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Hesitant hybrid interior ideals in semigroups

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Abstract. Hesitant fuzzy sets are very useful for dealing with group decision-making problems when experts have hesitation among several possible memberships for an element in a set. Recently, many researchers have used these concepts in clustering analysis and decision -making. In this article, the notions of hesitant hybrid subsemigroups and hesitant hybrid left (resp., right) ideals in a semigroup are introduced, and several properties are investigated. The concept of hesitant hybrid product is also introduced, and properties of hesitant hybrid subsemigroups and hybrid left (resp., right) ideals are considered using the notion of hesitant hybrid product. Relations between hesitant hybrid intersection and hesitant hybrid product are exhibited. Additionally, we establish hesitant hybrid interior ideals in semigroups and study their properties. Furthermore, we prove that in regular and intra-regular semigroups, the hesitant hybrid ideal and the hesitant hybrid interior ideals are coincide.

Keywords: Semigroup, Ideals, Hesitant hybrid structure, Hesitant hybrid ideal, Hesitant hybrid interior ideal.
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1 Introduction

L. A. Zadeh [20] proposed the idea of fuzzy subsets in 1965 and investigated their properties. This idea was applied in group theory and semigroup theory by Rosenfeld [15]. X. Yang et al. [19] interpreted the level set, support, kernel for bipolar complex fuzzy (BCF) set, bipolar complex characteristic function, and BCF point. Also interpreted the BCF subgroup, BCF normal subgroup, BCF conjugate, normalizer for BCF subgroup, cosets, BCF abelian subgroup, and BCF factor group. U. Ur . Rehman et al. [14] expanded the notion of BCFS by giving a general algebraic structure for tackling bipolar complex fuzzy (BCF) data by fusing the conception of

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BCFS and semigroup. T. Mahmood et al. [11] developed the concept of bipolar complex fuzzy $\Gamma-$ subsemigroup, bipolar complex fuzzy $\Gamma-$ ideal, bipolar complex fuzzy $\Gamma-$ bi-ideal, bipolar complex fuzzy $\Gamma-(1,2)$ ideal, bipolar complex fuzzy left $\Gamma-$duo, bipolar complex fuzzy right $\Gamma-$ duo and their related outcomes in the environment of $\Gamma-$ semigroup. Numerous researchers have developed fuzzy set theory in a variety of directions, and it has stimulated the interest of mathematicians working in a variety of fields. Fuzzy set theory has a broad range of applications, from robot design, water resource planning, and computer simulation to engineering.

As a fresh method, D. Molodtsov [12]introduced the soft set theory. The connections between fuzzy sets and soft sets have been examined by Y. B. Jun et al. [8]. The ideas of hybrid subsemigroups, hybrid ideals in semigroup were presented by S. Anis et al. in [2], and a number of features were examined. B. Elavarasan et al. [5] were the first to discuss hybrid generalised bi-ideal in semigroup. K. Porselvi et al. [13] introduced the notion of hybrid interior ideals and hybrid characteristic interior ideals of a semigroup. J. Catherine Grace John et al. [3] explored the concept of hybrid interior ideals, hybrid bi-ideals and the hybrid characteristic interior ideals of a ternary semigroup. M. Deepika et al. [4] examined ternary semigroups in the context of hybrid quasi-ideals and hybrid A-ideals.

The introduction of hesitant fuzzy sets was motivated by the observation that it can be challenging to determine the membership of an element belongs to a set in some circumstances, and that in some cases, this difficulty is caused by uncertainty surrounding a few possible values, e.g, two decision makers debate the membership degree of x into A, and one wishing to allocate 0.5 but the other wants to assign 0.8. The range of the possible values uncertainty consequently becomes somewhat restricted. In such situations, Torra [17] and Torra and Narukawa [18], developed the hesitant fuzzy set, a novel generalisation of the fuzzy set. The hesitant fuzzy set is used to manage reluctant problems that were challenging for prior enlargements to handle. The hesitant fuzzy set enables a collection of potential values ranging from zero to one to be used to express the degree of membership of an element to a set. It is not necessary for potential memberships to just have crisp values in the range [0,1] during the review process. The values might include intervals. A. Ali et al. [1] investigated AG-groupoids hesitant fuzzy sets. Y. B. Jun et al. [9] utilised semigroups to apply the idea of hesitant fuzzy sets. In addition, they proposed the concepts of hesitant fuzzy bi-ideals and hesitant fuzzy generalised bi-ideal in semigroup, as well as hesitant fuzzy left ideal and hesitant fuzzy right ideal. In semigroup, A.F. Talee et al. [16] studied some properties of hesitant fuzzy ideals and hesitant fuzzy bi-ideals and discussed their properties. Also proposed the idea of hesitant fuzzy interior ideals. T. Mahmood et al. [10] introduced cubic hesitant fuzzy set and defined internal (external) cubic hesitant fuzzy set, P(R)-union and P(R)-intersection of cubic hesitant fuzzy sets. Furthermore they defined P(R)-addition and P(R)-multiplication of cubic hesitant fuzzy sets.

Following [16], we are going to study the hesitant hybrid ideals in semigroups. The short review of the composition of the article is as: In section 2, the basic notion of semigroup, its related theory are reviewed. In section 3, we establish hesitant hybrid structure in semigroup and discuss their related properties. We initiate hybrid interior ideals in semigroup and look at their features. A hesitant hybrid ideal is a hesitant hybrid interior ideal in a semigroup, but the converse is not true, as we showed. Furthermore, we illustrate that the hesitant hybrid ideals and the hesitant hybrid interior ideal coincide in regular and intra-regular semigroups.

2 Preliminaries

A semigroup is an algebraic structure consisting of a non-empty set $\mathbb{F}$ together with an associative binary operation [7]. Semigroups are important in many areas of mathematics and the concepts of semigroups have been studied by several researchers [6]. We collect a few fundamental definitions in this section that are related to semigroup, which we will use in this paper.

Definition 1. *[6] Let $\mathbb{F}$ be a semigroup and $\mathcal{R}$, $\mathcal{B} \subseteq \mathbb{F}$. Then, multiplication of $\mathcal{R}$ and $\mathcal{B}$ can be defined as $\mathcal{R}\mathcal{B} = \{ab : a \in \mathcal{R}$ and $b \in \mathcal{B}\}$.*

Definition 2. *[6] A semigroup $\mathbb{F}$ is described as*

(i) a regular if each element is regular, meaning that for each element $c \in \mathbb{F}$, there exists an element $g \in \mathbb{F}$ such that $g = gcg$.

(ii) an intra-regular if each element is an intra-regular, meaning that for every element $c \in \mathbb{F}$, there exists an element $g, m \in \mathbb{F}$ such that $c = gm^2c$.

(iii) a completely regular if each element is competely regular, meaning that for every element $c \in \mathbb{F}$, there exists an element $g \in \mathbb{F}$ such that $c = cgc$ and $cg = gc$.

(iv) a right (resp., left) zero if $gm = m(resp., gm = g)$ for all $g, m \in \mathbb{F}$.

Definition 3. *[6] If an element c of a semigroup $\mathbb{F}$ satisfies the form $g = cg^2(resp.,\ g = g^2c)$, for some $g \in \mathbb{F}$, then the element is termed to be left (resp.,right)regular.*

Definition 4. *[6] Let $\mathcal{R}(\neq \phi) \subseteq \mathbb{F}$. Then*

(i) $\mathcal{R}$ is called a subsemigroup of $\mathbb{F}$ if $\mathcal{R}^2 \subseteq \mathcal{R}$.

(ii) $\mathcal{R}$ is known as a left (resp., right) ideal of $\mathbb{F}$ if $\mathbb{F}\mathcal{R} \subseteq \mathcal{R}$ (resp.,$\mathcal{R}\mathbb{F} \subseteq \mathcal{R}$).

(iii) $\mathcal{R}$ is referred to as a two-sided ideal or ideal of $\mathbb{F}$ if it is both a left and a right ideal of $\mathbb{F}$.

(iv) $\mathcal{R}$ is referred to as a interior ideal of $\mathbb{F}$ if $\mathbb{F}\mathcal{R}\mathbb{F} \subseteq \mathcal{R}$.

Definition 5. *[9] If every left (resp., right) ideal of $\mathbb{F}$ is a two-sided ideal of $\mathbb{F}$, then $\mathbb{F}$ is known as a left (resp., right) duo. If $\mathbb{F}$ is both a right and a left duo, then it is known as a duo.*

Note: For $g \in \mathbb{F}$, the principal

(i) left ideal generated by g is provided by $\langle g\rangle_L = \{g\} \cup \mathbb{F}g$.

(ii) right ideal generated by g is provided by $\langle g\rangle_R = \{g\} \cup g\mathbb{F}$.

(iii) two -sided ideal generated by g is provided by $\langle g\rangle = \{g\} \cup g\mathbb{F} \cup \mathbb{F}g$.

3 Hesitant Hybrid subsemigroups and ideals

Traditional fuzzy sets do not account for this complexity, leading to inaccurate representations of the decision-making process. To address this issue, hesitant fuzzy sets was introduced. The concept of hesitant fuzzy ideals was first introduced to semigroup by Y. B. Jun et al. [9]. Here we describe a brief definition of the hesitant hybrid ideals in semigroups.

Definition 6. *Let $[0, 1]$ be a unit interval, $\mathscr{P}(\mathbb{W})$ denotes the power set of a universal set $\mathbb{W}$ and $\mathscr{P}([0, 1])$ denotes the power set of $[0, 1]$. A hesitant hybrid structure in $\mathbb{F}$ over $\mathbb{W}$ is defined to be a mapping*

$$\tilde{\mathfrak{h}}_\alpha := (\tilde{\mathfrak{h}}, \alpha) : \mathbb{F} \to \mathscr{P}(\mathbb{W}) \times \mathscr{P}([0,1]),\ g \mapsto (\tilde{\mathfrak{h}}_g, \alpha_g)$$

where $\tilde{\mathfrak{h}} : \mathbb{F} \to \mathscr{P}(\mathbb{W})$ *and* $\alpha : \mathbb{F} \to \mathscr{P}([0,1])$ *are mappings.*

Let the collection of all hesitant hybrid structures in $\mathbb{F}$ over $\mathbb{W}$ is denoted by $\mathbb{HH}(\mathbb{F})$. For $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$ and $a, b, c \in \mathbb{F}$, we use the notations $(\tilde{\mathfrak{h}}_\alpha)_a = \tilde{\mathfrak{h}}_\alpha(a)$, $(\tilde{\mathfrak{h}}_\alpha)_a^b = (\tilde{\mathfrak{h}}_\alpha)_a \Cap (\tilde{\mathfrak{h}}_\alpha)_b$, $(\tilde{\mathfrak{h}}_\alpha)_a^b(c) = (\tilde{\mathfrak{h}}_\alpha)_a \Cap (\tilde{\mathfrak{h}}_\alpha)_b \Cap (\tilde{\mathfrak{h}}_\alpha)_c$. Obviously, $(\tilde{\mathfrak{h}_\alpha})_a^b = (\tilde{\mathfrak{h}_\alpha})_b^a$. Construct an order relation $\ll$ in $\mathbb{HH}(\mathbb{F})$ as:

$$\left(\forall\ \tilde{\mathfrak{h}}_\alpha, \tilde{y}_\varrho \in \mathbb{HH}(\mathbb{F})\right)\left(\tilde{\mathfrak{h}}_\alpha \ll \tilde{y}_\varrho \Leftrightarrow \tilde{\mathfrak{h}} \tilde{\subseteq} \tilde{y}, \alpha \tilde{\supseteq} \varrho\right)$$

where $\tilde{\mathfrak{h}} \tilde{\subseteq} \tilde{y}$ means $\tilde{\mathfrak{h}}_g \subseteq \tilde{y}_g$ and $\alpha \tilde{\supseteq} \varrho$ means that $\alpha_g \supseteq \varrho_g\ \forall g \in \mathbb{F}$. Then, the set $(\mathbb{HH}(\mathbb{F}), \ll)$ is partially ordered.

Definition 7. *For* $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$, $\Gamma \in \mathscr{P}(\mathbb{W})$ *and* $\xi \in \mathscr{P}([0,1])$, *the set*

$$\tilde{\mathfrak{h}}_\alpha[\Gamma, \xi] := \{g \in \mathbb{F} : \tilde{\mathfrak{h}}(g) \supseteq \Gamma, \alpha(g) \subseteq \xi\}$$

is described as $[\Gamma, \xi]-$ *hesitant hybrid cut of* $\tilde{\mathfrak{h}}_\alpha$.

Definition 8. *Let* $\tilde{y}_\varrho, \tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$. *Then*

(i) *the hesitant hybrid intersection of* $\tilde{y}_\varrho$ *and* $\tilde{\mathfrak{h}}_\alpha$ *in* $\mathbb{F}$ *is defined as* $\tilde{y}_\varrho \Cap \tilde{\mathfrak{h}}_\alpha = (\tilde{y} \cap \tilde{h},\ \varrho \cup \alpha)$ *in* $\mathbb{F}$ *over* $\mathbb{W}$, *where*

$$\tilde{y}_\varrho \Cap \tilde{\mathfrak{h}}_\alpha : \mathbb{F} \to \mathscr{P}(\mathbb{W}) \times \mathscr{P}([0,1]), g \mapsto ((\tilde{y} \tilde{\cap} \tilde{\mathfrak{h}})_g, (\varrho \cup \alpha)_g)$$

$$\begin{aligned} \textit{where}\ \ \tilde{y} \tilde{\cap} \tilde{\mathfrak{h}} &: \mathbb{F} \to \mathscr{P}(\mathbb{W}), g \mapsto \tilde{y}_g \cap \tilde{\mathfrak{h}}_g, \\ \varrho \cup \alpha &: \mathbb{F} \to \mathscr{P}([0,1]), g \mapsto \varrho_g \cup \alpha_g \end{aligned}$$

for all $g \in \mathbb{F}$.

(ii) *the hesitant hybrid union of* $\tilde{y}_\varrho$ *and* $\tilde{\mathfrak{h}}_\alpha$ *in* $\mathbb{F}$ *is defined as* $\tilde{y}_\varrho \Cup \tilde{\mathfrak{h}}_\alpha = (\tilde{y} \cup \tilde{h},\ \varrho \cap \alpha)$ *in* $\mathbb{F}$ *over* $\mathbb{W}$, *where*

$$\tilde{y}_\varrho \Cup \tilde{\mathfrak{h}}_\alpha : \mathbb{F} \to \mathscr{P}(\mathbb{W}) \times \mathscr{P}([0,1]), g \mapsto ((\tilde{y} \tilde{\cup} \tilde{\mathfrak{h}})_g, (\varrho \cap \alpha)_g)$$

$$\begin{aligned} \textit{where}\ \ \tilde{y} \tilde{\cup} \tilde{h} &: \mathbb{F} \to \mathscr{P}(\mathbb{W}), g \mapsto \tilde{y}_g \cup \tilde{h}_g, \\ \varrho \cap \alpha &: \mathbb{F} \to \mathscr{P}([0,1]), g \mapsto \varrho_g \cap \alpha_g \end{aligned}$$

for all $g \in \mathbb{F}$.

(iii) *the hesitant hybrid product of* $\tilde{y}_\varrho$ *and* $\tilde{\mathfrak{h}}_\alpha$ *in* $\mathbb{F}$ *is defined as* $\tilde{y}_\varrho \odot \tilde{\mathfrak{h}}_\alpha = (\tilde{y} \tilde{\circ} \tilde{h},\ \varrho \tilde{\circ} \alpha)$ *in* $\mathbb{F}$ *over* $\mathbb{W}$, *where*

$$(\tilde{y} \tilde{\circ} \tilde{\mathfrak{h}})_g = \begin{cases} \bigcup\limits_{g=mn} \{\tilde{y_m} \cap \tilde{\mathfrak{h}_n}\} & \textit{if } g = mn \\ \phi & \textit{otherwise,} \end{cases}$$

$$(\varrho \tilde{\circ} \alpha)_g = \begin{cases} \bigcap\limits_{g=mn} \{\varrho_m \cup \alpha_n\} & \textit{if } g = mn \\ [0,1] & \textit{otherwise.} \end{cases}$$

Definition 9. *Let $\mathcal{R}(\neq \phi) \subseteq \mathbb{F}$ and $\epsilon, \delta \in \mathscr{P}(\mathbb{W})$ with $\epsilon \supsetneq \delta$, and $v, t \in \mathscr{P}([0,1])$ with $v \subsetneq t$. Consider the hesitant hybrid structure $\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(v,t)}(\tilde{\mathfrak{h}}_\alpha)\right) = \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})}, \chi^{(v,t)}_{\mathcal{R}(\alpha)}\right)$ where*

$$\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})} : \mathbb{F} \to \mathscr{P}(\mathbb{W}), g \mapsto \begin{cases} \epsilon & \textit{if } g \in \mathcal{R} \\ \delta & \textit{otherwise,} \end{cases} \quad \chi^{(v,t)}_{\mathcal{R}(\alpha)} : \mathbb{F} \to P([0,1]), g \mapsto \begin{cases} v & \textit{if } g \in \mathcal{R} \\ t & \textit{otherwise,} \end{cases}$$

which is known as the $\frac{(\epsilon,\delta)}{(v,t)}$ characteristic hesitant hybrid structure in $\mathbb{F}$ over $\mathbb{W}$. The hesitant hybrid structure $\left(\chi^{(\epsilon,\delta)}_{\mathbb{F}(v,t)}(\tilde{\mathfrak{h}}_\alpha)\right)$ is called the $\frac{(\epsilon,\delta)}{(v,t)}$ identity hesitant hybrid structure in $\mathbb{F}$ over $\mathbb{W}$.

Note: $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1]))}(\tilde{\mathfrak{h}}_\alpha)\right) := \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}, \chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)$ is called the hesitant hybrid characteristic structure in $\mathbb{F}$ over $\mathbb{W}$. If $\mathbb{F} = \mathbb{W}$, then $\left(\chi^{(\mathbb{W})}_{F\{\phi\}}(\tilde{\mathfrak{h}}_\alpha)\right) = \left(\chi^{(\mathbb{W})}_{F(\tilde{\mathfrak{h}})}, \chi^{\{\phi\}}_{F(\alpha)}\right)$.

Definition 10. *A hesitant hybrid structure $\tilde{\mathfrak{h}}_\alpha$ in $\mathbb{F}$ over $\mathbb{W}$ is referred to as a hesitant hybrid subsemigroup of $\mathbb{F}$ over $\mathbb{W}$ if it meets the following criteria:*

$$(\forall\ a, b \in \mathbb{F}) \quad (\tilde{\mathfrak{h}}_{ab} \supseteq \tilde{\mathfrak{h}}^b_a \ \textit{ and } \ \alpha_{ab} \subseteq \alpha^b_a).$$

The below is the example of hesitant hybrid subsemigroup.

Example 1. Let $\mathbb{F} = \{0, 1, 2, 3, 4, 5\}$ be a semigroup with the following Cayley table (Table 1):

Table 1: Cayley table

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Define a hesitant hybrid structure $\tilde{\mathfrak{h}}_\alpha$ in $\mathbb{F}$ over $\mathbb{W} = Z$ which is given by :

$$\tilde{\mathfrak{h}}_b = \begin{cases} Z & if \ \ b \ = \ 0 \\ Z & if \ \ b \ = \ 1 \\ 4Z & if \ \ b \ = \ 2 \\ 2Z & if \ \ b \ = \ 3 \\ 4N & if \ \ b \ = \ 4 \\ 4N & if \ \ b \ = \ 5, \end{cases} \quad \text{and } \alpha_b = \begin{cases} [0, 0.3] & if \ \ b \ = \ 0 \\ [0, 0.3] & if \ \ b \ = \ 1 \\ [0, 1] & if \ \ b \ = \ 2 \\ [0, 0.3) \cup [0.3, 0.8] & if \ \ b \ = \ 3 \\ [0, 0.9] & if \ \ b \ = \ 4 \\ [0, 1] & if \ \ b \ = \ 5. \end{cases}$$

Then $\tilde{\mathfrak{h}}_\alpha$ is a hesitant hybrid subsemigroup in $\mathbb{F}$ over $\mathbb{W} = Z$.

Definition 11. *A hesitant hybrid structure $\tilde{\mathfrak{h}}_\alpha$ in $\mathbb{F}$ over $\mathbb{W}$ is referred to as a hesitant hybrid left (resp., right) ideal of $\mathbb{W}$ if it meets the following criteria:*

$$(\forall\ a, b \in \mathbb{F}) \quad (\tilde{\mathfrak{h}}_{ab} \supseteq \tilde{\mathfrak{h}}_b \ \textit{ and } \ \alpha_{ab} \subseteq \alpha_b \ (resp., \tilde{\mathfrak{h}}_{ab} \supseteq \tilde{\mathfrak{h}}_a \ \textit{ and } \ \alpha_{ab} \subseteq \alpha_a)).$$

A hesitant hybrid structure $\tilde{\mathfrak{h}}_\alpha$ in $\mathbb{F}$ over $\mathbb{W}$ is classified to be a hesitant hybrid two-sided ideal in $\mathbb{F}$ over $\mathbb{W}$ if it has a hesitant hybrid left ideal and a hesitant hybrid right ideal in $\mathbb{F}$ over $\mathbb{W}$.

Example 2. Consider a semigroup $\mathbb{F} = \{r, j, k, m\}$ with the binary operation $*$ as in table 2 .

Table 2: Tabular representation of $*$ in $\mathbb{F}$

$*$	r	j	k	m
r	r	r	r	r
j	r	r	r	r
k	r	r	j	r
m	r	r	j	j

Define the hesitant hybrid structure $\tilde{\mathfrak{h}}_\alpha$ in $\mathbb{F}$ over $\mathbb{W} = \{u_1, u_2, u_3, u_4, u_5\}$ is given by Table 3.

Table 3: Tabular representation of the hesitant hybrid structure $\tilde{\mathfrak{h}}_\alpha$

$\mathbb{F}$	$\tilde{\mathfrak{h}}$	α
r	$\mathbb{W}$	$\{0.2\}$
j	$\{u_2, u_3, u_4\}$	$[0, 0.5]$
k	$\{u_3\}$	$[0, 1]$
m	$\{u_2, u_3\}$	$[0, 0.5] \cup [0.5, 0.7]$

Then $\tilde{\mathfrak{h}}_\alpha$ is a hesitant hybrid subsemigroup and hybrid ideal in $\mathbb{F}$ over $\mathbb{W}$.

Note: It is clear that, all hesitant hybrid right(resp., left) ideal in $\mathbb{F}$ over $\mathbb{W}$ is a hesitant hybrid subsemigroup on $\mathbb{F}$. The opposite, however, is not always true. In fact, the hesitant hybrid subsemigroup $\tilde{\mathfrak{h}}_\alpha$ in Example 1 is not a hesitant hybrid right ideal of $\mathbb{F}$ over $\mathbb{W} = Z$ since $\tilde{\mathfrak{h}}(3 * 4) = \tilde{\mathfrak{h}}(2) = 4Z \not\supseteq \tilde{\mathfrak{h}}(3) = 2Z$ and $\alpha(3 * 4) = \alpha(2) = [0, 1] \nsubseteq \alpha(3) = [0, 0.3) \cup [0.3, 0.8]$.

Theorem 1. *Let $\mathcal{R}, \mathcal{B} \in \mathscr{P}(\mathbb{W}) \setminus \{\phi\}$ and $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$. Then the below conditions are hold:*

(i) $\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}_{(v,t)}}(\tilde{\mathfrak{h}}_\alpha)\right) \Cap \left(\chi^{(\epsilon,\delta)}_{\mathcal{B}_{(v,t)}}(\tilde{\mathfrak{h}}_\alpha)\right) = \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}\cap\mathcal{B}_{(v,t)}}(\tilde{\mathfrak{h}}_\alpha)\right).$

(ii) $\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}_{(v,t)}}(\tilde{\mathfrak{h}}_\alpha)\right) \odot \left(\chi^{(\epsilon,\delta)}_{\mathcal{B}_{(v,t)}}(\tilde{\mathfrak{h}}_\alpha)\right) = \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}\mathcal{B}_{(v,t)}}(\tilde{\mathfrak{h}}_\alpha)\right).$

Proof. (i) Let $g \in \mathbb{F}$. If $g \in \mathcal{R} \cap \mathcal{B}$, then

$$\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})} \tilde{\cap} \chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g = \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_g \cap \left(\chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g = \epsilon = \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}\cap\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g,$$

$$\left(\chi^{(v,t)}_{\mathcal{R}(\alpha)} \cup (\chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_g = \left(\chi^{(v,t)}_{\mathcal{R}(\alpha)}\right)_g \cup \left(\chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_g = v = \left(\chi^{(v,t)}_{\mathcal{R}\cap\mathcal{B}(\alpha)}\right)_g.$$

If $g \notin \mathcal{R} \cap \mathcal{B}$, then $g \notin \mathcal{R}$ or $g \notin \mathcal{B}$. Hence

$$\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})} \tilde{\cap} \chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g = \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_g \cap \left(\chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g = \delta = \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}\cap\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g,$$

$$\left(\chi^{(v,t)}_{\mathcal{R}(\alpha)} \cup \chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_g = \left(\chi^{(v,t)}_{\mathcal{R}(\alpha)}\right)_g \cup \left(\chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_g = t = \left(\chi^{(v,t)}_{\mathcal{R}\cap\mathcal{B}(\alpha)}\right)_g.$$

It follows that

$$\begin{aligned}\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}_{(v,t)}}(\tilde{\mathfrak{h}}_\alpha) \Cap \chi^{(\epsilon,\delta)}_{\mathcal{B}_{(v,t)}}(\tilde{\mathfrak{h}}_\alpha)\right)_g &= \left\{\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})} \tilde{\cap} \chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g, \left(\chi^{(v,t)}_{\mathcal{R}(\alpha)} \cup \chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_g\right\} \\ &= \left\{\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}\tilde{\cap}\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g, \left(\chi^{(v,t)}_{\mathcal{R}\cup\mathcal{B}(\alpha)}\right)_g\right\} \\ &= \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}\cap\mathcal{B}_{(v,t)}}(\tilde{\mathfrak{h}}_\alpha)\right).\end{aligned}$$

(ii) For any $g \in \mathbb{F}$, if $g \in \mathcal{R}\mathcal{B}$, then $g = rb$ for some $r \in \mathcal{R}$ and $b \in \mathcal{B}$. Now,

$$\begin{aligned}\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})} \tilde{\circ} \chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g &= \bigcup_{g=rb}\left\{\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_r \cap \left(\chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_b\right\} \\ &\supseteq \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_r \cap \left(\chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_b = \epsilon, \\ \left(\chi^{(v,t)}_{\mathcal{R}(\alpha)} \tilde{\circ} \chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_g &= \bigcap_{g=rb}\left\{\left(\chi^{(v,t)}_{\mathcal{R}(\alpha)}\right)_r \cup \left(\chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_b\right\} \\ &\subseteq \left(\chi^{(v,t)}_{\mathcal{R}(\alpha)}\right)_r \cap \left(\chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_b = v.\end{aligned}$$

Since $g \in \mathcal{R}\mathcal{B}$, we get $\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g = \epsilon$ and $\left(\chi^{(v,t)}_{\mathcal{R}\mathcal{B}(\alpha)}\right)_g = v$.
Suppose $g \notin \mathcal{R}\mathcal{B}$. Then $g \neq rb$ for all $r \in \mathcal{R}$ and $b \in \mathcal{B}$. If $g = cx$ for some $c, x \in \mathbb{F}$, then $c \notin \mathcal{R}$ *or* $x \notin \mathcal{B}$. Now,

$$\begin{aligned}\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})} \tilde{\circ} \chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g &= \bigcup_{g=cx}\left\{\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_c \cap \left(\chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_x\right\} = \delta = \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g, \\ \left(\chi^{(v,t)}_{\mathcal{R}(\alpha)} \tilde{\circ} (\chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_g &= \bigcap_{g=cx}\left\{\left(\chi^{(v,t)}_{\mathcal{R}(\alpha)}\right)_c \cup \left(\chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_x\right\} = t = \left(\chi^{(v,t)}_{\mathcal{R}\mathcal{B}(\alpha)}\right)_g.\end{aligned}$$

In any case, we have
$\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})} \tilde{\circ} \chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g = \left(\chi^{(\epsilon,\delta)}_{\mathcal{R}\mathcal{B}(\tilde{\mathfrak{h}})}\right)_g$ and $\left(\chi^{(v,t)}_{\mathcal{R}(\alpha)} \tilde{\circ} (\chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)_g = \left(\chi^{(v,t)}_{\mathcal{R}\mathcal{B}(\alpha)}\right)_g$ for all $g \in \mathbb{F}$. Hence,

$$\begin{aligned}\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(v,t)}(\tilde{\mathfrak{h}}_\alpha) \odot \chi^{\epsilon,\delta}_{\mathcal{B}(v,t)}(\tilde{\mathfrak{h}}_\alpha)\right) &= \left\{\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}(\tilde{\mathfrak{h}})} \tilde{\circ} \chi^{(\epsilon,\delta)}_{\mathcal{B}(\tilde{\mathfrak{h}})}\right)_, \left(\chi^{(v,t)}_{\mathcal{R}(\alpha)} \tilde{\circ} \chi^{(v,t)}_{\mathcal{B}(\alpha)}\right)\right\} \\ &= \left\{\left(\chi^{(\epsilon,\delta)}_{\mathcal{R}\mathcal{B}(\tilde{\mathfrak{h}})}\right), \left(\chi^{(v,t)}_{\mathcal{R}\mathcal{B}(\alpha)}\right)\right\} = \left(\chi^{(v,t)}_{\mathcal{R}\mathcal{B}(\epsilon,\delta)}(\tilde{\mathfrak{h}_\alpha})\right).\end{aligned}$$

□

Theorem 2. *For the identity hesitant hybrid structure $(\chi^{(\epsilon,\delta)}_{\mathbb{F}(v,t)}(\tilde{y}_\varrho))$, let $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$ such that $\tilde{\mathfrak{h}}_g \subseteq \epsilon$ and $\alpha_g \supseteq v$ for all $g \in \mathbb{F}$, the below statements are equivalent:*
(i) $\tilde{\mathfrak{h}}_\alpha$ is a hesitant hybrid left ideal of $\mathbb{F}$,
(ii)$\Big(\chi^{(\epsilon,\delta)}_{\mathbb{F}(v,t)}(\tilde{y}_\varrho)\Big) \odot \tilde{\mathfrak{h}}_\alpha \ll \tilde{\mathfrak{h}}_\alpha$.

Proof. $(i) \Longrightarrow (ii)$ Assume that $\tilde{\mathfrak{h}}_\alpha$ of $\mathbb{F}$ is a hesitant hybrid left ideal and $g \in \mathbb{F}$. If $g = cx$ for some $c, x \in \mathbb{F}$, then

$$\left(\chi^{(\epsilon,\delta)}_{\mathbb{F}(\tilde{y})}\tilde{\circ}\tilde{\mathfrak{h}}\right)_g = \bigcup_{g=cx}\left\{(\chi^{(\epsilon,\delta)}_{\mathbb{F}(\tilde{y})})_c \cap (\tilde{\mathfrak{h}})_x\right\} \subseteq \bigcup_{g=cx}\{\epsilon \cap \tilde{\mathfrak{h}}_{cx}\} = \tilde{\mathfrak{h}}_g,$$

$$\left(\chi^{(v,t)}_{\mathbb{F}(\varrho)}\tilde{\circ}\alpha\right)_g = \bigcap_{g=cx}\{(\chi^{(v,t)}_{\mathbb{F}(\varrho)})_c \cup \alpha_x\} \supseteq \bigcap_{g=cx}\{v \cup \alpha_{cx}\} = \alpha_g.$$

Otherwise, we have $\left(\chi^{(\epsilon,\delta)}_{\mathbb{F}(\tilde{y})}\tilde{\circ}\tilde{\mathfrak{h}}\right)_g = \phi \subseteq \tilde{\mathfrak{h}}_g$ and $\left(\chi^{(v,t)}_{\mathbb{F}(\varrho)}\tilde{\circ}\alpha\right)_g = [0,1] \supseteq \alpha_g$.
So, $\Big(\chi^{(\epsilon,\delta)}_{\mathbb{F}(v,t)}(\tilde{y}_\varrho)\Big) \odot \tilde{\mathfrak{h}}_\alpha \ll \tilde{\mathfrak{h}}_\alpha$.
$(ii) \Longrightarrow (i)$ Assume that $\Big(\chi^{(\epsilon,\delta)}_{\mathbb{F}(v,t)}(\tilde{y}_\varrho)\Big) \odot \tilde{\mathfrak{h}}_\alpha \ll \tilde{\mathfrak{h}}_\alpha$. For any $c, x \in \mathbb{F}$, we have

$$\tilde{\mathfrak{h}}_{cx} \supseteq \left(\chi^{(\epsilon,\delta)}_{\mathbb{F}(\tilde{y})}\tilde{\circ}\tilde{\mathfrak{h}}\right)_{cx} \supseteq \left(\chi^{(\epsilon,\delta)}_{\mathbb{F}(\tilde{y})}\right)_c \cap \tilde{\mathfrak{h}}_x = \epsilon \cap \tilde{\mathfrak{h}}_x = \tilde{\mathfrak{h}}_x,$$

$$\alpha_{cx} \subseteq \left(\chi^{(v,t)}_{\mathbb{F}(\varrho)}\tilde{\circ}\alpha\right)_{cx} \subseteq \left(\chi^{(v,t)}_{\mathbb{F}(\varrho)}\right)_c \cup \alpha_x = v \cup \alpha_x = \alpha_x.$$

Hence, $\tilde{\mathfrak{h}}_\alpha$ is a hesitant hybrid left ideal of $\mathbb{F}$ over $\mathbb{W}$. □

Theorem 3. *For the identity hesitant hybrid structure $(\chi^{(\epsilon,\delta)}_{\mathbb{F}(v,t)}(\tilde{y}_\varrho))$, and $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$ such that $\tilde{\mathfrak{h}}_g \subseteq \epsilon$ and $\alpha_g \supseteq v$ for all $g \in \mathbb{F}$, the below statements are equivalent:*
(i) $\tilde{\mathfrak{h}}_\alpha$ is a hesitant hybrid right ideal of $\mathbb{F}$,
(ii) $\tilde{\mathfrak{h}}_\alpha \odot \Big(\chi^{(\epsilon,\delta)}_{\mathbb{F}(v,t)}(\tilde{y}_\varrho)\Big) \ll \tilde{\mathfrak{h}}_\alpha$.

Proof. This theorem's proof is similar to that of Theorem 2. □

The subsequent theorem results from the combined result of Theorems 2 and 3.

Theorem 4. *For the identity hesitant hybrid structure $(\chi^{(\epsilon,\delta)}_{\mathbb{F}(v,t)}(\tilde{y}_\varrho))$ and $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$ such that $\tilde{\mathfrak{h}}_g \subseteq \epsilon$ and $\alpha_g \supseteq v$ for all $g \in \mathbb{F}$, the following assertions are equivalent:*
(i) $\tilde{\mathfrak{h}}_\alpha$ is a hesitant hybrid two-sided ideal of $\mathbb{F}$,
(ii)$\Big(\chi^{(\epsilon,\delta)}_{\mathbb{F}(v,t)}(\tilde{y}_\varrho)\Big) \odot \tilde{\mathfrak{h}}_\alpha \ll \tilde{\mathfrak{h}}_\alpha$ and $\tilde{\mathfrak{h}}_\alpha \odot \Big(\chi^{(\epsilon,\delta)}_{\mathbb{F}(v,t)}(\tilde{y}_\varrho)\Big) \ll \tilde{\mathfrak{h}}_\alpha$.

Theorem 5. *For $\mathcal{R}(\neq \phi) \subseteq \mathbb{F}$ and $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$, the below assertions are equivalent:*
(i) $\mathcal{R}$ is a subsemigroup of $\mathbb{F}$,
(ii) $\Big(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\alpha)\Big)$ is a hesitant hybrid subsemigroup of $\mathbb{F}$.

Proof. $(i) \Longrightarrow (ii)$ Let $\mathcal{R}$ be a subsemigroup of $\mathbb{F}$ and $g, b \in \mathbb{F}$. Then we now prove that

$$\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\alpha)\right)_{gb} \tilde{\supseteq} \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\alpha)\right)^b_g \qquad (*).$$

We consider the following three cases:
Case 1: If $g, b \in \mathcal{R}$, then $gb \in \mathcal{R}$. So $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_g = \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_b = \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{gb} = \mathbb{W}$ and $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_g = \left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_b = \left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_{gb} = \phi$.
Case 2: If $g \notin \mathcal{R}$ or $b \notin \mathcal{R}$ and $gb \notin \mathcal{R}$, then $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_g = \phi$ and $\left(\chi^{(\{\phi\},[0,1])}_{\mathcal{R}(\alpha)}\right)_g = [0,1]$ or $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_b = \phi$ and $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_b = [0,1]$ imply that $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{gb} = \phi$, $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_{gb} = [0,1]$.
Case 3: If $g \notin \mathcal{R}$ or $b \notin \mathcal{R}$, and $gb \in \mathcal{R}$, then $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{gb} = \mathbb{W}$, $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_{gb} = \phi$. So $(*)$ satisfied.
$(ii) \Longrightarrow (i)$ Assume that $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\alpha)\right)$ of $\mathbb{F}$ is a hesitant hybrid subsemigroup and let $g, b \in \mathcal{R}$, then $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_g = \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_b = \mathbb{W}$ and $\left(\chi^{(\{\phi\},[0,1])}_{\mathcal{R}(\alpha)}\right)_g = \left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_b = \phi$. So $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{gb} \supseteq \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)^b_g = \mathbb{W}$ and $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_{gb} \subseteq \left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)^b_g = \phi$ which imply $gb \in \mathcal{R}$. Thus $\mathcal{R}$ is a subsemigroup of $\mathbb{F}$. □

Theorem 6. *For $\mathcal{R}(\neq \phi) \subseteq \mathbb{F}$ and $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$, the below assertions are equivalent:*
(i) $\mathcal{R}$ is a left (resp., right, two sided)ideal of $\mathbb{F}$,
(ii) $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\alpha)\right)$ is a hesitant hybrid left (resp., right, two-sided) ideal of $\mathbb{F}$.

Proof. $(i) \Longrightarrow (ii)$ Assume that $\mathcal{R}$ is a left ideal of $\mathbb{F}$ and $g, b \in \mathbb{F}$. We now prove that

$$\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\alpha)\right)_{gb} \tilde{\supseteq} \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\alpha)\right)_b \qquad (*)$$

If $b \notin \mathcal{R}$, then $\left(\chi^{(W,\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_b = \phi \subseteq \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{gb}$ and $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_b = [0,1] \supseteq \left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_{gb}$.
If $b \in \mathcal{R}$ and $g \in \mathbb{F}$, then $gb \in \mathcal{R}$. Hence $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{gb} = \mathbb{W}$ and $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_{gb} = \phi$. Hence $(*)$ is satisfied.
$(ii) \Longrightarrow (i)$ Assume that $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\alpha)\right)$ of $\mathbb{F}$ is a hybrid left ideal and $g \in \mathbb{F}, b \in \mathcal{R}$. Then $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_b = \mathbb{W}$ and $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_b = \phi$, and so $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{gb} \supseteq \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_b = \mathbb{W}$ and $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_{gb} \subseteq \left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\alpha)}\right)_b = \phi$. Hence $gb \in \mathcal{R}$. So $\mathcal{R}$ is a left ideal of $\mathbb{F}$. Similarly, we can prove the other cases. □

Theorem 7. *Let $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$. Then, the below statements are equivalent:*
(i) $\tilde{\mathfrak{h}}_\alpha$ is a hesitant hybrid subsemigroup of $\mathbb{F}$,
(ii) For $\Gamma \in \mathscr{P}(\mathbb{W})$ and $\xi \in \mathscr{P}([0,1])$, $\tilde{\mathfrak{h}}_\alpha[\Gamma, \xi](\neq \phi)$ is a subsemigroup of $\mathbb{F}$.

Proof. $(i) \Longrightarrow (ii)$ Let $a, b \in \tilde{\mathfrak{h}}_\alpha[\Gamma, \xi]$. Then $\tilde{\mathfrak{h}}_a^b = \tilde{\mathfrak{h}}_a \cap \tilde{\mathfrak{h}}_b \subseteq \tilde{\mathfrak{h}}_{ab} \supseteq \Gamma$ and $\alpha_a^b = \alpha_a \cup \alpha_b \supseteq \alpha_{ab} \subseteq \xi$. So $ab \in \tilde{\mathfrak{h}}_\alpha[\Gamma, \xi]$ and $\tilde{\mathfrak{h}}_\alpha[\Gamma, \xi]$ is a subsemigroup of $\mathbb{F}$.
$(ii) \Longrightarrow (i)$ Suppose that for any $\Gamma \in \mathscr{P}(\mathbb{W})$ *and* $\xi \in \mathscr{P}([0,1])$, $\tilde{\mathfrak{h}}_\alpha[\Gamma, \xi](\neq \phi)$ is a subsemigroup of $\mathbb{F}$. Let $x, c \in \mathbb{F}$. Then $\tilde{\mathfrak{h}}_x = \Gamma_1, \alpha_x = \xi_1$ and $\tilde{\mathfrak{h}}_c = \Gamma_2, \alpha_c = \xi_2$ for some $\Gamma_1, \Gamma_2 \in \mathscr{P}(\mathbb{W})$ and $\xi_1, \xi_2 \in \mathscr{P}([0,1])$. If $\Gamma = \Gamma_1 \cap \Gamma_2$ and $\xi = \xi_1 \cup \xi_2$, then $\tilde{\mathfrak{h}}_x \supseteq \Gamma, \alpha_x \subseteq \xi$ and $\tilde{\mathfrak{h}}_c \supseteq \Gamma, \alpha_c \subseteq \xi$, so $x, c \in \tilde{\mathfrak{h}}_\alpha[\Gamma, \xi]$. Since $\tilde{\mathfrak{h}}_\alpha[\Gamma, \xi]$ is a subsemigroup, we have $xc \in \tilde{\mathfrak{h}}_\alpha[\Gamma, \xi]$, which gives $\tilde{\mathfrak{h}}_{xc} \supseteq \tilde{\mathfrak{h}}_x^c \supseteq \Gamma = \Gamma_1 \cap \Gamma_2 = \tilde{\mathfrak{h}}_x \cap \tilde{\mathfrak{h}}_c$ and $\alpha_{xc} \subseteq \alpha_x^c \subseteq \xi = \xi_1 \cup \xi_2 = \alpha_x \cup \alpha_c$. So $\tilde{\mathfrak{h}}_\alpha$ is a hesitant hybrid subsemigroup of $\mathbb{F}$. □

The idea of a hesitant fuzzy duo was presented to semigroup theory by A. F. Talee et al. [16]. The notion of a hesitant hybrid duo in a semigroup is now presented below:

Definition 12. *If each hesitant hybrid right (resp., left) ideal of* $\mathbb{F}$ *is a hesitant hybrid two-sided ideal of* $\mathbb{F}$*, then* $\mathbb{F}$ *is known as a right (resp., left) duo. If* $\mathbb{F}$ *is both a hesitant hybrid right and a hesitant hybrid left duo, then* $\mathbb{F}$ *is known as a hesitant hybrid duo.*

Theorem 8. *The below statements are equivalent for a regular semigroup* $\mathbb{F}$*:*
(i) $\mathbb{F}$ *is left(resp., right, two-sided) duo,*
(ii) $\mathbb{F}$ *is a hesitant hybrid left(resp., hesitant hybrid right, hesitant hybrid two-sided) duo.*

Proof. $(1) \Longrightarrow (2)$ Let $\tilde{\mathfrak{h}}_\alpha$ of $\mathbb{F}$ be a hesitant hybrid left ideal and $a, b \in \mathbb{F}$. As the left ideal $\mathbb{F}a$ is a two-sided ideal of $\mathbb{F}$, and $\mathbb{F}$ is regular, we have $ab \in (a\mathbb{F}a)b \subseteq (\mathbb{F}a)\mathbb{F} \subseteq \mathbb{F}a$. This indicates that $\exists\, g \in \mathbb{F}$ such that $ab = ga$. Since $\tilde{\mathfrak{h}}_\alpha$ of $\mathbb{F}$ is a hesitant hybrid left ideal, we have $\tilde{\mathfrak{h}}_{ab} = \tilde{\mathfrak{h}}_{ga} \supseteq \tilde{\mathfrak{h}}_a$, $\alpha_{ab} = \alpha_{ga} \subseteq \alpha_a$. This means that $\tilde{\mathfrak{h}}_\alpha$ of $\mathbb{F}$ is a hesitant hybrid right ideal, and so $\tilde{\mathfrak{h}}_\alpha$ of $\mathbb{F}$ is a hesitant hybrid two-sided ideal. $(2) \Longrightarrow (1)$ Let $\mathbb{F}$ be a hesitant hybrid left duo and $\mathcal{R}$ be a left ideal of $\mathbb{F}$. By Theorem 6, $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}_\alpha})\right)$ of $\mathcal{R}(\neq \phi)$ is a hesitant hybrid left ideal of $\mathbb{F}$. By our assumption $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}_\alpha})\right)$ is a hesitant hybrid ideal of $\mathbb{F}$. Since $\mathcal{R} \neq \phi$, it follows from Theorem 6 that $\mathcal{R}$ is an ideal of $\mathbb{F}$. Hence $\mathbb{F}$ is left duo. Likewise, we can verify another claim. □

Theorem 9. *The following statements are equivalent for a regular semigroup* $\mathbb{F}$:
(i) *A left (resp., right) zero semigroup of* $\mathbb{F}$ *is formed by the set of all idempotent elements* G *of* $\mathbb{F}$*,*
(ii) *For every hesitant hybrid left (resp., right) ideal* $\tilde{\mathfrak{h}_\alpha}$ *of* $\mathbb{F}$*,* $(\tilde{\mathfrak{h}_\alpha})_b = (\tilde{\mathfrak{h}_\alpha})_m$ *holds for* b *and* m *of* $\mathbb{F}$*.*

Proof. $(i) \Longrightarrow (ii)$ Assume that the left zero semigroup of $\mathbb{F}$ is formed by the set G of all idempotents and let b, m be any elements of $\tilde{\mathfrak{h}}_\alpha$ be any hesitant hybrid left ideal of $\mathbb{F}$. Since $bm = b$ and $mb = m$, we have $\tilde{\mathfrak{h}}_b = \tilde{\mathfrak{h}}_{bm} \supseteq \tilde{\mathfrak{h}}_m = \tilde{\mathfrak{h}}_{mb} \supseteq \tilde{\mathfrak{h}}_b$, $\alpha_b = \alpha_{bm} \subseteq \alpha_m = \alpha_{mb} = \alpha_b$ and so $(\tilde{\mathfrak{h}_\alpha})_b = (\tilde{\mathfrak{h}_\alpha})_m$.
$(ii) \Longrightarrow (i)$ Suppose that for every hesitant hybrid left ideal $\tilde{\mathfrak{h}_\alpha}$ of $\mathbb{F}$, $(\tilde{\mathfrak{h}_\alpha})_b = (\tilde{\mathfrak{h}_\alpha})_m$ for all idempotent elements b and m of $\mathbb{F}$. Let $\langle b \rangle_L = \{b\} \cup \mathbb{F}b$. Since $\langle b \rangle_L$ is a left ideal of $\mathbb{F}$ then by Theorem 6 we have, $\left(\chi^{(\mathbb{W},\phi)}_{\langle b \rangle_L(\phi,[0,1])}(\tilde{\mathfrak{h}}_\alpha)\right)$ for $\tilde{\mathfrak{h}_\alpha}$ is hesitant hybrid left ideal of $\mathbb{F}$. Since $b \in \langle b \rangle_L$ implies $\left(\chi^{(\mathbb{W},\phi)}_{\langle b \rangle_L(\tilde{\mathfrak{h}})}\right)_m = \left(\chi^{(\mathbb{W},\phi)}_{\langle b \rangle_L(\tilde{\mathfrak{h}})}\right)_b = \mathbb{W}$ and $\left(\chi^{(\phi,[0,1])}_{\langle b \rangle_L(\alpha)}\right)_m = \left(\chi^{(\phi,[0,1])}_{\langle b \rangle_L(\alpha)}\right)_b = \phi$ and so $m \in \langle b \rangle_L =$

$\mathbb{F}b$. Then for some $q \in \mathbb{F}$, we have $m = qb = qbb = mb$. Hence the set of all idempotent elements G of $\mathbb{F}$ is a left zero semigroup. $\square$

Theorem 10. *Let $\mathbb{F}$ be a monoid with identity e. For $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$, the smallest hesitant hybrid left ideal $(\tilde{\mathfrak{h}_\alpha})_l$ on $\mathbb{F}$ containing $\tilde{\mathfrak{h}}_\alpha$ is described below:*

$$(\tilde{\mathfrak{h}}_\alpha)_l : \mathbb{F} \longrightarrow \mathscr{P}(\mathbb{W}) \times \mathscr{P}([0,1]),$$

$$\begin{aligned} \textit{where}\quad (\tilde{\mathfrak{h}})_l : \mathbb{F} \to \mathscr{P}(\mathbb{W}), m &\longrightarrow \bigcup \left\{ \tilde{\mathfrak{h}}_g | m = cg, c, g \in \mathbb{F} \right\}, \\ (\alpha)_l : \mathbb{F} \to \mathscr{P}([0,1]), m &\longrightarrow \bigcap \left\{ \alpha_g | m = cg, c, g \in \mathbb{F} \right\}. \end{aligned}$$

Proof. Let $\tilde{k}_\tau$ be a hesitant hybrid structure in $\mathbb{F}$ over $\mathbb{W}$ defined as follows:
$\tilde{k}_m := \bigcup \left\{ \tilde{\mathfrak{h}}_g | m = cg,\ c, g \in \mathbb{F} \right\}$, $\tau_m := \bigcap \left\{ \alpha_g | m = cg,\ c, g \in \mathbb{F} \right\}$ for any $m \in \mathbb{F}$. Since $m = em$, we have $(\tilde{k_\tau})_m \tilde{\supseteq} (\tilde{\mathfrak{h}_\alpha})_m$ and so $(\tilde{k_\tau}) \tilde{\supseteq} (\tilde{\mathfrak{h}_\alpha})$. For any $c, g \in \mathbb{F}$, we have

$$\begin{aligned} \tilde{k}_{cg} &:= \bigcup \left\{ \tilde{\mathfrak{h}}_{c_2} | cg = c_1 c_2,\quad c_1, c_2 \in \mathbb{F} \right\} \\ &\supseteq \bigcup \left\{ \tilde{\mathfrak{h}}_{p_2} | cg = (cp_1) p_2,\quad g = p_1, p_2 \in \mathbb{F} \right\} \\ &\supseteq \bigcup \left\{ \tilde{\mathfrak{h}}_{p_2} | g = p_1 p_2,\quad p_1, p_2 \in \mathbb{F} \right\} = \tilde{k}_g, \\ \tau_{cg} &:= \bigcap \left\{ \tau_{c_2} | cg = c_1 c_2,\quad c_1, c_2 \in \mathbb{F} \right\} \\ &\subseteq \bigcap \left\{ \tau_{p_2} | cg = (cp_1) p_2,\quad g = p_1, p_2 \in \mathbb{F} \right\} \\ &\subseteq \bigcap \left\{ \tau_{p_2} | g = p_1 p_2,\quad p_1, p_2 \in \mathbb{F} \right\} = \tau_g. \end{aligned}$$

Thus $\tilde{k}_\tau$ of $\mathbb{F}$ is any hesitant hybrid left ideal. Assume that $\tilde{y}_\varrho$ of $\mathbb{F}$ is a hesitant hybrid left ideal so $\tilde{\mathfrak{h}_\alpha} \tilde{\subseteq} \tilde{y}_\varrho$. Then $(\tilde{\mathfrak{h}}_\alpha)_m \tilde{\subseteq} (\tilde{y}_\varrho)_m\ \forall\ m \in \mathbb{F}$.

$$\begin{aligned} \tilde{k}_m &:= \bigcup \left\{ \tilde{\mathfrak{h}}_{c_2} | m = c_1 c_2,\quad c_1, c_2 \in \mathbb{F} \right\} \\ &\subseteq \bigcup \left\{ \tilde{y}_{c_2} | m = c_1 c_2,\quad c_1, c_2 \in \mathbb{F} \right\} \\ &\subseteq \bigcup \left\{ \tilde{y}_{c_1 c_2} | m = c_1 c_2,\quad c_1, c_2 \in \mathbb{F} \right\} = \tilde{y}_m, \end{aligned}$$

$$\begin{aligned} \tau_m &:= \bigcap \left\{ \tau_{c_2} | m = c_1 c_2,\quad g_1, g_2 \in \mathbb{F} \right\} \\ &= \bigcap \left\{ \tau_{c_2} | m = c_1 c_2,\quad c_1, c_2 \in \mathbb{F} \right\} \\ &= \bigcap \left\{ \tau_{c_1 c_2} | m = c_1 c_2,\quad c_1, c_2 \in \mathbb{F} \right\} = \varrho_m. \end{aligned}$$

Thus $\tilde{k}_\tau \tilde{\subseteq} \tilde{y}_\varrho$ and hence $(\tilde{\mathfrak{h}_\alpha})_l = \tilde{k}_\tau$. $\square$

The right dual of Theorem 10 is as described below.

Theorem 11. *Let $\mathbb{F}$ be a monoid with identity e. For $\tilde{\mathfrak{h}}_\alpha \in \mathbb{HH}(\mathbb{F})$, the smallest hesitant hybrid right ideal $[\tilde{\mathfrak{h}_\alpha}]_r$ on $\mathbb{F}$ containing $\tilde{\mathfrak{h}}_\alpha$ is described below:*

$$(\tilde{\mathfrak{h}}_\alpha)_r : \mathbb{F} \longrightarrow \mathscr{P}(\mathbb{W}) \times \mathscr{P}([0,1]),$$

$$\begin{aligned} where \quad (\tilde{\mathfrak{h}})_r : \mathbb{F} \to \mathscr{P}(\mathbb{W}), m \longrightarrow \bigcup \left\{ \tilde{\mathfrak{h}}_c | m = cg, c, g \in \mathbb{F} \right\}, \\ (\alpha)_l : \mathbb{F} \to \mathscr{P}([0,1]), m \longrightarrow \bigcap \left\{ \alpha_c | m = cg, c, g \in \mathbb{F} \right\}. \end{aligned}$$

$(\tilde{\mathfrak{h}}_\alpha)_g = (\tilde{\mathfrak{h}}_\alpha)_c \ \forall \ g, c \in \mathcal{R}$.

The concept of hesitant fuzzy interior ideals was first introduced to semigroup by A.F. Talee et al. [18]. Here we define the notion of the hesitant hybrid interior ideals in semigroups.

Definition 13. *A hesitant hybrid subsemigroup $\tilde{\mathfrak{h}}_\eta$ in $\mathbb{F}$ over $\mathbb{W}$ is known as hesitant hybrid interior ideal in $\mathbb{F}$ over $\mathbb{W}$ if it satisfies*

$$(\forall \ a, b, c \in \mathbb{F}) \quad (\tilde{\mathfrak{h}}_{abc} \supseteq \tilde{\mathfrak{h}}_b \quad and \quad \eta_{abc} \subseteq \eta_b).$$

Example 3. Consider a $\mathbb{F} = \{m_1, m_2, m_3, m_4\}$ semigroup with the below multiplication table (Table 4).

Table 4: Multiplication table of a semigroup $\tilde{\mathfrak{h}}_\alpha$

$*$	m_1	m_2	m_3	m_4
m_1	m_1	m_1	m_1	m_1
m_2	m_1	m_1	m_1	m_1
m_3	m_1	m_1	m_2	m_1
m_4	m_1	m_1	m_2	m_2

Let $\tilde{\mathfrak{h}}_\eta$ be a hesitant hybrid structure in $\mathbb{F}$ over $\mathbb{W} = [0,1]$ defined by $\tilde{\mathfrak{h}}_{(m_1)} = [0,1], \tilde{\mathfrak{h}}_{(m_2)} = [0,0.7], \tilde{\mathfrak{h}}_{(m_3)} = [0,0.4], \tilde{\mathfrak{h}}_{(m_4)} = [0,0.2]$ and η be any constant mapping from $\mathbb{F}$ to I. Then $\tilde{\mathfrak{h}}_\eta$ is a hesitant hybrid interior ideal in $\mathbb{F}$ over $\mathbb{W}$.

Theorem 12. *For $\tilde{\mathfrak{h}}_\eta \in \mathbb{HH}(\mathbb{F})$. If $\tilde{\mathfrak{h}}_\eta$ is a hesitant hybrid ideal of $\mathbb{F}$, then $\tilde{\mathfrak{h}}_\eta$ is a hesitant hybrid interior ideal of $\mathbb{F}$.*

Proof. Consider a hesitant hybrid ideal $\tilde{\mathfrak{h}}_\eta$ of $\mathbb{F}$. For any $a, b, c \in \mathbb{F}$, we have $\tilde{\mathfrak{h}}_{abc} = \tilde{\mathfrak{h}}_{a(bc)} \supseteq \tilde{\mathfrak{h}}_{bc} \supseteq \tilde{\mathfrak{h}}_b$ and $\eta_{abc} = \eta_{a(bc)} \subseteq \eta_{bc} \subseteq \eta_b$. So, $\tilde{\mathfrak{h}}_\eta$ is a hesitant hybrid interior ideal of $\mathbb{F}$. □

The example below illustrates the above theorem is not true in general.

Example 4. Consider $\mathbb{F} = \{2b \cup \{0\} | b \ is \ a$ natural number$\}$. Then $\mathbb{F}$ forms a semigroup with respect to the usual multiplication. Consider a hesitant hybrid structure $\tilde{\mathfrak{h}}_\eta$ in $\mathbb{F}$ over $\mathbb{W} = [0,1]$ by:

$$\tilde{\mathfrak{h}}_b = \begin{cases} [0, 0.82] & if \quad b \ = \ 0 \\ [0, 0.58] & if \quad b \ = \ 2 \\ [0, 0.43] & if \quad b \ = \ 4, 12 \\ [0, 0.79] & otherwise, \end{cases} \quad \text{and } \eta : \mathbb{F} \longrightarrow [0,1] \text{ is constant.}$$

Then $\tilde{\mathfrak{h}}_\eta$ is a hesitant hybrid interior ideal of $\mathbb{F}$ but it is not a hesitant hybrid ideal of $\mathbb{F}$ as $\tilde{\mathfrak{h}}(6\times 2) \not\supseteq \tilde{\mathfrak{h}}(2)$.

Theorem 13. *For $\mathcal{R}(\neq \phi) \subseteq \mathbb{F}$ and $\tilde{\mathfrak{h}}_\eta \in \mathbb{HH}(\mathbb{F})$, the below statements are equivalent:*
(i) $\mathcal{R}$ is an interior ideal of $\mathbb{F}$,
(ii) $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\eta)\right)$ is hesitant hybrid interior ideal of $\mathbb{F}$.

Proof. $(i) \Longrightarrow (ii)$ Assume that $\mathcal{R}$ is an interior ideal of $\mathbb{F}$. For any $g,p,m \in \mathbb{F}$.

$$\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\eta)\right)_{gpm} \tilde{\supseteq} \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\eta)\right)_{p} \qquad (*)$$

If $p \in \mathcal{R}$, then $gpn \in \mathbb{F}\mathcal{R}\mathbb{F} \subseteq \mathcal{R}$. This implies $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{gpm} = \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{p} = \mathbb{W}$ and $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\eta)}\right)_{gpm} = \left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\eta)}\right)_{p} = \phi$.

If $p \notin \mathcal{R}$,then $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{p} = \phi \subseteq \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{gpm}$ and $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\eta)}\right)_{p} = [0,1] \supseteq \left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\eta)}\right)_{gpm}$. Thus $(*)$ is satisfied. $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\eta)\right)$ in $\mathbb{F}$ over $\mathbb{W}$ is hesitant hybrid subsemigroup of $\mathbb{F}$ by Theorem 5. Hence $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\eta)\right)$ in $\mathbb{F}$ over $\mathbb{W}$ is hesitant hybrid interior ideal of $\mathbb{F}$.

$(ii) \Longrightarrow (i)$ Assume that $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\phi,[0,1])}(\tilde{\mathfrak{h}}_\eta)\right)$ in $\mathbb{F}$ over $\mathbb{W}$ is a hesitant hybrid interior ideal of $\mathbb{F}$. Let $gpm \in \mathbb{F}\mathcal{R}\mathbb{F}$ such that $g,m \in \mathbb{F}, p \in \mathcal{R}$. Then $\left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{gpm} \supseteq \left(\chi^{(\mathbb{W},\phi)}_{\mathcal{R}(\tilde{\mathfrak{h}})}\right)_{p} = \mathbb{W}$ and $\left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\eta)}\right)_{gpm} \subseteq \left(\chi^{(\phi,[0,1])}_{\mathcal{R}(\eta)}\right)_{p} = \phi$ which indicates $gpm \in \mathcal{R}$ and $\mathbb{F}\mathcal{R}\mathbb{F} \subseteq \mathbb{F}$. So $\mathcal{R}$ is a subsemigroup of $\mathbb{F}$ by Theorem 5 and $\mathcal{R}$ is an interior ideal are of $\mathbb{F}$. □

The following theorem shows that in regular semigroup the concept of hesitant hybrid interior ideal and hesitant hybrid ideal coincide.

Theorem 14. *Let $\mathbb{F}$ be a regular semigroup and $\tilde{\mathfrak{h}}_\eta \in \mathbb{HH}(\mathbb{F})$. Then the below assertions are equivalent:*
(i) $\tilde{\mathfrak{h}}_\eta$ is a hesitant hybrid ideal of $\mathbb{F}$,
(ii) $\tilde{\mathfrak{h}}_\eta$ is a hesitant hybrid interior ideal of $\mathbb{F}$.

Proof. $(i) \Longrightarrow (ii)$ By Theorem 12, every hesitant hybrid ideal is a hesitant hybrid interior ideal of $\mathbb{F}$.
$(ii) \Longrightarrow (ii)$ Let $a,b \in \mathbb{F}$. Since $\mathbb{F}$ is regular, $\exists\ p,m \in \mathbb{F}$ such that $a = apa, b = bmb$. As $\tilde{\mathfrak{h}}_\eta$ is a hybrid interior ideal of $\mathbb{F}$, we have
$\tilde{\mathfrak{h}}_{ab} = \tilde{\mathfrak{h}}_{(apa)b} = \tilde{\mathfrak{h}}_{(ap)ab} \supseteq \tilde{\mathfrak{h}}_a$ and $\eta_{ab} = \eta_{(apa)b} = \eta_{(ap)ab} \subseteq \eta_a$.
$\tilde{\mathfrak{h}}_{ab} = \tilde{\mathfrak{h}}_{a(bmb)} = \tilde{\mathfrak{h}}_{ab(mb)} \supseteq \tilde{\mathfrak{h}}_b$ and $\eta_{ab} = \eta_{a(bmb)} = \eta_{ab(mb)} \subseteq \eta_b$.
Therefore, $\tilde{\mathfrak{h}}_\eta$ is a hesitant hybrid ideal of $\mathbb{F}$. □

Next theorem shows that the notions of hesitant hybrid interior ideal and hesitant hybrid ideal in an intra-regular semigroup are coincide.

Theorem 15. *Let $\mathbb{F}$ be an intra regular semigroup and $\tilde{\mathfrak{h}}_\eta \in \mathbb{HH}(\mathbb{F})$. Then the following conditions are equivalent:*

(i) $\tilde{\mathfrak{h}}_\eta$ *is a hesitant hybrid ideal of* $\mathbb{F}$,

(ii) $\tilde{\mathfrak{h}}_\eta$ *is a hesistant hybrid interior ideal of* $\mathbb{F}$.

Proof. $(i) \Longrightarrow (ii)$ By Theorem 12, every hesitant hybrid ideal is a hesitant hybrid interior ideal.
$(ii) \Longrightarrow (ii)$ Let $a, b \in \mathbb{F}$. Since $\mathbb{F}$ is intra - regular, $\exists\ p, m, r, s \in \mathbb{F}$ such that $a = pa^2m, b = rb^2s$. As $\tilde{\mathfrak{h}}_\eta$ is hybrid interior ideal, we have
$\tilde{\mathfrak{h}}_{ab} = \tilde{\mathfrak{h}}_{(pa^2m)b} = \tilde{\mathfrak{h}}_{(pa)a(mb)} \supseteq \tilde{\mathfrak{h}}_a$ and $\eta_{ab} = \eta_{(pa^2m)b} = \eta_{(pa)a(mb)} \subseteq \eta_a$.
$\tilde{\mathfrak{h}}_{ab} = \tilde{\mathfrak{h}}_{a(rb^2s)} = \tilde{\mathfrak{h}}_{(ar)b(bs)} \supseteq \tilde{\mathfrak{h}}_b$ and $\eta_{ab} = \eta_{a(rb^2s)} = \eta_{(ar)b(bs)} \subseteq \eta_b$.
Therefore, $\tilde{\mathfrak{h}}_\beta$ is a hesitant hybrid ideal of $\mathbb{F}$. □

Theorem 16. *Let $\mathbb{F}$ be an intra - regular semigroup. Then for each hesitant hybrid interior ideal $\tilde{\mathfrak{h}}_\beta$ of $\mathbb{F}$, $(\tilde{\mathfrak{h}}_\beta)_g = (\tilde{\mathfrak{h}}_\beta)_{g^2}$ and $(\tilde{\mathfrak{h}}_\beta)_{gb} = (\tilde{\mathfrak{h}}_\beta)_{bg}$ holds for all $g, b \in \mathbb{F}$.*

Proof. Let $\tilde{\mathfrak{h}}_\beta$ be a hesitant hybrid ideal of $\mathbb{F}$ and $g \in \mathbb{F}$. Since $\mathbb{F}$ is intra-regular semigroup, $\exists\ x, p \in \mathbb{F}$ such that $g = xg^2p$. Thus
$\tilde{\mathfrak{h}}_{g^2} \supseteq \tilde{\mathfrak{h}}_g = \tilde{\mathfrak{h}}_{xg^2p} \supseteq \tilde{\mathfrak{h}}_{g^2}$ and $\eta_{g^2} \subseteq \eta_g = \eta_{xg^2p} \subseteq \eta_{g^2}$. So we have $(\tilde{\mathfrak{h}}_\eta)_g = (\tilde{\mathfrak{h}}_\eta)_{g^2}$. Since $gb \in \mathbb{F}$, we have $\tilde{\mathfrak{h}}_{gb} = \tilde{\mathfrak{h}}_{(gb)^2} = \tilde{\mathfrak{h}}_{(g(bg)b)} \supseteq \tilde{\mathfrak{h}}_{bg}$ and $\beta_{gb} = \eta_{(gb)^2} = \eta_{(g(bg)b)} \subseteq \eta_{bg}$. Again, $\tilde{\mathfrak{h}}_{bg} = \tilde{\mathfrak{h}}_{(bg)^2} = \tilde{\mathfrak{h}}_{(b(gb)g)} \supseteq \tilde{\mathfrak{h}}_{gb}$ and $\eta_{bg} = \eta_{(bg)^2} = \eta_{(b(gb)g)} \subseteq \eta_{gb}$. So, $(\tilde{\mathfrak{h}}_\eta)_{gb} = (\tilde{\mathfrak{h}}_\eta)_{bg}$. □

4 Conclusion

In the current work, we proposed the notion of hesitant hybrid subsemigroups, hybrid left (resp., right) ideals, and hesitant hybrid product, hesitant hybrid interior ideals and looked into their associated features. Some important ideas for future work are as follows: (1) to develop strategies for obtaining more valuable results and (2) to apply these notions and results for studying related notions in other algebraic structures such as rings, hemirings, BL-algebras, MTL-algebras, R0-algebras, MV-algebras, EQalgebras, d-algebras, Q-algebras, and lattice implication algebras.

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