\alpha^2+\beta^2 = 1$.

2 Notations and Preliminaries Results

Throughout this paper, the word algebra refers to a non-necessarily associative algebra over the field of real numbers $\mathbb{R}$.

Definition 1. *Let A be an arbitrary algebra.*

i) A is called a normed algebra (resp, absolute valued algebra) if it is endowed with a space norm: $\|.\|$ such that $\|xy\| \le \|x\|\|y\|$ (resp, $\|xy\| = \|x\|\|y\|$), for all $x, y \in A$.

ii) A is called a pre-Hilbert algebra if it is endowed with a space norm comes from an inner product (./.) such that

$$(./.) : A \times A \longrightarrow \mathbb{R} \qquad (x, y) \longmapsto (x/y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

iii) We mean by a left omnipresent unit e, an idempotent ($e^2 = e$) which is contained in every two-dimensional subalgebra of A, and $ex = x$ for all $x \in A$.

The most natural examples of AVA are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the Quaternion of Hamilton) and $\mathbb{O}$ (the algebra of Cayley numbers), with norms equal to their usual absolute values(see [8] and [17]). The algebras $^*\mathbb{C}$, $\mathbb{C}^*$, and $\overset{*}{\mathbb{C}}$ (obtained by endowing the space $\mathbb{C}$ with the products defined by $x * y = \bar{x}y$, $x * y = x\bar{y}$ and $x * y = \bar{x}\bar{y}$, respectively) where $x \to \bar{x}$ is the standard conjugation of $\mathbb{C}$. Note that the algebras $\mathbb{C}$ and $^*\mathbb{C}$ are the only 2-dimensional AVA with left unit.
We need the following theorems which proved respectively in [15] and [18]:

Theorem 2. *The norm of any AVA A with left unit e comes from an inner product (./.), satisfying $(xy/z) = -(y/xz)$ and $x(xy) = -\|x\|^2 y$ for all $x, y, z \in A$ with x orthogonal to e.*

Theorem 3. *Every algebra in which $x^2 = 0$ only if $x = 0$ contains a nonzero idempotent.*

Lemma 1. *Let A be a finite-dimensional AVA and e be a nonzero idempotent in A, then $(xy/yx) = -(x^2/y^2)$ for all $x, y \in e^\perp$ such that $(x/y) = 0$.*

Proof. We get this identity by a simple linearisation of the identity $\|x^2\| = \|x\|^2$. □

Lemma 2. *The 2-dimensional AVA with left unit are isomorphic to $\mathbb{C}$, or $^*\mathbb{C}$.*

Proof. See [10]. □

3 New Models of 4-dimensional absolute valued algebras with left omnipresent unit

Now we construct all 4-dimensional absolute valued algebras with left omnipresent unit.

3.1 Construction of $\mathbb{M}_1(\alpha, \beta)$ and $\mathbb{M}_2(\alpha, \beta)$

Let $F = \{e, i, j, k\}$ be an orthonormal basis (where e is a nonzero idempotent), and let $\mathbb{M}_1(\alpha, \beta)$ and $\mathbb{M}_2(\alpha, \beta)$, with $(\alpha, \beta) \in \mathbb{R}^2$ such that $\alpha^2 + \beta^2 = 1$, be the real pre-Hilbert algebras defined by the multiplication tables relatively to the basis F.

Table 2: $\mathbb{M}_1(\alpha,\beta)$

	e	i	j	k
e	e	i	j	k
i	i	-e	k	-j
j	$-\alpha j-\beta k$	$-\beta j+\alpha k$	$\alpha e+\beta i$	$\beta e-\alpha i$
k	$\beta j-\alpha k$	$-\alpha j-\beta k$	$-\beta e+\alpha i$	$\alpha e+\beta i$

Table 3: $\mathbb{M}_2(\alpha,\beta)$

	e	i	j	k
e	e	i	j	k
i	i	-e	k	-j
j	$-\alpha j-\beta k$	$-\beta j+\alpha k$	$\alpha e+\beta i$	$\beta e-\alpha i$
k	$-\beta j+\alpha k$	$\alpha j+\beta k$	$\beta e-\alpha i$	$-\alpha e-\beta i$

Proposition 1. *The algebras* $\mathbb{M}_1(\alpha,\beta)$ *and* $\mathbb{M}_2(\alpha,\beta)$ *are AVAs with left omnipresent unit* e*.*

Proof. It suffices to show that $\mathbb{M}_1(\alpha,\beta)$ is AVA with left omnipresent unit e, and in the same way we prove the other one.
Let $x=\alpha_1 e+\beta_1 i+\gamma_1 j+\delta_1 k\in\mathbb{M}_1(\alpha,\beta)$. Then, we have

$$\begin{aligned}
\|xe\|^2 &= \|\alpha_1 e+\beta_1 i+\gamma_1 je+\delta_1 ke\|^2\\
&= \|\alpha_1 e+\beta_1 i+(-\gamma_1\alpha+\delta_1\beta)j+(-\gamma_1\beta-\delta_1\alpha)k\|^2\\
&= \alpha_1^2+\beta_1^2+(-\gamma_1\alpha+\delta_1\beta)^2+(-\gamma_1\beta-\delta_1\alpha)^2\\
&= \alpha_1^2+\beta_1^2+\gamma_1^2\alpha^2+\delta_1^2\beta^2-2\beta_1\gamma_1\alpha\delta_1+\gamma_1^2\beta^2+\delta_1^2\alpha^2+2\gamma_1\beta\delta_1\alpha\\
&= \alpha_1^2+\beta_1^2+\gamma_1^2+\delta_1^2 \qquad (\alpha^2+\beta^2=1)\\
&= \|x\|^2,
\end{aligned}$$

$$\begin{aligned}
\|xi\|^2 &= \|\alpha_1 i-\beta_1 e+\gamma_1 ji+\delta_1 ki\|^2\\
&= \|-\beta_1 e+\alpha_1 i+(-\gamma_1\beta-\delta_1\alpha)j+(\gamma_1\alpha-\delta_1\beta)k\|^2\\
&= \beta_1^2+\alpha_1^2+(-\gamma_1\beta-\delta_1\alpha)^2+(\gamma_1\alpha-\delta_1\beta)^2\\
&= \alpha_1^2+\beta_1^2+\gamma_1^2\beta^2+\delta_1^2\alpha^2+2\gamma_1\beta\delta_1\alpha+\gamma_1^2\alpha^2+\delta_1^2\beta^2-2\gamma_1\alpha\delta_1\beta\\
&= \alpha_1^2+\beta_1^2+\gamma_1^2+\delta_1^2 \qquad (\alpha^2+\beta^2=1)\\
&= \|x\|^2,
\end{aligned}$$

$$\begin{aligned}
\|xj\|^2 &= \|\alpha_1 ej+\beta_1 ij+\gamma_1 j^2+\delta_1 kj\|^2\\
&= \|(\gamma_1\alpha-\delta_1\beta)e+(\gamma_1\beta+\delta_1\alpha)i+\alpha_1 j+\beta_1 k\|^2\\
&= (\gamma_1\alpha-\delta_1\beta)^2+(\gamma_1\beta+\delta_1\alpha)^2+\alpha_1^2+\beta_1^2\\
&= \gamma_1^2\alpha^2+\delta_1^2\beta^2-2\gamma_1\alpha\delta_1\beta+\gamma_1^2\beta^2+\delta_1^2\alpha^2+2\gamma_1\beta\delta_1\alpha+\beta_1^2\\
&= \alpha_1^2+\beta_1^2+\gamma_1^2+\delta_1^2 \qquad (\alpha^2+\beta^2=1)\\
&= \|x\|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|xk\|^2 &= \|\alpha_1 ek + \beta_1 ik + \gamma_1 jk + \delta_1 k^2\|^2 \\
&= \|(\gamma_1\beta + \delta_1\alpha)e + (-\gamma_1\alpha + \delta_1\beta)i - \beta_1 j + \alpha_1 k\|^2 \\
&= (\gamma_1\beta + \delta_1\alpha)^2 + (-\gamma_1\alpha + \delta_1\beta)^2 + \beta_1^2 + \alpha_1^2 \\
&= \gamma_1^2\beta^2 + \delta_1^2\alpha^2 + 2\gamma_1\beta\delta_1\alpha + \gamma_1^2\alpha^2 + \delta_1^2\beta^2 - 2\gamma_1\alpha\delta_1\beta + \beta_1^2 + \alpha_1^2 \\
&= \alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2 \qquad (\alpha^2 + \beta^2 = 1) \\
&= \|x\|^2.
\end{aligned}$$

Moreover, we have

$$xe = \alpha_1 e + \beta_1 i + (-\gamma_1\alpha + \delta_1\beta)j + (-\gamma_1\beta - \delta_1\alpha)k, \tag{1}$$

$$xi = -\beta_1 e + \alpha_1 i + (-\gamma_1\beta - \delta_1\alpha)j + (\gamma_1\alpha - \delta_1\beta)k, \tag{2}$$

$$xj = (\gamma_1\alpha - \delta_1\beta)e + (\gamma_1\beta + \delta_1\alpha)i + \alpha_1 j + \beta_1 k, \tag{3}$$

and

$$xk = (\gamma_1\beta + \delta_1\alpha)e + (-\gamma_1\alpha + \delta_1\beta)i - \beta_1 j + \alpha_1 k. \tag{4}$$

The equalities (3.1) and (3.2) imply

$$\begin{aligned}
(xe/xi) &= -\alpha_1\beta_1 + \beta_1\alpha_1 + (-\gamma_1\alpha + \delta_1\beta)(-\gamma_1\beta - \delta_1\alpha) \\
&+ (-\gamma_1\beta - \delta_1\alpha)(\gamma_1\alpha - \delta_1\beta) \\
&= 0.
\end{aligned}$$

The equalities (3.1) and (3.3) give us

$$\begin{aligned}
(xe/xj) &= \alpha_1(\gamma_1\alpha - \delta_1\beta) + \beta_1(\gamma_1\beta + \delta_1\alpha) + \alpha_1(-\gamma_1\alpha + \delta_1\beta) \\
&+ \beta_1(-\gamma_1\beta - \delta_1\alpha) \\
&= 0.
\end{aligned}$$

By the equalities (3.1) and (3.4), we get

$$\begin{aligned}
(xe/xk) &= \alpha_1(\gamma_1\beta + \delta_1\alpha) - \beta_1(-\gamma_1\alpha + \delta_1\beta) + \beta_1(-\gamma_1\alpha + \delta_1\beta) \\
&+ \alpha_1(-\gamma_1\beta - \delta_1\alpha) \\
&= 0.
\end{aligned}$$

Also the equalities (3.2) and (3.3) give us

$$\begin{aligned}
(xi/xj) &= -\beta_1(\gamma_1\alpha - \delta_1\beta) + \alpha_1(\gamma_1\beta + \delta_1\alpha) + \alpha_1(-\gamma_1\beta - \delta_1\alpha) \\
&+ \beta_1(\gamma_1\alpha - \delta_1\beta) \\
&= 0.
\end{aligned}$$

According to equalities (3.2) and (3.4) we have

$$(xi/xk) = -\beta_1(\gamma_1\beta + \delta_1\alpha) + \alpha_1(-\gamma_1\alpha + \delta_1\beta) - \beta_1(-\gamma_1\beta - \delta_1\alpha)$$

$$\begin{array}{rl} + & \alpha_1(\gamma_1\alpha - \delta_1\beta) \\ = & 0. \end{array}$$

Similarly, the equalities (3.3) and (3.4) entail that

$$\begin{array}{rcl} (xj/xk) & = & (\gamma_1\alpha - \delta_1\beta)(\gamma_1\beta + \delta_1\alpha) + (\gamma_1\beta + \delta_1\alpha)(-\gamma_1\alpha + \delta_1\beta) \\ & - & \beta_1\alpha_1 + \beta_1\alpha_1 \\ & = & 0. \end{array}$$

Finally, if $y = \alpha_2 e + \beta_2 i + \gamma_2 j + \delta_2 k \in \mathbb{M}_1(\alpha, \beta)$, we have

$$\begin{array}{rcl} \|xy\|^2 & = & (xy/xy) \\ & = & (\alpha_2 xe + \beta_2 xi + \gamma_2 xj + \delta_2 xk / \alpha_2 xe + \beta_2 xi + \gamma_2 xj + \delta_2 xk) \\ & = & \alpha_2^2\|xe\|^2 + \beta_2^2\|xi\|^2 + \gamma_2^2\|xj\|^2 + \delta_2^2\|xk\|^2 \\ & = & \alpha_2^2\|x\|^2 + \beta_2^2\|x\|^2 + \gamma_2^2\|x\|^2 + \delta_2^2\|x\|^2 \\ & = & \|x\|^2(\alpha_2^2 + \beta_2^2 + \gamma_2^2 + \delta_2^2) \\ & = & \|x\|^2\|y\|^2. \end{array}$$

On the other hand if B is a subalgebra of $\mathbb{M}_1(\alpha, \beta)$, then B contains a nonzero idempotent f (Theorem 3). We have $ef = f = f^2$, therefore, $e = f \in B$. This implies that e is a left omnipresent unit of $\mathbb{M}_1(\alpha, \beta)$.
By the same way, we may show that $\mathbb{M}_2(\alpha, \beta)$ is an absolute valued algebra with left omnipresent unit e. □

3.2 Construction of $^*\mathbb{M}_1(\alpha, \beta)$ and $^*\mathbb{M}_2(\alpha, \beta)$,

We set $\mathbb{M}(\alpha, \beta)$, with $(\alpha, \beta) \in \mathbb{R}^2$ such that $\alpha^2 + \beta^2 = 1$, one of principal AVAs $\mathbb{M}_1(\alpha, \beta)$ or $\mathbb{M}_2(\alpha, \beta)$ and $^*\mathbb{M}(\alpha, \beta)$ the standard isotope of $\mathbb{M}(\alpha, \beta)$, other than $\mathbb{M}(\alpha, \beta)$, that is the algebra having $\mathbb{M}(\alpha, \beta)$ as vectorial space and product given by $x * y = \bar{x}y$, where $x \to \bar{x}$ is the standard conjugation of $\mathbb{M}(\alpha, \beta)$. We denote these new algebras by $^*\mathbb{M}_1(\alpha, \beta)$ and $^*\mathbb{M}_2(\alpha, \beta)$.

Proposition 2. *The algebras $^*\mathbb{M}_1(\alpha, \beta)$ and $^*\mathbb{M}_2(\alpha, \beta)$ are some AVAs with left omnipresent unit e.*

Proof. Let $\mathbb{M}(\alpha, \beta)$ be one of the principal AVAs $\mathbb{M}_1(\alpha, \beta)$ or $\mathbb{M}_2(\alpha, \beta)$, and $x, y \in {}^*\mathbb{M}(\alpha, \beta)$. We have

$$\|x * y\| = \|\bar{x}y\| = \|\bar{x}\|\|y\| = \|x\|\|y\|.$$

Therefore, $^*\mathbb{M}_1(\alpha, \beta)$ and $^*\mathbb{M}_2(\alpha, \beta)$ are AVA. Moreover, if B is a two-dimensional subalgebra of $\mathbb{M}(\alpha, \beta)$, then $^*B$ is a two-dimensional subalgebra of $^*\mathbb{M}(\alpha, \beta)$. Which means that $\mathbb{M}(\alpha, \beta)$ and $^*\mathbb{M}(\alpha, \beta)$ have same left omnipresent unit e, hence The algebras $^*\mathbb{M}_1(\alpha, \beta)$ and $^*\mathbb{M}_2(\alpha, \beta)$ are AVAs with left omnipresent unit e. □

Remark 1. *Let $F = \{e, i, j, k\}$ be the orthonormal basis of the algebra $\mathbb{M}(\alpha, \beta)$, where e is a left omnipresent unit of $\mathbb{M}(\alpha, \beta)$. The multiplication tables relatively to the basis F of $^*\mathbb{M}_1(\alpha, \beta)$ and $^*\mathbb{M}_2(\alpha, \beta)$ are given by*

Table 4: $^*\mathbb{M}_1(\alpha,\beta)$

	e	i	j	k
e	e	i	j	k
i	-i	e	-k	j
j	$\alpha j+\beta k$	$\beta j-\alpha k$	$-\alpha e-\beta i$	$-\beta e+\alpha i$
k	$-\beta j+\alpha k$	$\alpha j+\beta k$	$\beta e-\alpha i$	$-\alpha e-\beta i$

Table 5: $^*\mathbb{M}_2(\alpha,\beta)$

	e	i	j	k
e	e	i	j	k
i	-i	e	-k	j
j	$\alpha j+\beta k$	$\beta j-\alpha k$	$-\alpha e-\beta i$	$-\beta e+\alpha i$
k	$\beta j-\alpha k$	$-\alpha j-\beta k$	$-\beta e+\alpha i$	$\alpha e+\beta i$

4 Main results

In this section we classify algebraically all 4-dimensional AVAs A, contains a left omnipresent unit e.
Let B denote a 2-dimensional sub-algebra of A, we know by Lemma 2 that B is isomorphic to $\mathbb{C}$, or $^*\mathbb{C}$. So we distinguish the two following case.

4.1 B isomorphic to $\mathbb{C}$

Theorem 4. *Let A be a four-dimensional absolute valued algebra with left omnipresent unit e. If B is isomorphic to $\mathbb{C}$, then A is isomorphic to $\mathbb{M}_1(\alpha,\beta)$, $\mathbb{M}_2(\alpha,\beta)$, $\mathbb{M}_1(0,1)$, $\mathbb{M}_1(0,-1)$, $\mathbb{M}_2(0,1)$ or $\mathbb{M}_2(0,-1)$.*

Proof. We pose $B=A(e,i)$, such that

$$i^2=-e \ \ and \ \ ie=ei=i.$$

According to Theorem 2, A is an inner product space, hence there exists an orthonormal subset $\{e,i\}$ which can be extended to an orthonormal basis $F=\{e,i,j,k\}$ of A. We have

$$(ij/e)=-(j/ie)=-(j/i)=0,$$

$$(ij/i)=-(j/i^2)=(j/e)=0,$$

and $(ij/j)=-(j/ij)$ from where $(ij/j)=0$, therefore $ij=\pm k$. We can assume that $ij=k$, this last implies $i(ij)=ik$ that is $ik=-\|i\|^2j=-j$. Always by Theorem 2 we have

$$(j^2/j)=-(j/j^2)=0.$$

Since

$$\|j^2+k\|=\|j^2+ij\|=\|(j+i)j\|=\|j+i\|\|j\|=\|j+i\|$$

we have

$$(j^2/k) = (j^2/ij) = (j/i) = 0.$$

So $j^2 = \alpha e + \beta i$, where $(\alpha, \beta) \in \mathbb{R}^2$ such that $\alpha^2 + \beta^2 = 1$. Also,

$$(je/e) = (je/e^2) = (j/e) = 0,$$

and

$$(je/i) = (je/ie) = (j/i) = 0.$$

So $je = \alpha_1 j + \beta_1 k$, where $(\alpha_1, \beta_1) \in \mathbb{R}^2$ such that $\alpha_1^2 + \beta_1^2 = 1$. Since

$$(ji/e) = -(ji/i^2) = -(j/i) = 0 \quad and \quad (ji/i) = (ji/ei) = (j/e) = 0$$

we have $ji = \alpha_2 j + \beta_2 k$, where $(\alpha_2, \beta_2) \in \mathbb{R}^2$ such that $\alpha_2^2 + \beta_2^2 = 1$. Similarly,

$$(jk/j) = -(jk/ik) = -(j/i) = 0 \quad and \quad (jk/k) = -(k/jk) = 0.$$

Thus, $jk = \alpha_3 e + \beta_3 i$, where $(\alpha_3, \beta_3) \in \mathbb{R}^2$ such that $\alpha_3^2 + \beta_3^2 = 1$. By the same way, we have

$$(k^2/j) = -(k^2/ik) = -(k/i) = 0,$$

and

$$(k^2/k) = -(k/k^2) = 0.$$

Then $k^2 = \alpha_4 e + \beta_4 i$, where $(\alpha_4, \beta_4) \in \mathbb{R}^2$ such that $\alpha_4^2 + \beta_4^2 = 1$. Also,

$$(ke/e) = (ke/e^2) = (k/e) = 0,$$

and

$$(ke/i) = (ke/ie) = (k/i) = 0.$$

So, $ke = \alpha_5 j + \beta_5 k$, where $(\alpha_5, \beta_5) \in \mathbb{R}^2$ such that $\alpha_5^2 + \beta_5^2 = 1$. Further,

$$(ki/e) = -(ki/i^2) = -(k/i) = 0,$$

and

$$(ki/i) = (ki/ei) = (k/e) = 0.$$

Hence, $ki = \alpha_6 j + \beta_6 k$, where $(\alpha_6, \beta_6) \in \mathbb{R}^2$ such that $\alpha_6^2 + \beta_6^2 = 1$. Since

$$(kj/j) = -(j/kj) = 0,$$

$$(kj/k) = (kj/ij) = (k/i) = 0.$$

We obtain $kj = \alpha_7 e + \beta_7 i$, where $(\alpha_7, \beta_7) \in \mathbb{R}^2$ such that $\alpha_7^2 + \beta_7^2 = 1$.
According to the previous equalities and Theorem 2, we have the following relations

$$\alpha = (j^2/e) = -(j/je) = -\alpha_1$$

$$\beta = (j^2/i) = -(j/ji) = -\alpha_2,$$

$$\beta_1 = (je/k) = -(e/jk) = -\alpha_3,$$

and

$$\beta_2 = (ji/k) = -(i/jk) = -\beta_3.$$

Using Lemma 1, we have

$$\beta_2 = (ji/k) = (ji/ij) = -(j^2/i^2) = (j^2/e) = \alpha.$$

We conclude that

$$\begin{aligned} j^2 &= \alpha e + \beta i. \\ je &= -\alpha j + \beta_1 k. \\ ji &= -\beta j + \alpha k. \\ jk &= -\beta_1 e - \alpha i. \end{aligned}$$

Similarly,

$$\alpha_4 = (k^2/e) = -(k/ke) = -\beta_5.$$

$$\beta_4 = (k^2/i) = -(k/ki) = -\beta_6.$$

$$\alpha_5 = (ke/j) = -(e/kj) = -\alpha_7.$$

$$\alpha_6 = (ki/j) = -(i/kj) = -\beta_7.$$

By Lemma 1, we have

$$\alpha_6 = (ki/j) = -(ki/ik) = (k^2/i^2) = -(k^2/e) = -\alpha_4.$$

We obtain

$$\begin{aligned} k^2 &= \alpha_4 e + \beta_4 i. \\ ke &= \alpha_5 j - \alpha_4 k. \\ ki &= -\alpha_4 j - \beta_4 k. \\ kj &= -\alpha_5 e + \alpha_4 i. \end{aligned}$$

From the equality $k^2 = \alpha_4 e + \beta_4 i$, we get

$$\begin{aligned} jk^2 &= \alpha_4 je + \beta_4 ji \\ &= \alpha_4(-\alpha j + \beta_1 k) + \beta_4(-\beta j + \alpha k) \\ &= (-\alpha_4\alpha - \beta_4\beta)j + (\alpha_4\beta_1 + \beta_4\alpha)k. \end{aligned}$$

But since

$$0 = (k/j) = (k^2/jk) = -(jk^2/k) = -(\alpha_4\beta_1 + \beta_4\alpha) = 0.$$

That is

$$\begin{aligned} j(jk^2) &= (-\alpha_4\alpha - \beta_4\beta)j^2 \\ -k^2 &= (-\alpha_4\alpha - \beta_4\beta)j^2 \end{aligned}$$

Hence $k^2 = \pm j^2$ ($\|k\| = \|j\| = 1$), we have the following two cases

1. If $k^2 = j^2$, then $\alpha_4 = \alpha$ and $\beta_4 = \beta$. By Lemma 1, we have

$$(jk/kj) = -(j^2/k^2) = -1,$$

then $jk = -kj$. Indeed

$$\|jk + ki\|^2 = 2 + 2(jk/kj) = 0.$$

Hence $jk = -kj$ that is $-\beta_1 = \alpha_5$, this last imply that

$$\begin{array}{rcl} j^2 &=& \alpha e + \beta i, \\ je &=& -\alpha j + \beta_1 k, \\ ji &=& -\beta j + \alpha k, \\ jk &=& -\beta_1 e - \alpha i, \end{array}$$

and

$$\begin{array}{rcl} k^2 &=& \alpha e + \beta i, \\ ke &=& -\beta_1 j - \alpha k, \\ ki &=& -\alpha j - \beta k, \\ kj &=& \beta_1 e + \alpha i. \end{array}$$

Since

$$0 = (j/k) = (j^2/jk) = (\alpha e + \beta i / - \beta_1 e - \alpha i) = -\alpha\beta_1 - \beta\alpha,$$

we get $\alpha(\beta + \beta_1) = 0$ and we have

i) If $\alpha \neq 0$, then $\beta_1 = -\beta$. Hence A is isomorphic to $\mathbb{M}_1(\alpha, \beta)$.

ii) If $\alpha = 0$ then $\beta^2 = \beta_1^2 = 1$, we have the following cases

a) If $\beta = \beta_1$, then

$$\begin{array}{rcl} j^2 &=& \beta i, \\ je &=& \beta k, \\ ji &=& -\beta j, \\ jk &=& -\beta e. \end{array}$$

And

$$\begin{array}{rcl} k^2 &=& \beta i, \\ ke &=& -\beta j, \\ ki &=& -\beta k, \\ kj &=& \beta e. \end{array}$$

Since

$$(i + j)e = i + \beta k \;\; and \;\; (i + j)j = k + \beta i = \beta(\beta k + i).$$

So $(i + j)j = \beta(i + j)e$, that is $j = \beta e$ which is absurd (A without divisors of zero). Hence $\beta \neq \beta_1$

b) If $\beta = 1$ and $\beta_1 = -1$, then

$$\begin{array}{rcl} j^2 & = & i, \\ je & = & -k, \\ ji & = & -j, \\ jk & = & e. \end{array}$$

And

$$\begin{array}{rcl} k^2 & = & i, \\ ke & = & j, \\ ki & = & -k, \\ kj & = & -e. \end{array}$$

Hence A is isomorphic to $\mathbb{M}_1(0,1)$.

c) If $\beta = -1$ and $\beta_1 = 1$, then

$$\begin{array}{rcl} j^2 & = & -i, \\ je & = & k, \\ ji & = & j, \\ jk & = & -e. \end{array}$$

And

$$\begin{array}{rcl} k^2 & = & -i, \\ ke & = & -j, \\ ki & = & k, \\ kj & = & e. \end{array}$$

Hence A is isomorphic to $\mathbb{M}_1(0,-1)$.

2. If $k^2 = -j^2$, then $\alpha_4 = -\alpha$ and $\beta_4 = -\beta$. By lemma 1, we have

$$(jk/kj) = -(j^2/k^2) = 1.$$

then $jk = kj$, that is $\beta_1 = \alpha_5$, this last imply that

$$\begin{array}{rcl} j^2 & = & \alpha e + \beta i, \\ je & = & -\alpha j + \beta_1 k, \\ ji & = & -\beta j + \alpha k, \\ jk & = & -\beta_1 e - \alpha i, \end{array}$$

$$\begin{aligned} k^2 &= -\alpha e - \beta i, \\ ke &= \beta_1 j + \alpha k, \\ ki &= \alpha j + \beta k, \\ kj &= -\beta_1 e - \alpha i. \end{aligned}$$

Since

$$0 = (j/k) = (j^2/jk) = (\alpha e + \beta i / - \beta_1 e - \alpha i) = -\alpha\beta_1 - \beta\alpha,$$

we get $\alpha(\beta + \beta_1) = 0$. In a similar manner, we have

iii) If $\alpha \neq 0$, then $\beta_1 = -\beta$. Hence A is isomorphic to $\mathbb{M}_2(\alpha, \beta)$.

iv) If $\alpha = 0$ then $\beta^2 = \beta_1^2 = 1$, according to ii)*) $\beta \neq \beta_1$. We have the following cases

a) If $\beta = 1$ and $\beta_1 = -1$, then

$$\begin{aligned} j^2 &= i, \\ je &= -k, \\ ji &= -j, \\ jk &= e. \end{aligned}$$

And

$$\begin{aligned} k^2 &= -i, \\ ke &= -j, \\ ki &= k, \\ kj &= e. \end{aligned}$$

Therefore, A is isomorphic to $\mathbb{M}_2(0, 1)$.

b) If $\beta = -1$ and $\beta_1 = 1$, then

$$\begin{aligned} j^2 &= -i, \\ je &= k, \\ ji &= j, \\ jk &= -e. \end{aligned}$$

And

$$\begin{aligned} k^2 &= i, \\ ke &= j, \\ ki &= -k, \\ kj &= -e. \end{aligned}$$

So A is isomorphic to $\mathbb{M}_2(0, -1)$.

□

4.2 B isomorphic to $^*\mathbb{C}$

Theorem 5. *Let A be a four-dimensional absolute valued algebra with left omnipresent unit e. If B is isomorphic to $^*\mathbb{C}$, then A is isomorphic to $^*\mathbb{M}_1(\alpha,\beta)$, $^*\mathbb{M}_2(\alpha,\beta)$, $^*\mathbb{M}_1(0,1)$, $^*\mathbb{M}_1(0,-1)$, $^*\mathbb{M}_2(0,1)$ or $^*\mathbb{M}_2(0,-1)$.*

Proof. If we define a new multiplication on A by $x * y = \bar{x}y$, then we obtain an algebra $^*A$ which contains a subalgebra isomorphic to $\mathbb{C}$. Therefore, applying Theorem 4, $^*A$ is isomorphic to $\mathbb{M}_1(\alpha,\beta)$, $\mathbb{M}_2(\alpha,\beta)$, $\mathbb{M}_1(0,1)$, $\mathbb{M}_1(0,-1)$, $\mathbb{M}_2(0,1)$ or $\mathbb{M}_2(0,-1)$. Consequently, A is isomorphic to $^*\mathbb{M}_1(\alpha,\beta)$, $^*\mathbb{M}_2(\alpha,\beta)$, $^*\mathbb{M}_1(0,1)$, $^*\mathbb{M}_1(0,-1)$, $^*\mathbb{M}_2(0,1)$ or $^*\mathbb{M}_2(0,-1)$. □

Remark 2. *Assume that $ij = -k$, if we substitute $-k = t$ we get $ij = t$, that is we again get the same multiplication tables previously.*

The following theorem summarizes our study.

Theorem 6. *Let's A be a 4-dimensional AVA with left omnipresent unit e and B a 2-dimensional sub-algebra of A. The following table specifies the isomorphisms classes.*

Table 6

B isomorphic to	A isomorphic to
$\mathbb{C}$	$\mathbb{M}_1(\alpha,\beta)$, $\mathbb{M}_2(\alpha,\beta)$, $\mathbb{M}_1(0,\pm1)$, $\mathbb{M}_2(0,\pm1)$
$^*\mathbb{C}$	$^*\mathbb{M}_1(\alpha,\beta)$, $^*\mathbb{M}_2(\alpha,\beta)$, $^*\mathbb{M}_1(0,\pm1)$, $^*\mathbb{M}_2(0,\pm1)$

With $(\alpha,\beta) \in \mathbb{R}^2$ such that $\alpha^2 + \beta^2 = 1$.

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References

[1] A. A. Albert, *Absolute valued real algebras* , Ann. Math., **48** (1947), 495-501.

[2] M. Benslimane, A. Moutassim and A. Rochdi, *Sur les algébres absolument valuées contenant un elément central non nul*, Advances in Applied Cliford Algebras, **20** (2010), 13-21.

[3] M. Benslimane and A. Moutassim, *Some new class of absolute valued algebras with left unit*, Advances in Applied Cliford Algebras, **21** (2011), 31-40.

[4] A. Calderón, A. Kaidi, C. Martín, A. Morales, M. I. Ramírez and A. Rochdi, *Finite-dimensional absolute valued algebras*, Isr. J. Math., **184** (2011), 193-220.

[5] M. L. El-Mallah, *On finite dimensional absolute valued algebras satisfying* $(x,x,x)=0$, Arch Math., **49** (1987), 16-22.

[6] M. L. El-Mallah, *Absolute valued algebras satisfying* $(x^2,x,x)=0$, Arch. Math., **77** (2001), 378-382.

[7] S. S. Gashiti, *Some results on nilpotent lie algebras*, Journal of Algebra and Related Topics, (2) **11** (2023), 99-103.

[8] F. Hirzebruch, M. Koecher and R. Remmert, *Numbers*, Springer Verlag, 1991.

[9] A. Kaidi, M. I. Ramírez and A. Rodrǵuez, *Absolute valued algebraic algebras are finite dimensional*, J. Algebra, **195** (1997), 295-307.

[10] N. Motya, H. Mouanis, and A. Moutassim, *On pre-Hilbert algebras containing a nonzero central idempotent f such that* $\|x^2\| \leq \|x\|^2$*, and* $\|fx\| = \|x\|$, Journal of Southwest Jiaotong University, (6) **57** (2022), 289-295.

[11] N. Motya, and A. Moutassim,*On four dimensional absolute valued algebras with non zero omnipresent idempotent*, Wseas Transactions on Mathematics, **22** (2023), 562-569.

[12] A. Moutassim, *Quelques résultats sur les algèbres flexibles préhilbertiennes sans diviseurs de zéro vérifiant* $\|x^2\| = \|x\|^2$, Advances in Applied Clifford Algebras, **18** (2008), 255-267.

[13] A. Moutassim and A. Rochdi, *Sur les algèbres préhilbertiennes vérifiant* $\|a\|^2 \leq \|a\|^2$, Advances in Applied Clifford Algebras, **18** (2008), 269-278.

[14] M. I. Ramŕez, *On four-dimensional absolute valued algebras*, Proceedings of the International Conference on Jordan Structures (Málaga, 1997), univ. Málaga, (1999), 169-173.

[15] A. Rodríguez, *One-sided division absolute-valued algebras*, Publ. Math, **36** (1992), 925-954.

[16] A. Rodríguez, *Absolute valued algebras of degree two*, In Nonassociative Algebra and its applications, **303** (1994), 350-356.

[17] A. Rodríguez, *Absolute-valued algebras and absolute-valuable Banach spaces*, Advanced courses of mathematical analysis I, World Sci. Publ., Hackensack, NJ, (2004), 99-155.

[18] B. Segre, *La teoria delle algebre ed alcune questione di realta*, Univ. Roma, Ist. Naz. Alta. Mat., Rend. Mat. E Appl., (5) **13** (1954), 157-188.

[19] M. F. Smiley, *Real Hilbert algebras with identity*, Proc. Amer. Math. Soc., **16** (1965), 440-441.

[20] K. Urbanik and F. B. Wright, *Absolute valued algebras*, Proc. Amer. Math. Soc., **11** (1960), 861-866.

[21] B. Zalar, *On Hilbert spaces with unital multiplication*, Proc. Amer. Math. Soc., **123** (1995), 1497-1501.